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Online Weakly DR-Submodular Optimization Under Stochastic Cumulative Constraints

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Abstract: In this paper, we study a class of online continuous optimization problems. At each round, the utility function is the sum of a weakly diminishing-returns (DR) submodular function and a concave function, certain cost associated with the action will occur, and the problem has total limited budget. Combining the two methods, the penalty function and Frank-Wolfe strategies, we present an online method to solve the considered problem. Choosing appropriate stepsize and penalty parameters, the performance of the online algorithm is guaranteed, that is, it achieves sub-linear regret bound and certain mild constraint violation bound in expectation.

Key words: online maximization; weakly DR-submodular; regret; stochastic

1 Introduction

In the era of information, a large number of data are produced rapidly, and it is urgent to make a relatively optimal choice for the future based on the history information. Repeat in this way, this is online optimization. Generally speaking, at each round $t \in [T]$, first the learner makes a decision $x_t \in \mathcal{X}$, where $\mathcal{X}$ is the fixed constraint set, then the environment feeds back the utility function $U_t : \mathcal{X} \to \mathbb{R}$. The goal is to make a sequence of decisions such that the following quantity is as small as possible:

$$\mathcal{R}_T := \max_{x \in \mathcal{X}} \sum_{t=1}^{T} U_t(x) - \sum_{t=1}^{T} U_t(x_t).$$

It means that we pursuit to get decisions $\{x_t\}_{t\in[T]}$, which can produce total utility as good as the largest utility we can get at the fixed point in hindsight to some extent. At a glance, since we make current decision only with history information, it seems impossible to get guaranteed result. However, some rigorous results about online optimization have been obtained, especially in convex setting[1, 2].

In real world applications, the involved functions may not have convex (concave) structure. It is well known that nonconvex optimization is hard to solve, both in offline and online settings. Submodularity, which is neither convex nor concave property, has been studied extensively in recent years, such as in machine learning[3, 4], etc. In the offline setting, consider the following problem:

$$\max_{x \in \mathcal{X}} U(x),$$

where $U : \mathcal{X} \to \mathbb{R}_+$ is of submodular property. There have been some guaranteed approximation algorithms, deterministic or stochastic type, under certain assumptions. That is, the algorithm outputs vector x_t after t iterations with

$$U(x_t) \geqslant r \max_{x \in \mathcal{X}} U(x) - \mathrm{loss}(t),$$

where $r > 0$ is the approximation ratio, and $\mathrm{loss}(t)$ reprensents the loss term[4–12] . It is natural to introduce certain factor r for the comparator in the online setting, that is, to measure the following:

$$\mathcal{R}_T^r := r \max_{x \in \mathcal{X}} \sum_{t=1}^{T} U_t(x) - \sum_{t=1}^{T} U_t(x_t).$$

It is obvious that we are devoted to design algorithms

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with higher factor r and lower upper bound on $\mathcal{R}_T^r$ which is needed at least sub-linear with respect to T, meaning that the total utility of the algorithm produced is as good as the utility at the fixed benchmark point in handsight multiplied by r in average. Guaranteed results for online continuous diminishing-returns (DR) submodular optimization can be found in Refs. [13–15], etc.

In the case of limited budget available, in addition to make actions to maximize the utility, we need to consider the consumption occured by decisions in every round such that the true cost does not violate much from the budget. Mathematically, the corresponding offline problems is

$$\begin{aligned} \max\ & \sum_{t=1}^{T} U_t(x_t), \\ \text{s.t.,}\ & \sum_{t=1}^{T} c_t(x_t) \leqslant B, \\ & x_t \in \mathcal{X},\ \forall t \in [T], \end{aligned}$$

where we have one more constraint. Note that, in the online setting, c_t may be given before or arrive in online form. Compared with the unconstrained form online problem, we need to add the measure for constraint violation, that is,

$$C_T := \sum_{t=1}^{T} c_t(x_t) - B.$$

For some certain settings of U_t and c_t, different online algorithms have been proposed[16–19].

Since the problem in real world is not of perfect property, it is necessary to study problems that cover more utility functions and constraint functions appeared in real application. In this work, we consider a class of online continuous weakly DR-submodular maximization problems with stochastic linear long term budget constraint. At each round, the utility function is the sum of two terms: one is weakly DR submodular, the other is concave, the revealed linear constraint vectors are stochastic and independent identically distributed with certain unknown distribution. By the approach of penalty function method and Frank-Wolfe method, we propose our online algorithm. The sub-linear regret bound is guaranteed in expectation under mild assumptions, as well as certain bound for constraint violation.

2 Preliminary

For any integer T, $[T]$ denotes the set of $\{t \in \mathbb{N} : 1 \leqslant t \leqslant T\}$. For any two vectors $x = (x_i)_{i\in[n]}$, $y = (y_i)_{i\in[n]} \in \mathbb{R}^n$, $x \preceq y$ means that $x_i \leqslant y_i$, $\forall i \in [n]$. $x \vee y$ and $x \wedge y$ denote the element wise maximum and minimum vector, respectively, $\forall i \in [n]$, that is

$$(x \vee y)_i = \max\{x_i, y_i\},\ (x \wedge y)_i = \min\{x_i, y_i\}.$$

Given nonempty set $S \subseteq \mathbb{R}^n$. If S is closed and convex, we use $\mathcal{P}_S$ to denote the metric projective operator of $\mathbb{R}^n$ onto S. In particular, we use $[\cdot]_+ := \mathcal{P}_{\mathbb{R}_+}$.

A function $f : S \to \mathbb{R}$ is said to be monotone if for any two vectors $x, y \in S$ with $x \preceq y$, it holds that $f(x) \leqslant f(y)$. Suppose that $f : S \to \mathbb{R}$ is differentiable, it is called DR-submodular function if $\nabla f(x) \succeq \nabla f(y)$, $\forall x, y \in S$, with $x \preceq y$. Moreover, if f is monotone, $\nabla_i f(\cdot) \geqslant 0$, $\forall i \in [n]$, set $\rho = \sup\{\rho_1 \geqslant 0 : \nabla f(x) \succeq \rho_1 \nabla f(y), \forall x \preceq y\}$, then $0 \leqslant \rho \leqslant 1$ is well defined, and we call that f is ρ-weakly DR-submodular at this time.

3 Problem Model

In this paper, we consider the online continuous maximization problem with linear stochastic cumulative constraint. The corresponding offline problem is as follows:

$$\begin{aligned} \max\ & \sum_{t=1}^{T} U_t(x_t) := f_t(x_t) + \theta_t(x_t), \\ \text{s.t.,}\ & \sum_{t=1}^{T} \langle c, x_t \rangle \leqslant B, \\ & x_t \in \mathcal{X},\ \forall t \in [T] \end{aligned} \quad (1)$$

where $B > 0$ and $\mathcal{X} \subseteq \mathbb{R}^n$ are known budget and constraint set, respectively. Under online environment, it means that integer T is time horizon, at each round $t \in [T]$, two actions will occur in order. (1) The user executes $x_t \in \mathcal{X}$; (2) the utility functions f_t, $\theta_t : \mathcal{X} \to \mathbb{R}$ are revealed, as well as the stochastic constraint sample vector c_t with certain unknown distribution: expectation c and covariance matrix $\boldsymbol{N}$. The target of online optimization is to maximize the total utility $\sum_{t=1}^{T} U_t(x_t)$ with decision vectors satisfying the linear constraint $\sum_{t=1}^{T} \langle c, x_t \rangle \leqslant B$. Since the algorithm gives $x_t \in \mathcal{X}$ without knowing information about $U_t(\cdot)$, naturally, it is unreasonable to measure the performance of online algorithm similar to the offline case. Before putting forward the measure criterion, we make the following clear assumptions about online optimization model (1).

Assumption 1 (1) About the constraint set $\mathcal{X}$: $0 \in \mathcal{X} \subseteq \mathbb{R}_+^n$, and $\mathcal{X}$ is compact and convex.

(2) At each round $t\in[T]$, the utility function $U_t:\mathcal{X}\to\mathbb{R}_+$ is of structure $U_t(\cdot):=f_t(\cdot)+\theta_t(\cdot)$, where $f_t,\theta_t:\mathcal{X}\to\mathbb{R}$ are monotone and differentiable, and the gradient of U_t is Lipschitz continuous. Moreover, f_t and θ_t are ρ-weakly DR-submodular and concave, respectively.

(3) The revealed stochastic vectors $\{c_t\}_{t\in[T]}$ are independent identically distributed with certain unknown distribution $\mathcal{D}$: expectation c and covariance matrix $\boldsymbol{N}$. Meanwhile, the sample vectors $\{c_t\}_{t\in[T]}$ are nonnegative and bounded.

Observation 1 Based on these conditions, there exist $D,L,C_1,C_2,C_3>0$, such that

(1) $D:=\max\limits_{x,y\in\mathcal{X}}\|x-y\|=\max\limits_{x\in\mathcal{X}}\|x\|$,

(2) $\{U_t\}_{t\in[T]}$ are L-smooth, that is $\forall t\in[T],x,y\in\mathcal{X}$,

$$\|\nabla U_t(x)-\nabla U_t(y)\|\leqslant L\|x-y\|,$$

$\{U_t\}_{t\in[T]}$ are C_1-Lipschitz continuous, that is $\forall t\in[T],x,y\in\mathcal{X}$,

$$|U_t(x)-U_t(y)|\leqslant C_1\|x-y\|,$$

(3) The sample vectors $\{c_t\}_{t\in[T]}$ has the following property:

$$\|c_t\|\leqslant C_2,\forall t\in[T],$$

$$C_3=\max_{p\sim\mathcal{D}(c,\boldsymbol{N}),x\in\mathcal{X}}\left|\langle p,x\rangle-\frac{B}{T}\right|<\infty.$$

For simplicity, set $C=\max\{C_1,C_2,C_3\}>0$.

Applying an algorithm to online form of optimization problem (1), it will output $\{x_t\}_{t\in[T]}$ at the end. We define two notions: regret and constraint violation to evaluate the performance of the algorithm.

Definition 1 Let $\{x_t\}_{t\in[T]}$ be outputs of an online algorithm for problem (1).

(1) Take $\tilde{\mathcal{X}}=\left\{x\in\mathcal{X},\sum\limits_{t=1}^{T}\langle c,x\rangle\leqslant B\right\}$. The $\left(1-\frac{1}{\mathrm{e}^{\rho}}\right)$-regret with respect to $\{x_t\}_{t\in[T]}$ is defined as

$$\mathcal{R}_T^{\rho}:=\left(1-\frac{1}{\mathrm{e}^{\rho}}\right)\max_{x\in\tilde{\mathcal{X}}}\sum_{t=1}^{T}U_t(x)-\sum_{t=1}^{T}U_t(x_t).$$

(2) The stochastic constraint violation with respect to $\{x_t\}_{t\in[T]}$ is defined as

$$C_T:=\sum_{t=1}^{T}\langle c,x_t\rangle-B.$$

4 Proposed Algorithm

Inspired by the penalty function method for constrained optimization and Frank-Wolfe algorithm, we propose our online algorithm for solving online maximization problem (1).

Algorithm 1 Online algorithm of primal-dual type

Input: The constraint set $\mathcal{X}$, time horizon T, integer K, stepsize $\alpha>0$, and penalty parameter $\beta>0$ with $\beta=\dfrac{1}{\alpha C^2}$.

Output: Action sequence $\{x_t\}_{t\in[T]}$.

Initialize $\mathcal{L}_0(\cdot)\equiv 0$, $v_0^{(k)}=0,\forall k\in[K]$.

for $t=1$ to T **do**

$x_t^{(1)}=0$.

for $k=1$ to K **do**

$$v_t^{(k)}=\mathcal{P}_{\mathcal{X}}(v_{t-1}^{(k)}+\alpha\nabla_x\mathcal{L}_{t-1}(x_{t-1}^{(k)},\lambda_{t-1}))\tag{2}$$

$$x_t^{(k+1)}=x_t^{(k)}+\tfrac{1}{K}v_t^{(k)}\tag{3}$$

end for

Set $x_t=x_t^{(K+1)}$ and act x_t.

Observe the utility function $U_t(\cdot)=f_t(\cdot)+\theta_t(\cdot)$ and the random constraint vector sampled as c_t.

Set $\tilde{c}_t=\dfrac{1}{t}\sum\limits_{i=1}^{t}c_i$, and construct function $\mathcal{L}_t:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ as $\mathcal{L}_t(x,\lambda)=f_t(x)+\theta_t(x)+\dfrac{1}{2\beta}\lambda^2-\lambda h_t(x)$, where $h_t(\cdot)=\langle\tilde{c}_t,\cdot\rangle-\dfrac{B}{T}$.

Compute $\lambda_t=\beta[h_t(x_t)]_+$.

end for

5 Performance Analysis

Lemma 1 Let $\{x_t\}_{t\in[T]}$ be the sequence generated by the Algorithm 1. Then we have, for any $x\in\mathcal{X}$,

$$\left(1-\frac{1}{\mathrm{e}^{\rho}}\right)\sum_{t=1}^{T}U_t(x)-\sum_{t=1}^{T}U_t(x_t)\leqslant$$
$$\frac{LD^2T}{2K}+\frac{D^2C^2\beta}{2}+\frac{T}{\beta}+\sum_{t=1}^{T}\lambda_t h_t(x)\tag{4}$$

Proof The proof is done in four steps.

Step 1: To show the following claim:

Suppose that $U(\cdot)=f(\cdot)+\theta(\cdot)$, and $f,\theta:\mathcal{X}\to\mathbb{R}$ are monotone and differentiable, and they are ρ-weakly DR-submodular and concave, respectively. Then $U(y)-U(x)\leqslant\dfrac{1}{\rho}\langle\nabla U(x),y\rangle$, $\forall x,y\in\mathcal{X}$.

Since f is ρ-weakly DR-submodular, then

$$f(y)-f(x)\leqslant\frac{1}{\rho}\langle\nabla f(x),y-x\rangle,\ \forall x,y\in\mathcal{X},y\geq x.$$

Thus, for any $x,y\in\mathcal{X}$, it follows from the monotonicity of f that

$$f(y)-f(x)\leqslant f(y\vee x)-f(x)\leqslant \frac{1}{\rho}\langle\nabla f(x), y\vee x-x\rangle = \frac{1}{\rho}\langle\nabla f(x), y-y\wedge x\rangle\leqslant\frac{1}{\rho}\langle\nabla f(x), y\rangle.$$

According to the concavity and monotonicity of θ, we get

$$\theta(y)-\theta(x)\leqslant\langle\nabla\theta(x), y-x\rangle\leqslant\frac{1}{\rho}\langle\nabla\theta(x), y\rangle,$$

thus we get the claim.

Step 2: To proof that for any $x\in\mathcal{X}$, $k\in[K]$, the following relational expression holds:

$$\sum_{t=1}^{T}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t), x-v_t^{(k)}\rangle\leqslant\frac{D^2}{2\alpha}+\alpha C^2T+\alpha C^2\sum_{t=1}^{T}|\lambda_t|^2 \tag{5}$$

Since the projection operator is nonexpansive, it follows from Formula (2) that, $\forall k\in[K], t\in[T]$,

$$\begin{aligned}&\|v_t^{(k)}-x\|^2\leqslant\\&\|v_{t-1}^{(k)}+\alpha\nabla_x\mathcal{L}_{t-1}(x_{t-1}^{(k)},\lambda_{t-1})-x\|^2=\\&\|v_{t-1}^{(k)}-x\|^2+\alpha^2\|\nabla_x\mathcal{L}_{t-1}(x_{t-1}^{(k)},\lambda_{t-1})\|^2+\\&2\alpha\langle\nabla_x\mathcal{L}_{t-1}(x_{t-1}^{(k)},\lambda_{t-1}), v_{t-1}^{(k)}-x\rangle,\end{aligned}$$

thus, for any $k\in[K], 1<t\in[T]$,

$$\begin{aligned}&2\alpha\langle\nabla_x\mathcal{L}_{t-1}(x_{t-1}^{(k)},\lambda_{t-1}), x-v_{t-1}^{(k)}\rangle\leqslant\\&\|v_{t-1}^{(k)}-x\|^2-\|v_t^{(k)}-x\|^2+\alpha^2\|\nabla_x\mathcal{L}_{t-1}(x_{t-1}^{(k)},\lambda_{t-1})\|^2=\\&\|v_{t-1}^{(k)}-x\|^2-\|v_t^{(k)}-x\|^2+\alpha^2\|\nabla_x U_{t-1}(x_{t-1}^{(k)})-\lambda_{t-1}\tilde{c}_{t-1}\|^2\leqslant\\&\|v_{t-1}^{(k)}-x\|^2-\|v_t^{(k)}-x\|^2+2\alpha^2C^2+2\alpha^2C^2|\lambda_{t-1}|^2.\end{aligned}$$

By summing up the above inequalities over indices t from 2 to $T+1$, we obtain that

$$\begin{aligned}&2\alpha\sum_{t=2}^{T+1}\langle\nabla_x\mathcal{L}_{t-1}(x_{t-1}^{(k)},\lambda_{t-1}), x-v_{t-1}^{(k)}\rangle\leqslant\\&\|v_1^{(k)}-x\|^2+2\alpha^2C^2T+2\alpha^2C^2\sum_{t=2}^{T+1}|\lambda_{t-1}|^2.\end{aligned}$$

By arranging the above formula, we get the Formula (5).

Step 3: To show that for any $x\in\mathcal{X}$, it holds that

$$\begin{aligned}&\sum_{t=1}^{T}U_t(x)-\sum_{t=1}^{T}U_t(x_t)\leqslant(1-\frac{\rho}{K})^K\sum_{t=1}^{T}U_t(x)+\\&\frac{1}{K}\sum_{t=1}^{T}\sum_{j=1}^{K}\left(1-\frac{\rho}{K}\right)^{K-j}\Big[\frac{LD^2}{2K}+\lambda_t h_t(x)+\lambda_t\cdot\frac{B}{T}-\\&\lambda_t\langle\tilde{c}_t, v_t^{(j)}\rangle-\langle\nabla_x\mathcal{L}_t(x_t^{(j)},\lambda_t), v_t^{(j)}-x\rangle\Big]\end{aligned} \tag{6}$$

Since $\mathcal{L}_t(\cdot)$ is L-smooth with respect to x variable, we conclude from Formula (3) that, $\forall k\in[K], t\in[T]$,

$$\begin{aligned}&\mathcal{L}_t(x_t^{(k)},\lambda_t)\leqslant\mathcal{L}_t(x_t^{(k+1)},\lambda_t)+\\&\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t), x_t^{(k)}-x_t^{(k+1)}\rangle+\\&\frac{L}{2}\|x_t^{(k)}-x_t^{(k+1)}\|^2=\mathcal{L}_t(x_t^{(k+1)},\lambda_t)-\\&\frac{1}{K}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t), v_t^{(k)}\rangle+\frac{L}{2K^2}\|v_t^{(k)}\|^2=\\&\mathcal{L}_t(x_t^{(k+1)},\lambda_t)-\frac{1}{K}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t), v_t^{(k)}-x\rangle+\\&\frac{L}{2K^2}\|v_t^{(k)}\|^2-\frac{1}{K}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t), x\rangle\end{aligned} \tag{7}$$

Notice that

$$\begin{aligned}&-\frac{1}{K}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t), x\rangle=-\frac{1}{K}\langle\nabla U_t(x_t^{(k)})-\lambda_t\tilde{c}_t, x\rangle\leqslant\\&-\frac{\rho}{K}(U_t(x)-U_t(x_t^{(k)}))+\frac{\lambda_t}{K}\langle\tilde{c}_t, x\rangle,\end{aligned}$$

where the inequality follows from the conclusion of Step 1.

Substituting it into Formula (7), and combining with the definition of $\mathcal{L}_t$, iterative Formula (3) and the bound of $v_t^{(k)}$, we obtain that

$$\begin{aligned}&U_t(x)-U_t(x_t^{(k+1)})\leqslant(1-\frac{\rho}{K})(U_t(x)-U_t(x_t^{(k)}))-\\&\frac{1}{K}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t), v_t^{(k)}-x\rangle+\frac{1}{K}\Big[\frac{LD^2}{2K}+\\&\lambda_t h_t(x)+\lambda_t\cdot\frac{B}{T}-\lambda_t\langle\tilde{c}_t, v_t^{(k)}\rangle\Big].\end{aligned}$$

Using the above recurrence formula, we get that

$$\begin{aligned}&U_t(x)-U_t(x_t)=U_t(x)-U_t(x_t^{(K+1)})\leqslant\\&\left(1-\frac{\rho}{K}\right)^K(U_t(x)-U_t(x_t^{(1)}))+\\&\frac{1}{K}\sum_{j=1}^{K}\left(1-\frac{\rho}{K}\right)^{K-j}\Big[\frac{LD^2}{2K}+\lambda_t h_t(x)+\lambda_t\cdot\frac{B}{T}-\\&\lambda_t\langle\tilde{c}_t, v_t^{(j)}\rangle-\langle\nabla_x\mathcal{L}_t(x_t^{(j)},\lambda_t), v_t^{(j)}-x\rangle\Big],\end{aligned}$$

thus we get Formula (6) by summing up the above inequalities over t.

Step 4: To show that for any $x\in\mathcal{X}$, it holds that

$$\begin{aligned}&\left(1-\frac{1}{e^{\rho}}\right)\sum_{t=1}^{T}U_t(x)-\sum_{t=1}^{T}U_t(x_t)\leqslant\\&\frac{LD^2T}{2K}+\frac{D^2}{2\alpha}+\alpha C^2T+\sum_{t=1}^{T}\lambda_t h_t(x)\end{aligned} \tag{8}$$

Notice that $x_t = \frac{1}{K}\sum_{j=1}^{K} v_t^{(j)}$, it follows from Formulas (5) and (6) that

$$\sum_{t=1}^{T} U_t(x) - \sum_{t=1}^{T} U_t(x_t) \leqslant \frac{1}{e^{\rho}}\sum_{t=1}^{T} U_t(x) + \frac{LD^2T}{2K} + \sum_{t=1}^{T}\lambda_t h_t(x) - \sum_{t=1}^{T}\lambda_t h_t(x_t) + \frac{D^2}{2\alpha} + \alpha C^2 T + \alpha C^2 \sum_{t=1}^{T}|\lambda_t|^2.$$

On the other hand,

$$\alpha C^2|\lambda_t|^2 - \lambda_t h_t(x_t) = \alpha C^2\beta^2[h_t(x_t)]_+^2 - \beta[h_t(x_t)]_+ h_t(x_t) = 0,$$

where the two equalities follows from the definitions of λ_t and operator $[\cdot]_+$, respectively. Thus we get Formula (4).

Using the relation $\beta = \frac{1}{\alpha C^2}$, we get Lemma 1. □

Lemma 2 Let $\{\tilde{c}_t\}_{t\in[T]}$ be random vectors defined in Algorithm 1. Then,

(1) For any $t \in [T]$, $\mathrm{CE}[\|\tilde{c}_t - c\|^2] \leqslant \frac{\mathrm{Tr}(\boldsymbol{N})}{t}$;

(2) $\mathrm{CE}\left[\sum_{t=1}^{T}\|\tilde{c}_t - c\|\right] \leqslant 2\sqrt{T+1}\sqrt{\mathrm{Tr}(\boldsymbol{N})}$.

Proof (1) Since $\tilde{c}_t = \frac{1}{t}\sum_{i=1}^{t} c_i$ and $\{c_t\}_{t\in[T]}$ are i.i.d. with distribution $\mathcal{D}(c,\boldsymbol{N})$, we obtain that $\mathrm{CE}[\tilde{c}_t] = c$ and $\mathrm{Cov}(\tilde{c}_t) = \frac{\boldsymbol{N}}{t}$. Thus we conclude that

$$\mathrm{CE}[\|\tilde{c}_t - c\|^2] = \mathrm{CE}[(\tilde{c}_t - c)^{\mathrm{T}}(\tilde{c}_t - c)] = \mathrm{CE}[\mathrm{Tr}((\tilde{c}_t - c)(\tilde{c}_t - c)^{\mathrm{T}})] = \mathrm{Tr}(\mathrm{CE}[(\tilde{c}_t - c)(\tilde{c}_t - c)^{\mathrm{T}}]) = \mathrm{Tr}(\mathrm{Cov}(\tilde{c}_t)) = \frac{\mathrm{Tr}(\boldsymbol{N})}{t}.$$

(2) It follows from the concavity of function $\sqrt{\cdot}$ that

$$\mathrm{CE}[\sqrt{\|\tilde{c}_t - c\|^2}] \leqslant \sqrt{\mathrm{CE}[\|\tilde{c}_t - c\|^2]}.$$

Hence

$$\mathrm{CE}\left[\sum_{t=1}^{T}\|\tilde{c}_t - c\|\right] = \sum_{t=1}^{T}\mathrm{CE}[\|\tilde{c}_t - c\|] \leqslant \sum_{t=1}^{T}\sqrt{\mathrm{CE}[\|\tilde{c}_t - c\|^2]} \leqslant \sum_{t=1}^{T}\sqrt{\frac{\mathrm{Tr}(\boldsymbol{N})}{t}} \leqslant 2\sqrt{T+1}\sqrt{\mathrm{Tr}(\boldsymbol{N})}.$$

□

Theorem 1 Take $K = \sqrt{T}$, $\beta = \sqrt[4]{T}$. Then we have the following upper bound of the regret in expectation sense,

$$\mathrm{CE}[\mathcal{R}_T^{\rho}] \leqslant \frac{LD^2}{2}\sqrt{T} + \frac{D^2C^2}{2}\sqrt[4]{T} + \sqrt[4]{T^3} + 2CD\sqrt{\mathrm{Tr}(\boldsymbol{N})}\sqrt[4]{T}\sqrt{T+1}.$$

Proof Take $x^* \in \arg\max_{x\in\tilde{\mathcal{X}}}\sum_{t=1}^{T} U_t(x)$. Substitute $x = x^*$ in Fomula (4), and take expectation in both sides of the inequality, we get that

$$\mathrm{CE}[\mathcal{R}_T^{\rho}] \leqslant \frac{LD^2T}{2K} + \frac{D^2C^2\beta}{2} + \frac{T}{\beta} + \mathrm{CE}[\mathrm{Term1}] + \lambda_t\cdot\mathrm{CE}\left[\sum_{t=1}^{T}\left(\langle c, x^*\rangle - \frac{B}{T}\right)\right] \leqslant \frac{LD^2T}{2K} + \frac{D^2C^2\beta}{2} + \frac{T}{\beta} + \mathrm{CE}[\mathrm{Term1}] \tag{9}$$

where $\mathrm{Term1} := \sum_{t=1}^{T}\lambda_t\left(h_t(x^*) - \langle c, x^*\rangle + \frac{B}{T}\right)$, and the last inequality follows from $x^* \in \tilde{\mathcal{X}}$.

Moreover, we observe that

$$\mathrm{Term1} = \sum_{t=1}^{T}\lambda_t\cdot\langle\tilde{c}_t - c, x^*\rangle \leqslant D\|\lambda\|_\infty\|g\|_1 \leqslant DC\beta\|g\|_1,$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_T)^{\mathrm{T}}$, $g = (\|\tilde{c}_1 - c\|, \|\tilde{c}_2 - c\|, \ldots, \|\tilde{c}_T - c\|)^{\mathrm{T}}$, and the last inequality is based on the definition of λ_t. Therefore, it comes from Lemma 2 (2) that

$$\mathrm{CE}[\mathrm{Term1}] \leqslant 2CD\sqrt{\mathrm{Tr}(\boldsymbol{N})}\beta\sqrt{T+1},$$

substituting into Formula (7), we get the conclusion of Theorem 1.

Theorem 2 Take $K = \sqrt{T}, \beta = \sqrt[4]{T}$. Then we have the following upper bound of the constraint violation in expectation sense,

$$\mathrm{CE}[\mathcal{C}_T] \leqslant \frac{CT}{B}\sqrt[4]{T^3} + \frac{T}{B}\sqrt{T} + \frac{LD^2T}{2B}\sqrt[4]{T} + \frac{D^2C^2}{2B}T + 2D\sqrt{T+1}\sqrt{\mathrm{Tr}(\boldsymbol{N})}.$$

Proof Take $x = 0$ as the fixed vector in Formula (4), and according to the definition of h_t, we conclude that

$$\sum_{t=1}^{T}\lambda_t\cdot\frac{B}{T} \leqslant \frac{LD^2T}{2K} + \frac{D^2C^2\beta}{2} + \frac{T}{\beta} + TC,$$

hence

$$\sum_{t=1}^{T}[h_t(x_t)]_+ = \sum_{t=1}^{T}\frac{\lambda_t}{\beta} \leqslant$$

$$\frac{LD^2T^2}{2K\beta B}+\frac{D^2C^2T}{2B}+\frac{T^2}{\beta^2 B}+\frac{T^2C}{\beta B}=$$
$$\frac{CT}{B}\sqrt[4]{T^3}+\frac{T}{B}\sqrt{T}+\frac{LD^2T}{2B}\sqrt[4]{T}+\frac{D^2C^2}{2B}T.$$

On the other hand, we have

$$C_T=\sum_{t=1}^{T}\langle c,x_t\rangle-B=$$
$$\sum_{t=1}^{T}h_t(x_t)+\sum_{t=1}^{T}\langle c-\tilde{c}_t,x_t\rangle\leqslant$$
$$\sum_{t=1}^{T}[h_t(x_t)]_+ +\sum_{t=1}^{T}\|c-\tilde{c}_t\|\|x_t\|.$$

Thus, the above two relationships together with Lemma 2 (2) yield the conclusion. □

6 Conclusion

We consider a certain class of nonconvex online continuous optimization problems with stochastic linear budget constraint, where the objective function at each round is composed of two parts: weakly DR-submodular and concave function. We present an online algorithm of primal-dual type to solve it. The expectation of the regret related with the weakly ratio achieves sub-linear bound with respect to the time horizon. Meanwhile, the violation constraint obtains certain bound in expectation.

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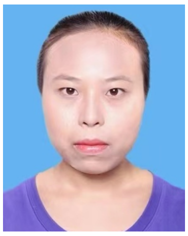

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