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A Note on Maximizing Regularized Submodular Functions Under Streaming

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Abstract: Recent progress in maximizing submodular functions with a cardinality constraint through centralized and streaming modes has demonstrated a wide range of applications and also developed comprehensive theoretical guarantees. The submodularity was investigated to capture the diversity and representativeness of the utilities, and the monotonicity has the advantage of improving the coverage. Regularized submodular optimization models were developed in the latest studies (such as a house on fire), which aimed to sieve subsets with constraints to optimize regularized utilities. This study is motivated by the setting in which the input stream is partitioned into several disjoint parts, and each part has a limited size constraint. A first threshold-based bicriteria $(1/3, 2/3)$-approximation for the problem is provided.

Key words: submodular optimization; regular model; streaming algorithms; threshold technique

1 Introduction

Submodular optimization is a remarkably popular topic in combinatorial optimization, theoretical computer science, game theory, machine learning, and many other fields. From the perspective of algorithm development, most of the submodular optimization problems with or without constraints are usually NP-hard. Two $(1 - 1/e)$[1] and $1/e$-approximation[2] respectively exist for the monotone and non-monotone settings for the well-studied cardinality-constrained model. Additionally, there are random $(1 - 1/e)$-approximations[3, 4], and determined breakout 0.5008-approximation[5] that are available for the highly expressive matroid-constrained models. Highly generalized results can be found in the k-matchoid[6] and k-independence system[7–9].

Considering the submodular maximization with a regularized modular term[10–12], an element ground set E is provided and defined on submodular and modular functions f and c, respectively. An important weak approximation concept can be restated as follows. Given $\rho \in (0, 1)$, an algorithm is called ρ-approximation if it outputs a feasible set S satisfying

$$f(S) - c(S) \geqslant \rho \cdot f(O) - c(O) \quad (1)$$

where O denotes the optimum solution for the discussed regularized problem. The highly expressive bicriteria approximation was introduced by Ref. [13] and restated in the current study. A polynomial-time algorithm is a bicriteria (ρ_1, ρ_2)-approximation if it returns a set S satisfying

$$f(S) - c(S) \geqslant \rho_1 \cdot f(O) - \rho_2 \cdot c(O) \quad (2)$$

where $\rho_1, \rho_2 \in (0, 1)$.

The current investigation is motivated by the studies in a streaming fashion. The elements in the streaming model are revealed in a sequential form. Recall that the input is assumed to be accessed in advance under a centralized setting. However, the streaming model forces the algorithms to visit the elements in sequences and store at most $(O(\log n))$ elements in memory complexity, where n represents the input size of the stream. Thus, finding a high-quality solution that uses memory space efficiently is crucial. The referred streaming algorithms

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often restrict the use of the memory to poly-logarithmic in the input parameters. An algorithm is called a single (one) pass if it is allowed to visit the input stream only once. In several other cases, one may be allowed to visit the stream numerous times to improve the solution quality. However, this algorithm is not optimized in the current discussion. A partition matroid-constrained regularized submodular maximization under a streaming mode is investigated in the current study, and a threshold-based algorithm that produced a bicriteria $(1/3, 2/3)$-approximation for the regularized problem is obtained.

The related literature to the current study is investigated in Section 2. The notations and models considered in this paper are then prepared in Section 3. The framework for maximizing the regularized submodular with a partition matroid and its theoretical guarantee is provided in Section 4. The regularized problem under a streaming mode is considered, and a distorted threshold-based algorithm with novelty theoretical guarantees is provided in Section 5. The conclusions are finally presented in Section 6.

2 Related Work

A large body of work on optimizing submodular functions for their wide applications and comprehensive theoretical performance guarantees is available. Regularized and streaming submodular maximizations are the most relevant concepts to the current work. The algorithms developed for the two settings below are investigated.

Regularized setting. Most previously listed pieces of literature in Refs. [14, 15] for submodular maximization types of constraints focused on the utilities to obtain only nonnegative values. A tight approximation result for the monotone submodular maximization problem was developed by Ref. [16]. The study in Ref. [12] first considered the summing utility of submodular and modular functions, and provided two randomized tight approximations. Feldman[10] studied the regularized submodular maximization over a solvable polytope, and presented a distorted continuous greedy that achieves the same weak approximation as Ref. [12]. The formally regularized model was introduced in Ref. [11], which investigated the cardinality constrained problem and obtained a tight weak $(1-1/\mathrm{e})$-approximation. An extended distorted greedy appeared in Ref. [17], which considered the setting with a non-monotone submodular utility and obtained a weak $1/\mathrm{e}$-approximation. References [18–20] can be used as a reference for optimizing the regularized submodular models.

Streaming setting. As previously mentioned, the threshold-based method is popular in handling streaming submodular optimization problems. A series of studies in Refs. [21–24] proposed the above threshold-based streaming algorithms for the cardinality constrained submodular maximization problem. General knapsack constrained submodular maximization appeared in Refs. [25, 26]. Researchers further enhanced the models of maximizing submodular functions with highly generalized constraints. The k-matroid[27, 28] and k-independence system[29] can be used as a reference. Some streaming algorithms have been developed for maximizing submodular functions based on graphs. A single pass 0.129-approximation can be used for the matching constrained submodular maximization[30]. Improved results with single pass 0.172-approximation were independently developed[31, 32]. Moreover, an extended streaming algorithm based on the local ratio technique for the matroid intersection constraint was studied[33]. Notably, the previous results were assumed that the visited order is arbitrary. References [34–36] summarized the submodular maximization under a random order. Additionally, streaming algorithms were originally developed for the regularized problems[13, 19, 37].

3 Preliminary

This study focuses on the regularized submodular maximization problem, which aims to choose a subset of representative elements from the ground set along with a regularized utility.

(1) Utility

Given an element ground set E, a set function f represents the revenue term, such that $f(S)$ is the revenue that can be attributed to S for each $S \subseteq E$. Given any pair (e, S), where element $e \in E$ and subset $S \subseteq E$, the marginal gain of e addition to S considering f is denoted by $f(e|S) = f(S \cup \{e\}) - f(S)$. Similarly, let $f(T|S) = f(S \cup T) - f(S)$ be the marginal gain of subset T to additional S considering f. Set function f is submodular if

$$f(e|S) \geqslant f(e|T), \forall S \subseteq T \subseteq E, e \notin T \tag{3}$$

For any pair (e, S), the function f is additionally monotone if the marginal gain $f(e|S)$ is always nonnegative. The regularized term is denoted by c :

$2^E \to \mathbf{R}^+$, such that $c(S)$ captures the penalty for each $S \subseteq E$ if S is chosen.

(2) Matroid

Recall that a pair (E, I) is defined as an independence system if $S \supseteq S' \in I$ for any $S \in I$. A subset $S \in I$ is referred to as an independent set and the maximum size among all independent sets is defined as the base. An independence system (E, I) is further defined as a matroid if an element $e \in T \setminus S$ satisfying $S \cup \{e\} \in I$ exists for any two independent sets $S, T \in I$ with $|T| > |S|$. A special partition matroid is restated as follows. Assume that E is partitioned into $\{E_i\}_{i=1}^{\ell}$ parts considering constraint sizes $\{k_i\}_{i=1}^{\ell}$. Let $I = \{S \subseteq E : |S \cap E_i| \leqslant k_i\}$ denote the collection of subsets, which mostly includes at most k_i elements from each part E_i.

The computation of submodular value and independence for subsets exponentially increase in the size of the ground set. Thus, the independence and value of $S \subseteq E$ can be obtained in oracles.

(3) Problem

In the regularized submodular maximization under a partition matroid, a regularized utility function $f - c$ is provided, where f is nonnegative monotone submodular, c is nonnegative modular, and a partition matroid (E, I) is observed in the same ground set. Formally, the problem is cast as follows:

$$\max_{S \in I : S \subseteq E} f(S) - c(S) \tag{4}$$

where $I = \{S \subseteq E : |S \cap E_i| \leqslant k_i, k = \sum_{i=1}^{\ell} k_i\}$. The above problem is reduced to the classic cardinality constrained submodular maximization if the partition matroid (E, I) decreases to $\ell = 1$.

4 Distorted Greedy

The regularized submodular maximization under a partition matroid constraint is considered in this section, and a distorted greedy algorithm with a theoretical guarantee is provided.

4.1 Algorithm description

The introduced algorithm, which is listed in Algorithm 1, obtains a parameter $\lambda = 2$ and starts with an empty set. Algorithm 1 uses sets S_t for certain iterations t from 1 to k. For some iteration t and part i with $|S_t \cap E_i| < k_i$, the algorithm greedily selects element $e_t = e_i$ with a maximum distorted marginal gain, that is,

$$e_i \in \arg\max_{e \in E_i} f(e|S_t) - \lambda \cdot c(e) \tag{5}$$

Algorithm 1 breaks if either all the distorted marginal gains are negative or the iterations reach to k.

Algorithm 1 Distorted-fair greedy

Input: Ground set E with a partition $\{E_i\}_{=1}^{\ell}$, input size constraint k_i with respect to E_i, parameter $\lambda = 2$

Output: Solution set S

Initialization. $S \leftarrow \varnothing, t_0 \leftarrow 0$;
for $t = 1 : k$ **do**
  for $i = 1 : \ell$ **do**
    if $|S_t \cap E_i| < k_i$ **then**
      Set $e_i \leftarrow \arg\max_{e \in E_i} f(e|S_t) - \lambda \cdot c(e)$;
      if $f(e_i|S_t) - \alpha(r)c(e) \leqslant 0$ **then**
        break;
      end if
      Set $S_t \leftarrow S_t + e_i$;
    end if
  end for
end for
return S_k

4.2 Analysis

The properties of the distorted greedy algorithm are summarized using the following theorem. Notably, an independent study for the general matroid constrained regularized submodular maximization is available[19].

Theorem 1 Algorithm 1 obtains a weak 0.5-approximation for maximizing regularized submodular functions under a partition matroid constraint.

Proof The details for proving Theorem 1 are omitted in this study because they mainly follow the framework of Theorem D.1.[19], we omit it here. ■

5 Streaming Algorithm

The regularized partition matroid-constrained submodular maximization under a streaming mode is studied in this section, and a distorted threshold based algorithm with theoretical guarantees is provided.

5.1 Algorithm description

The proposed algorithm, which is listed in Algorithm 2, obtains parameters $r = 2/3$; thus let $\alpha(r) = (2r + 1 - \sqrt{4r^2 + 1})/2 = 2$, and $h(r) = (1 - 1/\alpha(r))/(2 - 1/\alpha(r)) = 1/3$. The influence of the distorted marginal gains is intuitively investigated by introducing the above parameter $\alpha(r)$. Motivated by Ref. [37], a distorted alternative optimization problem is introduced as follows:

$$T \in \arg\max_{T' \in I, T' \subseteq E} h(r) f(T') - r \cdot c(T') \tag{6}$$

where parameters r and $h(r)$ are stated as before.

Algorithm 2 starts with empty solution S and buffer B. Let $m_t := h(r) f(e_t) - r c(e_t)$ be a distorted single element value. The lower bound of optimum O estimated

Algorithm 2 Distorted threshold-based algorithm

Input: Stream E is partitioned by $\{E_i\}_{=1}^{\ell}$ with constraints $\{k_i\}_{i=1}^{\ell}$. Setting $r = 2/3, \alpha(r) = 2$, and $h(r) = 1/3$

Output: Solution set S_τ

Initialization. $m_0 = L = 0$, $B \leftarrow \varnothing$, and $t \leftarrow 1$;
while there exists an element $e_t \in E_i$ arrives from E **do**
  Let $m_t \leftarrow \max\{m_{t-1}, h(r)f(e_t) - r \cdot c(e_t)\}$;
  Update R_i with respect to e_i using reservoir sampling;
  $O_t \leftarrow \{(1+\varepsilon)^j : (1+\varepsilon)^j \in [\frac{\max\{m_t, L\}}{k}, \frac{\alpha(r)}{r} m_t]\}$;
  Discard S_τ for all $\tau \notin O_t$;
  Initialize $S_\tau \leftarrow \varnothing$ for each newly added to O_t;
  for each $\tau \in O_t$ **do**
    if $|S_\tau \cap E_i| < k_i$ **then**
      if $f(e_t|S_\tau) - \alpha(r)c(e_t) \geqslant \tau$ **then**
        $S_\tau \leftarrow S_\tau + e_t$;
      else
        if $f(e_t|S_\tau) - \alpha(r)c(e_t) \geqslant \frac{L}{k}$ **then**
          $B \leftarrow B + e_t$;
        end if
      end if
    end if
    $L \leftarrow \max_{\tau \in O_t} f(S_\tau) - c(S_\tau)$;
  end for
end while

by candidate subsets is denoted as L and a set returned by reservoir sampling is denoted as set R_i considering $e_t = e_i$. For any revealed element $e_t \in E_i$ from E, we construct an increasing geometric progression $\{\tau = (1+\varepsilon)^j, j \in Z\}$ interval by m is constructed as follows:

$$O_t = \left\{(1+\varepsilon)^j : \tau \in \left[\frac{\max\{m_t, L\}}{k}, \frac{\alpha(r)m_t}{r}\right]\right\} \quad (7)$$

For each newly instantiated threshold $\tau \in O_t$, the algorithm starts with a corresponding candidate solution $S_\tau = \varnothing$. For each $\tau \in O_t$, Algorithm 2 updates S_τ with $e_t = e_i \in E_i$, if the cardinality of $S_\tau \cap E_i$ satisfies $|S_\tau \cap E_i| < k_i$ and the distorted marginal gain of e_i is larger than threshold τ, that is,

$$f(e_t|S_\tau) - \alpha(r)c(e_t) \geqslant \tau \quad (8)$$

Furthermore, Algorithm 2 updates buffer B if

$$f(e_t|S_\tau) - \alpha(r)c(e_t) \geqslant \frac{L}{k} \quad (9)$$

and returns the maximal lower bound $L = \max_\tau f(S_\tau) - c(S_\tau)$. The postprocessing is then followed to obtain good theoretical guarantees.

The postprocessing, which is listed as Algorithm 3, starts with identifying an index τ', such that τ' is the smallest τ if $|S_\tau \cap E_i| < k_i$ for each i; if some part i holds $|S_\tau \cap E_i| = k_i$ for each τ, then the largest $\tau \in O_n$ is denoted by τ'. For each $\tau \leqslant \tau'$, Algorithm 3 relies on the distorted greedy. Distorted greedy, which is listed as

Algorithm 3 Post processing

Input: Buffer B and samples $\{R_i\}_{i=1}^{\ell}$

Output: Solution set S_τ

Let $\tau' = \begin{cases} \min_{\tau \in O_n} \tau, & \text{if } |S_\tau \cap E_i| < k_i \text{ for any } i \in [\ell]; \\ \max_{\tau \in O_n} \tau, & \text{otherwise.} \end{cases}$
for any $\tau \leqslant \tau'$ in O_n **do**
  Run Algorithm 1 to add elements from buffer B and independently choose R_i for all $i \in [\ell]$ to S_τ with non-negative marginal gains;
end for
return $S_\tau \leftarrow \arg\max_{\tau \in O_n} f(S_\tau) - c(S_\tau)$

Algorithm 1, recalculates the elements in B and R_i, and adds them to S_τ if their marginal gains are nonnegative.

5.2 Analysis

Algorithms 2 and 3 together imply that the main result is summarized by the following theorem.

Lemma 1 below addresses to the set produced by stream processing without postprocessing.

Lemma 1 Assume we have access to $h(r)f(T) - rc(T))/(1+\varepsilon) \leqslant k\tau \leqslant h(r)f(T) - rc(T)$. Considering the following two cases of $|S_\tau| = k$ and $|S_\tau \cap E_i| < k_i$ for any part i, then

$$f(S_\tau) - c(S_\tau) \geqslant (1-\varepsilon)h(r)f(T) - r \cdot c(T) \quad (10)$$

Proof The proof is separately divided into the two subcases below.

• Case of $|S_\tau| = k$. $S_\tau^t = \{e_1, e_2, \ldots, e_t\}$ denotes set of the first $1 \leqslant t \leqslant k$ elements added to S_τ according to their visited order. Since any element e_t added to S_τ^{t-1} must be larger than or equal to τ, we have $f(S_\tau) - \alpha(r)c(S_\tau) = \sum_{t=1}^{k} f(e_t|S_\tau^{t-1}) - \alpha(r)c(e_t) \geqslant k\tau$. Following the assumption, the lower bound $k\tau$ can be further lower bounded by $(h(r)f(T) - r \cdot c(T))/(1+\varepsilon)$. Therefore, $f(S_\tau) - c(S_\tau) \geqslant f(S_\tau) - \alpha(r)c(S_\tau) \geqslant (h(r)f(T) - r \cdot c(T))/(1+\varepsilon) \geqslant (1-\varepsilon)h(r)f(T) - r \cdot c(T)$, where the first inequality obtains as $\alpha(r) \geqslant 1$ and the last two equalities readily hold.

• Case of $|S_\tau \cap E_i| < k_i$ for any $i \in [\ell]$. Consider any element $e = e_t \in E \setminus S_\tau$, $f(e|S_\tau^{t-1}) - \alpha(r)c(e) < \tau$ holds, and then $f(e|S_\tau) - \alpha(r)c(e) \leqslant \tau$ based on submodularity. Adding this inequality over all elements $e \in T \setminus S_\tau$, then yields the following: $f(T) - f(S_\tau) - \alpha(r)c(T) \leqslant \sum_{e \in T \setminus S_\tau} f(e|S_\tau) - \alpha(r)c(e) < k\tau \leqslant h(r)f(T) - r \cdot c(T)$. Rearranging the obtained inequality implies

$$f(S_\tau) \geqslant (1-h(r))f(T) - (\alpha(r) - r)c(T) \quad (11)$$

Moreover, $f(S_\tau) - \alpha(r)c(S_\tau) \geqslant 0$. Adding a $1/\alpha(r)$ fraction of this inequality to a $1 - 1/\alpha(r)$ fraction of

Formula (11) obtains $f(S_\tau)-c(S_\tau) \geqslant (1-1/\alpha(r))(1-h(r))f(T)-(1-1/\alpha(r))(\alpha(r)-r)c(T) = h(r)f(T)-r \cdot c(T)$, where the equality holds due to the following setting: $h(r) = (1-1/\alpha(r))(1-h(r))$ and $r = (1-1/\alpha(r))(\alpha(r)-r)$.

The proof is then completed. ■

Considering the setting of maximizing fair submodular maximization without regular term[38], the approximation ratio might drop to $O(1/k)$ when the partition budgets are exhausted but the others are not. Similar buffer and postprocessing procedures are introduced. This finding indicates that the algorithm still achieves good theoretical guarantees, as summarized by Lemma 2 in the following.

Lemma 2 Denote τ' as the selected of Algorithm 3. Then, we have $f(S_{\tau'}) - \frac{(1+\varepsilon)\alpha(r)}{2+\varepsilon}c(S_{\tau'}) \geqslant \frac{1-h(r)}{2+\varepsilon}f(T) - \frac{\alpha(r)-r}{2}c(T)$.

Proof The proof is separately divided into the two subcases below.

• Case of $|S_{\tau'} \cap E_i| < k_i$ for any $i \in [\ell]$. In this case, the elements of T are partitioned into the following three parts: (1) The set of elements of T included in $S_{\tau'}$ during the algorithms process is denoted by $T_1 := T \cap S_{\tau'}$; (2) The retained set of elements of T in B while not given to $S_{\tau'}$ is denoted by $T_2 := T \cap (B \setminus S_{\tau'})$; (3) The elements discarded during the streaming process are denoted by $T_3 := T \cap (E \setminus (B \cup S_{\tau'}))$. An element $e' \in S_{\tau'}$ exists for each $e \in T_2$, such that $f(e|S_{\tau'}) - \alpha(r)c(e) \leqslant f(e|S'_{\tau'}) - \alpha(r)c(e) \leqslant f(e'|S'_{\tau'}) - \alpha(r)c(e')$, where $S'_{\tau'} \subseteq S_{\tau'}$ denotes the partial solution at the time of encountering e'. For each $e \in T_3$, $f(e|S_{\tau'}) - \alpha(r)c(e) < \frac{L}{k} \leqslant \frac{h(r)f(T)-r\cdot c(T)}{k}$ holds. Hence, the following inequality can be obtained as $f(T) - f(S_{\tau'}) - \alpha(r)c(T) \leqslant \sum_{e\in T_2\cup T_3} f(e|S_{\tau'}) - \alpha(r)c(e)$. Moreover, the terms of $\sum_{e\in T_2} f(e|S_{\tau'}) - \alpha(r)c(e)$ and $\sum_{e\in T_3} f(e|S_{\tau'}) - \alpha(r)c(e)$ can be upper bounded by $\sum_{e'\in S_{\tau'}} f(e'|S'_{\tau'}) - \alpha(r)c(e') = f(S_{\tau'}) - \alpha(r)c(S_{\tau'})$ and $h(r)f(T)-r\cdot c(T)$, respectively. Therefore, $f(T)-f(S_{\tau'})-\alpha(r)c(T) \leqslant f(S_{\tau'})-\alpha(r)c(S_{\tau'})+h(r)f(T)-r\cdot c(T)$ holds. Rearranging the above inequality yields $f(S_{\tau'}) - \frac{\alpha(r)}{2}c(S_{\tau'}) \geqslant \frac{1-h(r)}{2}f(T) - \frac{\alpha(r)-r}{2}c(T)$.

• Case of τ' attaining the maximum in T. Notably, $\frac{\alpha(r)m_n}{(1+\varepsilon)r} \leqslant \tau' \leqslant \frac{\alpha(r)m_n}{r}$ in this case. The elements in T can also be similarly partitioned three parts: T_1, T_2, and T_3. The results readily hold for T_1 and T_3. Thus $e = e_t \in T_2$ is considered. Overall, $f(e|S_{\tau'}) - \alpha(r)c(e) \leqslant f(e) - \alpha(r)c(e) = \frac{1}{h(r)}(h(r)f(e) - \alpha(r)h(r)c(e)) = \frac{1}{h(r)}(h(r)f(e) - r\cdot c(e)) \leqslant \frac{m_n}{h(r)} \leqslant \frac{1}{h(r)}\frac{(1+\varepsilon)r\tau'}{\alpha(r)} = (1+\varepsilon)\tau' \leqslant (1+\varepsilon)(f(e'|S'_{\tau'}) - \alpha(r)c(e'))$. Then $f(S_{\tau'}) - \frac{(1+\varepsilon)\alpha(r)}{2+\varepsilon}c(S_{\tau'}) \geqslant \frac{1-h(r)}{2+\varepsilon}f(T) - \frac{\alpha(r)-r}{2+\varepsilon}c(T)$ is obtained.

Therefore, the proof in this part is completed. ■

The bicriteria approximation for the algorithm in Theorem 2 is provided.

Theorem 2 Let $r = 2/3$ and then $\alpha(r) = 2$, and $h(r) = 1/3$. A bicriteria streaming algorithm that produces a solution set S satisfying

$$f(S) - c(S) \geqslant \left(\frac{1}{3} - \varepsilon\right) f(O) - \frac{2}{3}c(O),$$

which is available for the regularized submodular maximization problem.

6 Conclusion

The regularized submodular maximization under a partition matroid is studied in this work, and a distorted greedy weak 1/2-approximation algorithm is presented. The regularized problem under the streaming model is then considered, and a threshold-based algorithm that obtained a bicriteria $(1/3, 2/3)$ approximation is provided. We believe that the state-of-the-art results for the matroid-constrained nonnegative monotone submodular maximization are single-pass 0.3178-approximation and multipass $(1-1/\mathrm{e})$-approximation[39]. The possible application of the above developed techniques for the unregular setting to regular cases will be further considered.

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