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Approximation Algorithms for Graph Partition into Bounded Independent Sets

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Abstract: The partition problem of a given graph into three independent sets of minimizing the maximum one is studied in this paper. This problem is NP-hard, even restricted to bipartite graphs. First, a simple $\frac{3}{2}$-approximation algorithm for any 2-colorable graph is presented. An improved $\frac{7}{5}$-approximation algorithm is then designed for a tree. The theoretical proof of the improved algorithm performance ratio is constructive, thus providing an explicit partition approach for each case according to the cardinality of two color classes.

Key words: graph partition; independent set; 2-colorable graph; approximation algorithm

1 Introduction

Bodlaender and Jansen[1] introduced the partition problem of a given graph into a bounded number of independent sets, such that the cardinality of the maximum one is as small as possible. This problem mainly focuses on determining whether a connected graph G can be partitioned into at most m independent sets with at most v vertices in each set for given m and v. Therefore, the restriction of G to the following classes, namely forests, split graphs, complements of bipartite graphs, and complements of interval graphs, involves polynomial algorithms[1]. However, the problem remains NP-hard when G is restricted to bipartite graphs (even for $m = 3$), interval graphs, and cographs[2].

This problem is motivated by an assignment problem of operations given in a flow graph to processors, which is a practical problem attributed to the manufacturing industry[3]. One related problem is the classic k-coloring problem[2]. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The k-coloring problem involves the assignment of a color (a number chosen in $\{1, 2, \ldots, k\}$) to each vertex of G, such that no edge has both endpoints with the same color. In other words, the k-coloring problem corresponds to the problem of finding a partition of the vertices into k independent sets. The introduced problem is equivalent to the m-coloring problem in G, where each color class can include at most v vertices. A graph is 2-colorable if we can color each of its vertices with one of the two colors (i.e., red and blue), such that no two red (blue) vertices are connected by an edge. References [4–6] present an overview of the coloring problem restricted to different graph classes. The work of Bodlaender et al.[3], which proposed a $\frac{5}{3}$-approximation algorithm for partitioning a vertex-weighted tree-like graph into three independent sets to minimize the total weight of the maximum one, is the most relevant to the problem under investigation. Many related studies have been conducted regarding conflict constraints, including parallel machine scheduling[7–9] to unrelated machine scheduling[10–12]. Several other studies focused on equitable coloring[13–15], weighted coloring[16, 17], and subgraph partitioning[18].

We believe that no approximation algorithm result prior to the current work exists for this problem even when $m = 3$ and G is restricted to 2-colorable graphs. In this paper, we first present a simple $\frac{3}{2}$-approximation algorithm for partitioning a graph in any 2-colorable graph into three independent sets, and then a slightly

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complex $\frac{7}{5}$-approximation algorithm in trees.

2 Preliminary

First, we provide a simple $\frac{3}{2}$-approximation algorithm for any 2-colorable graph. Given a 2-colorable graph $G=(V,E)$ with $|V|=n$, let R and B be the vertex set with red and blue colors, respectively. Without loss of generality, we assume that $1 \leqslant |R| \leqslant |B| \leqslant n-1$. The first simple approximation algorithm assigns a subset of the independent sets to each color class: one independent set I_1 is assigned to R, and two other independent sets I_2 and I_3 are assigned to B as evenly as possible. Let $Alg(G)=\max\{|I_1|,|I_2|,|I_3|\}$ and $OPT(G)$ denote the solution obtained by the proposed algorithm and the optimal solution, respectively. By simple calculations, we have $Alg(G)=\max\{|I_1|,|I_2|,|I_3|\} \leqslant k'$ (whether $n=2k'$ or $n=2k'+1$, $k' \in \mathbf{N}$); therefore, $Alg(G) \leqslant \frac{3}{2} \times \frac{n}{3} \leqslant \frac{3}{2} OPT(G)$ holds. The last inequality is due to $OPT(G) \geqslant \frac{n}{3}$.

A tree is an organized set of nodes in which each node has one parent, except for a node called root. If a node p is the parent of a node f, then f is the child of p; if f has no children, then it is a leaf. The nodes of a tree are distributed by level. Level 1 contains only the root, Level 2 contains its children, and so on. Compatibility indicates that two vertex sets V_1 and V_2 are compatible if any two vertices within $V_1 \cup V_2$ are non-adjacent.

Let $T_r=(V,E)$ be a tree rooted at vertex r with $|V|=n$ and $V=X \cup Y$, where X and Y denote the two color classes (without loss of generality, $1 \leqslant |X| \leqslant |Y| \leqslant n-1$). Let L_i be the vertex set of the i-th level in T_r; then $L_1, L_2, \dots, L_i, \dots$ alternately belong to X and Y. The following lemma is simple due to the property of a tree.

Lemma 1 For any two consecutive Levels L_i and L_{i+1} of T_r, let $F_i \subseteq L_i$ be the vertex set with at least one child in L_{i+1}. Then, $|F_i| \leqslant |L_{i+1}|$.

In the figures presented in the next section, circles and triangles are used for vertices in X and Y, respectively. A white circle or triangle represents a single vertex in its corresponding set, while a black one represents a number of vertices within the same level.

3 $\frac{7}{5}$-Approximation Algorithm

In this section, we present an improved $\frac{7}{5}$-approximation algorithm for a tree. The main result is the following theorem.

Theorem 1 Let $T_r=(V,E)$ be a tree rooted at vertex r. A solution to the problem can then be observed in polynomial time, whose cardinality of the maximum independent set is at most $\frac{7}{5}$ times the optimal one.

The proof is constructive according to the sizes of X and Y. For each subcase, we provide an explicit feasible partition approach, such that the cardinality of the maximum independent set is at most $\frac{7}{5}$ times the optimal one; that is, we prove $Alg(T_r)=\max\{|I_1|,|I_2|,|I_3|\} \leqslant \frac{7}{15}n$ without a better low bound of $OPT(T_r)$ than $\frac{n}{3}$.

Case 1: $\frac{2}{15}n \leqslant |X| \leqslant \frac{7}{15}n$

Partition Approach 1 We assign one independent set I_1 to X and two other independent sets I_2 and I_3 to Y as evenly as possible.

Suppose that $|I_2| \geqslant |I_3|$. When $|I_1|=|X| \leqslant \frac{7}{15}n$, $|I_2|=\left\lceil \frac{|Y|}{2} \right\rceil \leqslant \frac{7}{15}n$, and $|I_3|=\left\lfloor \frac{|Y|}{2} \right\rfloor \leqslant \frac{7}{15}n$, then $Alg(T_r) \leqslant \frac{7}{5} \times \frac{n}{3} \leqslant \frac{7}{5} OPT(T_r)$.

Case 2: $\frac{7}{15}n < |X| \leqslant \frac{1}{2}n$

When $\frac{7}{15}n < |X| \leqslant \frac{1}{2}n$, then $\frac{1}{2}n \leqslant |Y| < \frac{8}{15}n$. In this case, the partition approach mainly focuses on identifying two compatible vertex subsets, $X' \subset X$ and $Y' \subset Y$, such that $|X'| \geqslant \frac{1}{15}n$, $|Y'| \geqslant \frac{1}{15}n$, and $|X'|+|Y'| \leqslant \frac{7}{15}n$. If the aforementioned condition is met, then let $I_1 = X' \cup Y'$, $I_2 = X \setminus X'$, $I_3 = Y \setminus Y'$, yielding $Alg(T_r)=\max\{|I_1|,|I_2|,|I_3|\} \leqslant \frac{7}{5} \times \frac{n}{3} \leqslant \frac{7}{5} OPT(T_r)$.

Let $\Phi=\{\mu | L_\mu \subset X, |L_\mu| \geqslant \frac{1}{15}n\}$ and $\Psi=\{\nu | L_\nu \subset Y, |L_\nu| \geqslant \frac{1}{15}n\}$ for simplicity. The following four subcases are determined.

Subcase 1 $\Phi \neq \varnothing, \Psi \neq \varnothing$, and $\exists i \in \Phi, j \in \Psi$, such that $|i-j|>1$.

If $|i-j|>1$ holds for some i and j, then any vertex in L_i is not adjacent to that in L_j, indicating that L_i and L_j are compatible. Therefore, the following partition approach can be applied to obtain the results.

Partition Approach 2

Step 1: If $|L_i|+|L_j| \leqslant \frac{7}{15}n$, let $X'=L_i$ and $Y'=L_j$. Otherwise, let X' be any of the smallest subsets of L_i, such that $|X'| \geqslant \frac{1}{15}n$, let Y' be any of the smallest subsets of L_j such that $|Y'| \geqslant \frac{1}{15}n$.

Step 2: Let $I_1 = X' \cup Y'$, $I_2 = X \setminus X'$, and $I_3 = Y \setminus Y'$. Thus, $Alg(T_r)=\max\{|I_1|,|I_2|,|I_3|\}$ is outputted.

Subcase 2 $\Phi \neq \varnothing, \Psi \neq \varnothing$, and $\forall i \in \Phi, j \in \Psi$, $|i-j|=1$ holds.

$|i-j|=1$ implies that L_i is always adjacent to L_j, $\forall i \in \Phi, j \in \Psi$. Let $L_\alpha, L_{\alpha+1}, \dots, L_{\alpha+s-1}$ be an adjacent sequence, such that $|L_\gamma| \geqslant \frac{1}{15}n(\gamma=\alpha, \alpha+$

$1, \ldots, \alpha + s - 1$). Then we have the following lemma.

Lemma 2 The length of the adjacent sequence does not exceed 3, that is $s \leqslant 3$.

Proof Let $L_\alpha, L_{\alpha+1}, \ldots, L_{\alpha+s-1}$ be the adjacent sequence of T_r and $s \geqslant 4$. Suppose that $L_\alpha \subset X$, it is not hard to see that one of $L_{\alpha+s-2}$ and $L_{\alpha+s-1}$ must be a subset of Y and is not adjacent to L_α. This condition is a contradiction with the current subcase; therefore, Lemma 2 holds. ■

Based on Lemma 2, we know that the length of the adjacent sequence is either 2 or 3. In what follows, we distinguish these two subcases.

Subcase 2.1 The adjacent sequence is L_i and L_j, where $j = i + 1$.

Without loss of generality, we assume that $L_i \subset X$ and $L_j \subset Y$, then we have the following partition approach.

Partition Approach 3

Step 1: Sort each vertex of L_i in non-increasing order according to the number of its corresponding children. Denote the ordered t-th node in L_i as $L_i^{(t)}$ and its children as $L_j^{(t)}$.

Step 2: If $|L_j^{(1)}| < \frac{1}{15}n$, then let k be the smallest index, such that $\sum_{t=1}^{k} |L_j^{(t)}| \geqslant \frac{1}{15}n$, let $Y' = \sum_{t=1}^{k} L_j^{(t)}$. Otherwise, let Y' be any of the smallest subsets of $L_j^{(1)}$, such that $|Y'| \geqslant \frac{1}{15}n$.

Step 3: Identify all nodes in L_i, which are not adjacent to any node in Y', and denote them by a set N.

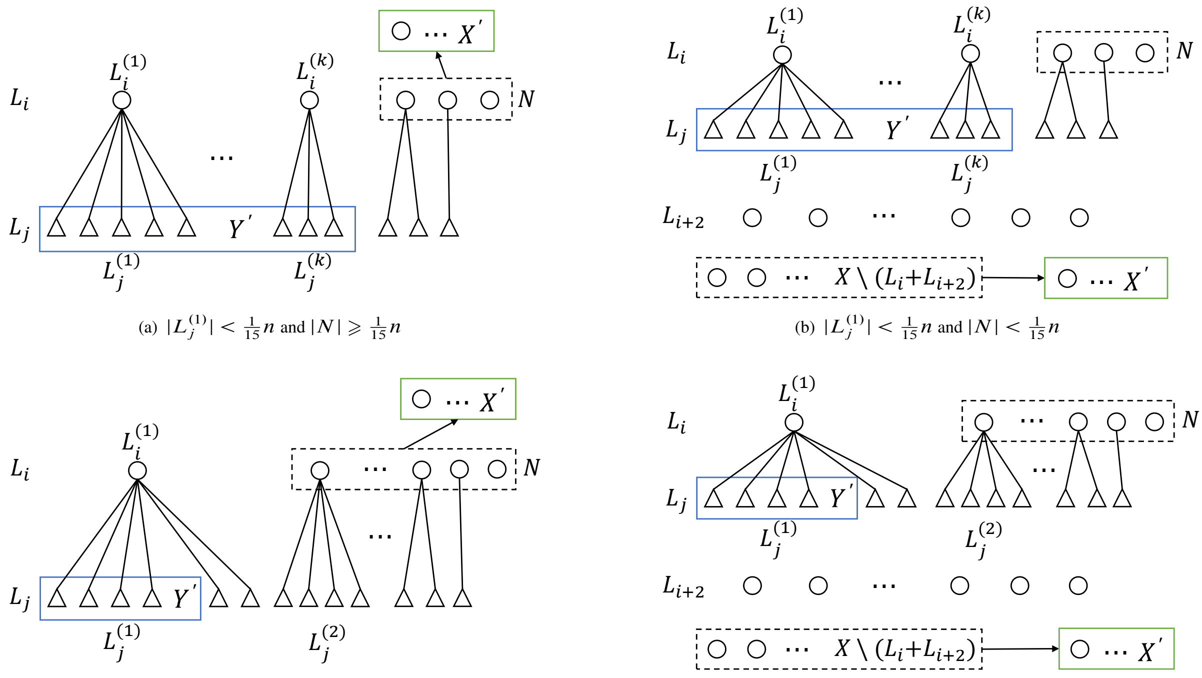
Step 4: If $|N| \geqslant \frac{1}{15}n$, let X' be any of the smallest subsets of N, such that $|X'| \geqslant \frac{1}{15}n$. Otherwise, let X' be any of the smallest subsets of $X \backslash \{L_i + L_{i+2}\}$, such that $|X'| \geqslant \frac{1}{15}n$. Then, let $I_1 = X' \cup Y'$, $I_2 = X \backslash X'$, and $I_3 = Y \backslash Y'$, thereby yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Lemma 3 Partition Approach 3 can produce a feasible partition solution, such that $Alg(T_r) \leqslant \frac{7}{5}OPT(T_r)$.

Proof If $|L_j^{(1)}| < \frac{1}{15}n$ (see Fig. 1), then $\forall t, L_j^{(t)} < \frac{1}{15}n$ because L_i has been sorted. In addition, $Y' = \sum_{t=1}^{k} L_j^{(t)}$, where k is the smallest index such that $\sum_{t=1}^{k} |L_j^{(t)}| \geqslant \frac{1}{15}n$; thus, $\frac{1}{15}n \leqslant |Y'| < \frac{2}{15}n$. If $|L_j^{(1)}| \geqslant \frac{1}{15}n$, then Y' will be the smallest set, such that $\frac{1}{15}n \leqslant |Y'| < \frac{2}{15}n$.

If $|N| \geqslant \frac{1}{15}n$, we can conclude that $\frac{1}{15}n \leqslant |X'| < \frac{2}{15}n$ similarly. If $|N| < \frac{1}{15}n$, then $|L_i| < \frac{3}{15}n$ because $|L_i \backslash N| \leqslant |Y'| < \frac{2}{15}n$. Combined with $|L_{i+2}| < \frac{1}{15}n$ and $|X| > \frac{7}{15}n$, we derive $|X \backslash \{L_i + L_{i+2}\}| > \frac{3}{15}n$. This condition implies the feasibility of yielding a subset X' from $X \backslash \{L_i + L_{i+2}\}$, such that $\frac{1}{15}n \leqslant |X'| < \frac{2}{15}n$. X' and Y' are compatible and can be merged into an independent set I_1. Therefore, $|I_1| = |X'| + |Y'| < \frac{4}{15}n$, $|I_2| = |X \backslash X'| < \frac{7}{15}n$, $|I_3| = |Y \backslash Y'| < \frac{7}{15}n$, and $Alg(T_r) < \frac{7}{15}n \leqslant \frac{7}{5}OPT(T_r)$. ■

Subcase 2.2 The adjacent sequence is L_i, L_j, and



(a) $|L_j^{(1)}| < \frac{1}{15}n$ and $|N| \geqslant \frac{1}{15}n$ (b) $|L_j^{(1)}| < \frac{1}{15}n$ and $|N| < \frac{1}{15}n$

(c) $|L_j^{(1)}| \geqslant \frac{1}{15}n$ and $|N| \geqslant \frac{1}{15}n$ (d) $|L_j^{(1)}| \geqslant \frac{1}{15}n$ and $|N| < \frac{1}{15}n$

Fig. 1 Overview of Partition Approach 3 (○ and △ represent single vertices in *X* and *Y*, respectively).

L_{i+2}, where $j = i + 1$. Then we can apply Partition Approach 3 to L_j and L_{i+2}. Similar to Lemma 3, the same result can be obtained.

Subcase 3 $\Psi = \varnothing, \Phi \neq \varnothing$.

Let L_i be the level satisfying $|L_i| \geqslant \frac{1}{15}n$, L_{j-2} and L_j be adjacent to L_i.

Partition Approach 4

Step 1: Let X' be any of the smallest subsets of L_i, such that $|X'| \geqslant \frac{1}{15}n$.

Step 2: Let Y' be any of the smallest subsets of $Y \backslash \{L_{j-2} + L_j\}$, such that $|Y'| \geqslant \frac{1}{15}n$.

Step 3: Let $I_1 = X' \cup Y'$, $I_2 = X \backslash X'$, and $I_3 = Y \backslash Y'$, thereby yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Lemma 4 Partition Approach 4 can produce a feasible partition solution, such that $Alg(T_r) \leqslant \frac{7}{5}OPT(T_r)$.

Proof As $|L_i| \geqslant \frac{1}{15}n$ (see Fig. 2), we can conclude that $\frac{1}{15}n \leqslant |X'| < \frac{2}{15}n$. Since $|L_{j-2}| < \frac{1}{15}n, |L_j| < \frac{1}{15}n$, and $|Y| > \frac{7}{15}n$, we obtain $|Y \backslash \{L_{j-2} + L_j\}| > \frac{5}{15}n$, making it feasible for Step 2. Therefore, $|I_1| = |X'| + |Y'| < \frac{4}{15}n$, $|I_2| = |X \backslash X'| < \frac{7}{15}n$, $|I_3| = |Y \backslash Y'| < \frac{7}{15}n$, and $Alg(T_r) < \frac{7}{15}n \leqslant \frac{7}{5}OPT(T_r)$. ■

Subcase 4 $\Phi = \Psi = \varnothing$.

Without loss of generality, we assume that L_i and L_j are the odd and even levels, respectively.

Partition Approach 5

Step 1: Let k be the smallest index, such that $\sum_{t=1}^{k} |L_{2t-1}| \geqslant \frac{1}{15}n$.

Step 2: If $|Y \backslash \sum_{t=1}^{k} L_{2t}| \geqslant \frac{1}{15}n$, then let $X' = \sum_{t=1}^{k} L_{2t-1}$ and Y' be any of the smallest subsets of $Y \backslash \sum_{t=1}^{k} L_{2t}$, such that $|Y'| \geqslant \frac{1}{15}n$. Afterward, proceed to Step 4.

Step 3: If $|Y \backslash \sum_{t=1}^{k} L_{2t}| < \frac{1}{15}n$, then let X' be any of the smallest subsets of $X \backslash \sum_{t=1}^{k} L_{2t-1}$, such that $|X'| \geqslant \frac{1}{15}n$, and Y' be any of the smallest subsets of $\sum_{t=1}^{k-1} L_{2t}$, such that $|Y'| \geqslant \frac{1}{15}n$. Afterward, proceed to Step 4.

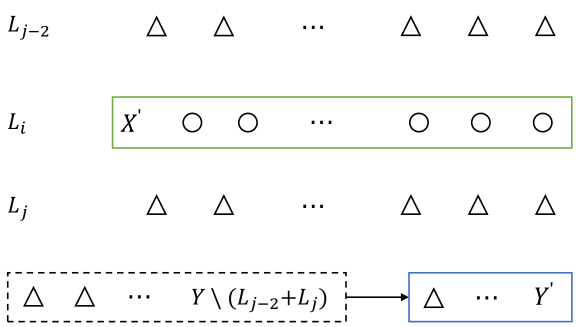


Fig. 2 Overview of Partition Approach 4 (○ and △ represent single vertices in *X* and *Y*, respectively).

Step 4: Let $I_1 = X' \cup Y'$, $I_2 = X \backslash X'$, and $I_3 = Y \backslash Y'$, thereby yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Lemma 5 Partition Approach 5 can produce a feasible partition solution, such that $Alg(T_r) \leqslant \frac{7}{5}OPT(T_r)$.

Proof Through Steps 1 and 2 (see Fig. 3), we can obtain compatible X' and Y' such that $|I_1| < \frac{4}{15}n$, and $|I_2|$ and $|I_3|$ are less than $\frac{7}{15}n$. In Step 3 (see Fig. 4), $|Y| \geqslant \frac{1}{2}n > \frac{7}{15}n$, $\sum_{t=1}^{k-1} |L_{2t}|$ must exceed $\frac{5}{15}n$ because $|Y \backslash \sum_{t=1}^{k} L_{2t}|$ and $|L_{2k}|$ are less than $\frac{1}{15}n$. As previously discussed, $\sum_{t=1}^{k} |L_{2t-1}| \geqslant \frac{1}{15}n$, where k is the smallest index, leading to $\sum_{t=1}^{k} |L_{2t-1}| < \frac{2}{15}n$. Accordingly, $|X \backslash \sum_{t=1}^{k} L_{2t-1}|$ and $\sum_{t=1}^{k-1} |L_{2t}|$ exceed $\frac{5}{15}n$, allowing Step 3 to yield compatible X' and Y'. Therefore, $|I_1| = |X'| + |Y'| < \frac{4}{15}n$, $|I_2| = |X \backslash X'| < \frac{7}{15}n$, $|I_3| = |Y \backslash Y'| < \frac{7}{15}n$, and $Alg(T_r) < \frac{7}{15}n \leqslant \frac{7}{5}OPT(T_r)$. ■

Case 3: $1 \leqslant |X| < \frac{2}{15}n$

Suppose that X and Y comprise levels with odd and even indexs, respectively. In this case, we first identify

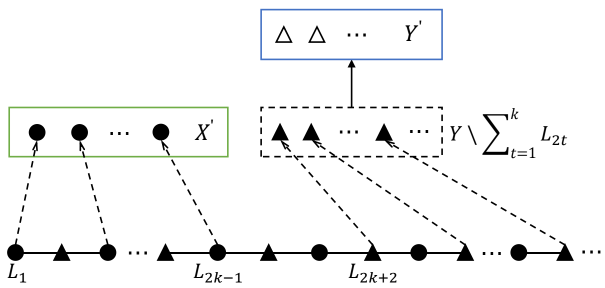


Fig. 3 When Step 2 is active in Partition Approach 5 (△ represents single vertex in *Y*; ● and ▲ represent a number of vertices in *X* and *Y*, respectively).

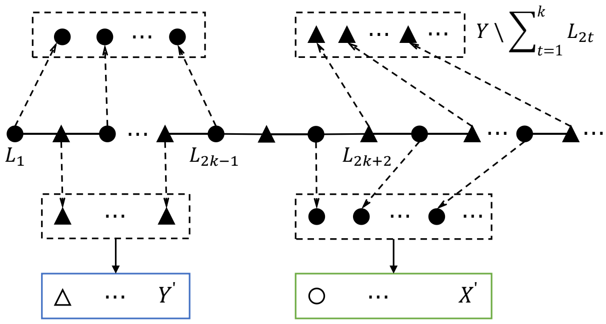


Fig. 4 When Step 3 is active in Partition Approach 5 (○ and △ represent single vertices in *X* and *Y*, respectively; ● and ▲ represent a number of vertices in *X* and *Y*, respectively).

the smallest index k, such that $\sum_{t=1}^{k}|L_{2t}| \geqslant \frac{5}{15}n$, then distinguish the two cases according to the size of $Y\setminus\sum_{t=1}^{k}L_{2t}$.

Subcase 5 $|Y\setminus\sum_{t=1}^{k}L_{2t}| \geqslant \frac{4}{15}n$.

Partition Approach 6

Step 1: Let Y_1 be any of the smallest subsets of $Y\setminus\sum_{t=1}^{k}L_{2t}$, such that $|Y_1| \geqslant \frac{4}{15}n$. Let $I_1 = Y_1 \cup \sum_{t=1}^{k}L_{2t-1}$.

Step 2: All non-leaf nodes in L_{2k} are denoted as set L_{2k}^{in}. If $1 \leqslant |X| \leqslant \frac{1}{15}n$, then let Y_2 be any of the smallest subsets of $\sum_{t=1}^{k}L_{2t}\setminus L_{2k}^{in}$, such that $|Y_2| \geqslant \frac{4}{15}n$. If $\frac{1}{15}n < |X| < \frac{2}{15}n$, then let Y_2 be any of the smallest subsets of $\sum_{t=1}^{k}L_{2t}\setminus L_{2k}^{in}$, such that $|Y_2| \geqslant \frac{3}{15}n$.

Step 3: Let $I_2 = Y_2 + X\setminus\sum_{t=1}^{k}L_{2t-1}$.

Step 4: Let $I_3 = Y\setminus\{Y_1, Y_2\}$, yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Lemma 6 Partition Approach 6 can produce a feasible partition solution, such that $Alg(T_r) \leqslant \frac{7}{5}OPT(T_r)$.

Proof When $\frac{1}{15}n < |X| < \frac{2}{15}n$ (see Fig. 5), then $|Y_1| < \frac{5}{15}n$ in Step 1. Together with $\sum_{t=1}^{k}|L_{2t-1}| < |X| < \frac{2}{15}n$, we derive $\frac{4}{15}n \leqslant |I_1| < \frac{7}{15}n$. In Step 2, as $|L_{2k+1}| < |X| < \frac{2}{15}n$, by Lemma 1, we then have $|L_{2k}^{in}| \leqslant |L_{2k+1}| < \frac{2}{15}n$ and $|\sum_{t=1}^{k}L_{2t}\setminus L_{2k}^{in}| > \frac{3}{15}n$, implying the feasibility of yielding Y_2. L_{2k}^{in} is not included in Y_2, which is incompatible with L_{2k+1}; thus, I_2 becomes an independent set. Similar to Y_1, we have $\frac{3}{15}n \leqslant |Y_2| < \frac{4}{15}n$. Together with $|X\setminus\sum_{t=1}^{k}L_{2t-1}| < |X| < \frac{2}{15}n$, we can conclude that $\frac{3}{15}n \leqslant |I_2| < \frac{6}{15}n$ and $|I_3| = |Y| - |Y_1| - |Y_2| < \frac{14}{15}n - \frac{4}{15}n - \frac{3}{15}n = \frac{7}{15}n$. Therefore, $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\} < \frac{7}{15}n \leqslant \frac{7}{5}OPT(T_r)$.

Similarly, we can prove the lemma when $1 \leqslant |X| \leqslant \frac{1}{15}n$. ■

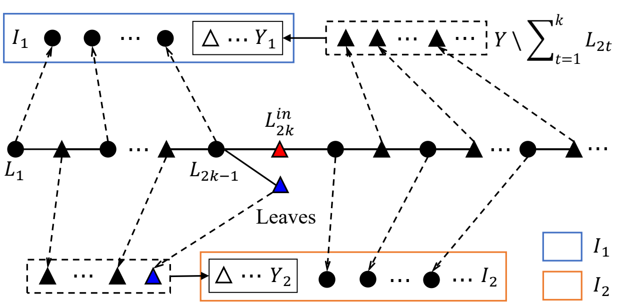


Fig. 5 Overview of Partition Approach 6 (△ represents single vertex in *Y*; ● and ▲ represent a number of vertices in *X* and *Y*, respectively; ▲ and ▲ represent non-leaf and leaf nodes of L_{2k}, respectively).

Subcase 6 $|Y\setminus\sum_{t=1}^{k}L_{2t}| < \frac{4}{15}n$.

Since $|Y| > \frac{13}{15}n$ and $\sum_{t=1}^{k-1}|L_{2t}| < \frac{5}{15}n$, it can be inferred that $|L_{2k}| > \frac{4}{15}n$. Sort each vertex in L_{2k-1} in non-increasing order according to the number of its corresponding children in L_{2k}. Denote the ordered t-th vertex in L_{2k-1} and its children, respectively, as $L_{2k-1}^{(t)}$ and $L_{2k}^{(t)}$.

Now we distinguish four subcases mainly according to the size of $|L_{2k}^{(1)}|$.

Subcase 7 $|L_{2k}^{(1)}| \geqslant \frac{10}{15}n$.

In this subcase, a superior low bound of the optimal solution is first provided and then used for the proof of the worst-case ratio.

Lemma 7 $OPT(T_r) \geqslant \left\lceil \frac{|L_{2k}^{(1)}|}{2} \right\rceil \geqslant \frac{n}{3}$.

Proof Since any vertex in $L_{2k}^{(1)}$ is adjacent to $L_{2k-1}^{(1)}$, then vertices in $L_{2k}^{(1)}$ cannot occupy all three independent sets in the optimal solution. Hence, we have $OPT(T_r) \geqslant \left\lceil \frac{|L_{2k}^{(1)}|}{2} \right\rceil \geqslant \frac{n}{3}$. ■

Partition Approach 7

Step 1: Denote arbitrary $\left\lceil \frac{|L_{2k}^{(1)}|}{2} \right\rceil$ nodes in $L_{2k}^{(1)}$ by set Y^*, and all the children of Y^* by set $\widetilde{Y}^*$.

Step 2: Let $I_1 = Y^* + X\setminus\{L_{2k-1}^{(1)} + \widetilde{Y}^*\}$.

Step 3: Assign $L_{2k}^{(1)}\setminus Y^*$ together with $\widetilde{Y}^*$ and the parent of $L_{2k-1}^{(1)}$ into I_2.

Step 4: Assign all the residual vertices into I_3, yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Lemma 8 Partition Approach 7 can produce a feasible partition solution, such that $Alg(T_r) \leqslant \frac{7}{5}OPT(T_r)$.

Proof In Step 2, the total number of vertices in $X\setminus\{L_{2k-1}^{(1)} + \widetilde{Y}^*\}$ is less than $\frac{2}{15}n$ due to $|X| < \frac{2}{15}n$. Thus, $|I_1| < \left\lceil \frac{|L_{2k}^{(1)}|}{2} \right\rceil + \frac{2}{15}n \leqslant OPT(T_r) + \frac{2}{5}OPT(T_r) = \frac{7}{5}OPT(T_r)$. In Step 3, since $|L_{2k-1}^{(1)} + \widetilde{Y}^*| \leqslant |X| < \frac{2}{15}n$, then the number of vertices in $\widetilde{Y}^*$ and the parent of $L_{2k-1}^{(1)}$ is less than $\frac{2}{15}n$. Thus, $|I_2| < \left\lceil \frac{|L_{2k}^{(1)}|}{2} \right\rceil + \frac{2}{15}n \leqslant OPT(T_r) + \frac{2}{5}OPT(T_r) = \frac{7}{5}OPT(T_r)$. In addition, $|I_3| \leqslant n - |L_{2k}^{(1)}| \leqslant \frac{5}{15}n$. Hence, $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\} < \frac{7}{5}OPT(T_r)$ holds (see Fig. 6). ■

Subcase 8 $\frac{5}{15}n \leqslant |L_{2k}^{(1)}| < \frac{10}{15}n$.

Let Y^* be any of the smallest subsets of $L_{2k}^{(1)}$ such that $|Y^*| \geqslant \frac{5}{15}n$, and $\widetilde{Y}^*$ are denoted as the children of Y^*. For completeness, we then distinguish three subcases and provide a partition approach for each one despite

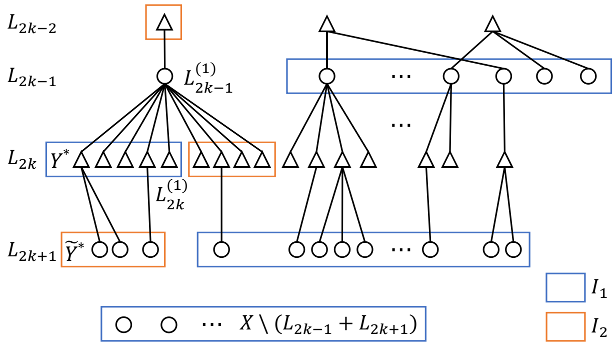


Fig. 6 Overview of Partition Approach 7 (○ and △ represent single vertices in *X* and *Y*, respectively).

the substantial similarity of the following three partition approaches.

Subcase 9 $1 \leqslant |X| \leqslant \frac{1}{15}n$ (see Fig. 7).

Partition Approach 8

Step 1: Let $I_1 = Y^* + X\backslash\{L^{(1)}_{2k-1} + \widetilde{Y}^*\}$.

Step 2: Assign $L^{(1)}_{2k}\backslash Y^*$ together with the parent of $L^{(1)}_{2k-1}$ into I_2.

Step 2.1: If $|I_2| < \frac{4}{15}n$, then add as few nodes as possible from $Y\backslash\{Y^* + I_2\}$ excluding the children of $\widetilde{Y}^*$, such that $|I_2| \geqslant \frac{4}{15}n$; that is, add minimum nodes in Y, which are compatible with $\widetilde{Y}^*$ to I_2 until $|I_2| \geqslant \frac{4}{15}n$.

Step 2.2: Assign $\widetilde{Y}^*$ into I_2.

Step 3: Assign all the residual vertices into I_3, yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Subcase 10 $\frac{1}{15}n < |X| < \frac{2}{15}n$, $|X\backslash\{L^{(1)}_{2k-1} + \widetilde{Y}^*\}| < \frac{1}{15}n$.

Partition Approach 9

Step 1: Let $I_1 = Y^* + X\backslash\{L^{(1)}_{2k-1} + \widetilde{Y}^*\}$.

Step 2: Assign $L^{(1)}_{2k}\backslash Y^*$ together with the parent of $L^{(1)}_{2k-1}$ into I_2.

Step 2.1: If $|I_2| < \frac{3}{15}n$, then add as few nodes as possible from $Y\backslash\{Y^* + I_2\}$ excluding the children of $\widetilde{Y}^*$, such that $|I_2| \geqslant \frac{3}{15}n$; that is, add minimum nodes in

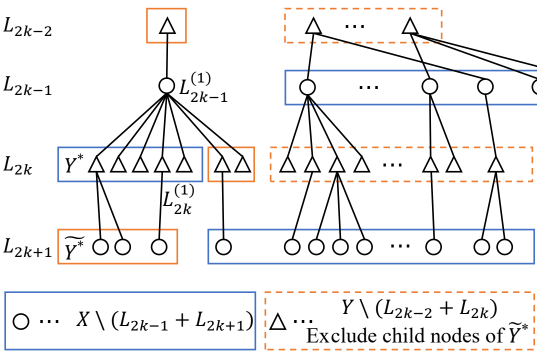


Fig. 7 Overview of Partition Approach 8 (○ and △ represent single vertices in *X* and *Y*, respectively).

Y, which are compatible with $\widetilde{Y}^*$ to I_2 until $|I_2| \geqslant \frac{3}{15}n$.

Step 2.2: Assign $\widetilde{Y}^*$ into I_2.

Step 3: Assign all the residual vertices into I_3, yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Subcase 11 $\frac{1}{15}n < |X| < \frac{2}{15}n$, $|X\backslash\{L^{(1)}_{2k-1} + \widetilde{Y}^*\}| \geqslant \frac{1}{15}n$.

Partition Approach 10

Step 1: Readjust the size of Y^* to be any of the smallest subsets of $L^{(1)}_{2k}$ such that $|Y^*| \geqslant \frac{4}{15}n$, and then update $\widetilde{Y}^*$ correspondingly.

Step 2: Let $I_1 = Y^* + X\backslash\{L^{(1)}_{2k-1} + \widetilde{Y}^*\}$.

Step 3: Assign $L^{(1)}_{2k}\backslash Y^*$ together with the parent of $L^{(1)}_{2k-1}$ into I_2.

Step 3.1: If $|I_2| < \frac{4}{15}n$, then add as few nodes as possible from $Y\backslash\{Y^* + I_2\}$ excluding the children of $\widetilde{Y}^*$, such that $|I_2| \geqslant \frac{4}{15}n$; that is, add minimum nodes in Y, which are compatible with $\widetilde{Y}^*$ to I_2 until $|I_2| \geqslant \frac{4}{15}n$.

Step 3.2: Assign $\widetilde{Y}^*$ into I_2.

Step 4: Assign all the residual vertices into I_3, yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

For simplicity, we only consider the proof of Subcase 11 in what follows because the two other subcases can be verified similarly.

Lemma 9 Partition Approach 10 can produce a feasible partition solution, such that $Alg(T_r) \leqslant \frac{7}{5}OPT(T_r)$.

Proof After Step 1, $\frac{4}{15}n \leqslant |Y^*| < \frac{5}{15}n$ holds. When $\frac{1}{15}n \leqslant |X\backslash\{L^{(1)}_{2k-1} + \widetilde{Y}^*\}| < |X| < \frac{2}{15}n$, we can infer $|I_1| < |Y^*| + \frac{2}{15}n < \frac{7}{15}n$ accordingly by Step 2.

If $|I_2| \geqslant \frac{4}{15}n$, then Step 3.1 is skipped. Since $|X\backslash\{L^{(1)}_{2k-1} + \widetilde{Y}^*\}| \geqslant \frac{1}{15}n$ and $|X| < \frac{2}{15}n$, $|L^{(1)}_{2k-1} + \widetilde{Y}^*| < \frac{1}{15}n$ holds. Accordingly, the total size of $\widetilde{Y}^*$ and the parent of $L^{(1)}_{2k-1}$ is less than $\frac{1}{15}n$. Based on $|L^{(1)}_{2k}| < \frac{10}{15}n$ and $|Y^*| \geqslant \frac{4}{15}n$, the size of $L^{(1)}_{2k}\backslash Y^*$ is estimated to be less than $\frac{6}{15}n$. Thus, $|I_2| = |L^{(1)}_{2k}\backslash Y^*| + |\widetilde{Y}^*| + 1 < \frac{6}{15}n + \frac{1}{15}n = \frac{7}{15}n$.

If $|I_2| < \frac{4}{15}n$, then Step 3.1 is activated. After some additional nodes are added to I_2, the size of I_2 will end up as $\frac{4}{15}n \leqslant |I_2| < \frac{5}{15}n$ because nodes are added individually to satisfy $|I_2| \geqslant \frac{4}{15}n$. From $|L^{(1)}_{2k-1} + \widetilde{Y}^*| < \frac{1}{15}n$, we know that $|\widetilde{Y}^*| < \frac{1}{15}n$. Therefore, $|I_2| < \frac{5}{15}n + |\widetilde{Y}^*| < \frac{6}{15}n$.

The aforementioned discussion proves that the upper bound of $|I_2|$ is $\frac{7}{15}n$ whether Step 3.1 is active or not. It should suffice to demonstrate the continuous success of the aforementioned approach in complementing the size of I_2 to satisfy $\frac{4}{15}n$ when Step 3.1 is active. Suppose that $|I_2| < \frac{4}{15}n$ after all triangles compatible with $\widetilde{Y}^*$ are put into I_2. Since $|Y^*| < \frac{5}{15}n$, $|I_2| < \frac{4}{15}n$, and $|Y| > \frac{13}{15}n$, the size of the children of $\widetilde{Y}^*$ must exceed $\frac{4}{15}n$, contradicting $|Y \setminus \sum_{t=1}^{k} L_{2t}| < \frac{4}{15}n$. This condition guarantees that at least $\frac{4}{15}n$ triangles are included in I_2.

Since I_1 and I_2 contain at least $\frac{4}{15}n$ triangles, we derive $|I_3| \leqslant \frac{7}{15}n$. Hence, $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\} \leqslant \frac{7}{5}OPT(T_r)$ holds. ■

Subcase 11 $|L_{2k}^{(1)}| < \frac{5}{15}n$, $|L_{2k}| > \frac{5}{15}n$.

First, identify the smallest index l, such that $\sum_{t=1}^{l} |L_{2k}^{(t)}| \geqslant \frac{5}{15}n$. Let $Y^* = \sum_{t=1}^{l-1} L_{2k}^{(t)}$, then add as few nodes as possible from $L_{2k}^{(l)}$ into Y^*, such that $|Y^*| \geqslant \frac{5}{15}n$. Let $\widetilde{Y}^*$ be the children of Y^*. For completeness, we then distinguish four subcases and provide a partition approach for each one despite the considerable similarity following the four approaches.

Subcase 12 $1 \leqslant |X| \leqslant \frac{1}{15}n$.

Partition Approach 11

Step 1: Let $I_1 = Y^* + X \setminus \{\sum_{t=1}^{l} L_{2k-1}^{(t)} + \widetilde{Y}^*\}$.

Step 2: Assign $\sum_{t=1}^{l} L_{2k}^{(t)} \setminus Y^*$ together with the parents of $\sum_{t=1}^{l} L_{2k-1}^{(1)}$ into I_2.

Step 2.1: If $|I_2| < \frac{4}{15}n$, then add as few nodes as possible from $Y \setminus \{Y^* + I_2\}$ excluding the children of $\widetilde{Y}^*$, such that $|I_2| \geqslant \frac{4}{15}n$; that is, add minimum nodes in Y, which are compatible with $\widetilde{Y}^*$ to I_2 until $|I_2| \geqslant \frac{4}{15}n$.

Step 2.2: Assign $\widetilde{Y}^*$ into I_2.

Step 3: Assign all the residual vertices into I_3, yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Subcase 13 $\frac{1}{15}n < |X| < \frac{2}{15}n$, $|X \setminus \{\sum_{t=1}^{l} L_{2k-1}^{(t)} + \widetilde{Y}^*\}| < \frac{1}{15}n$, and $|\sum_{t=1}^{l} L_{2k-1}^{(t)}| < \frac{1}{15}n$.

Partition Approach 12

Step 1: Let $I_1 = Y^* + X \setminus \{\sum_{t=1}^{l} L_{2k-1}^{(t)} + \widetilde{Y}^*\}$.

Step 2: Assign $\sum_{t=1}^{l} L_{2k}^{(t)} \setminus Y^*$ together with the parents of $\sum_{t=1}^{l} L_{2k-1}^{(1)}$ into I_2.

Step 2.1: If $|I_2| < \frac{3}{15}n$, then add as few nodes as possible from $Y \setminus \{Y^* + I_2\}$ excluding the children of $\widetilde{Y}^*$ such that $|I_2| \geqslant \frac{3}{15}n$; that is, add minimum nodes in Y, which are compatible with $\widetilde{Y}^*$ to I_2 until $|I_2| \geqslant \frac{3}{15}n$.

Step 2.2: Assign $\widetilde{Y}^*$ into I_2.

Step 3: Assign all the residual vertices into I_3, yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Subcase 14 $\frac{1}{15}n < |X| < \frac{2}{15}n$, $|X \setminus \{\sum_{t=1}^{l} L_{2k-1}^{(t)} + \widetilde{Y}^*\}| \geqslant \frac{1}{15}n$.

Partition Approach 13

Step 1: Readjust l to be the smallest index such that $\sum_{t=1}^{l} |L_{2k}^{(t)}| \geqslant \frac{4}{15}n$. Let $Y^* = \sum_{t=1}^{l-1} L_{2k}^{(t)}$, then add as few nodes as possible from $L_{2k}^{(l)}$ into Y^* such that $|Y^*| \geqslant \frac{4}{15}n$. Update $\widetilde{Y}^*$ correspondingly.

Step 2: Let $I_1 = Y^* + X \setminus \{\sum_{t=1}^{l} L_{2k-1}^{(t)} + \widetilde{Y}^*\}$.

Step 3: Assign $\sum_{t=1}^{l} L_{2k}^{(t)} \setminus Y^*$ together with the parents of $\sum_{t=1}^{l} L_{2k-1}^{(1)}$ into I_2.

Step 3.1: If $|I_2| < \frac{4}{15}n$, then add as few nodes as possible from $Y \setminus \{Y^* + I_2\}$ excluding the children of $\widetilde{Y}^*$ such that $|I_2| \geqslant \frac{4}{15}n$; that is, add minimum nodes in Y, which are compatible with $\widetilde{Y}^*$ to I_2 until $|I_2| \geqslant \frac{4}{15}n$.

Step 3.2: Assign $\widetilde{Y}^*$ into I_2.

Step 4: Assign all the residual vertices into I_3, yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Subcase 15 $\frac{1}{15}n < |X| < \frac{2}{15}n$, $|X \setminus \{\sum_{t=1}^{l} L_{2k-1}^{(t)} + \widetilde{Y}^*\}| < \frac{1}{15}n$, and $|\sum_{t=1}^{l} L_{2k-1}^{(t)}| \geqslant \frac{1}{15}n$.

Partition Approach 14

Step 1: Let $I_1 = Y^* + X \setminus \{\sum_{t=1}^{l} L_{2k-1}^{(t)} + \widetilde{Y}^*\}$.

Step 2: Assign $\sum_{t=1}^{l} L_{2k}^{(t)} \setminus Y^*$ together with the parents of $\sum_{t=1}^{l} L_{2k-1}^{(1)}$ into I_2.

Step 2.1: If $|I_2| < \frac{4}{15}n$, then add as few nodes as possible from $Y \setminus \{Y^* + I_2\}$ excluding the children of $\widetilde{Y}^*$, such that $|I_2| \geqslant \frac{4}{15}n$; that is, add minimum nodes in Y, which are compatible with $\widetilde{Y}^*$ to I_2 until $|I_2| \geqslant \frac{4}{15}n$.

Step 2.2: After Step 2.1, if $|I_2| < \frac{4}{15}n$ (i.e., nodes manipulated in Step 2.1 are insufficient), then reset I_2 to be empty and then assign the children of $\widetilde{Y}^*$ together with $\sum_{t=1}^{l} L_{2k-1}^{(t)}$ into I_2. Otherwise, assign $\widetilde{Y}^*$ into I_2 directly.

Step 3: Assign all the residual vertices into I_3, yielding $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

For simplicity, we only consider the proof of Subcase 15 in what follows because the three other subcases can be verified similarly.

Lemma 10 Partition Approach 14 can produce a feasible partition solution such that $Alg(T_r) \leqslant \frac{7}{5}OPT(T_r)$.

Proof $\forall t$, $|L_{2k}^{(t)}| < \frac{5}{15}n$ obviously holds. Since $|L_{2k}| > \frac{5}{15}n$, then the feasibility of yielding Y^* and index l can be verified. Before Step 1, we have $\frac{5}{15}n \leqslant |Y^*| < \frac{6}{15}n$. Since $|X| < \frac{2}{15}n$ and $|\sum_{t=1}^{l} L_{2k-1}^{(t)}| \geqslant \frac{1}{15}n$, then $|X \setminus \{\sum_{t=1}^{l} L_{2k-1}^{(t)} + \widetilde{Y}^*\}| < \frac{1}{15}n$ holds. Therefore, $|I_1| = |Y^*| + |X \setminus \{\sum_{t=1}^{l} L_{2k-1}^{(t)} + \widetilde{Y}^*\}| < \frac{6}{15}n + \frac{1}{15}n = \frac{7}{15}n$.

If $|I_2| \geqslant \frac{4}{15}n$, then Step 2.1 is skipped. Since $|\sum_{t=1}^{l} L_{2k}^{(t)} \setminus Y^*| \leqslant |L_{2k}^{(l)}| < \frac{5}{15}n$, the size of $\widetilde{Y}^*$ and the parents of $\sum_{t=1}^{l} L_{2k-1}^{(t)}$ adds up to no more than $|\widetilde{Y}^* + \sum_{t=1}^{l} L_{2k-1}^{(t)}|$. As $|\widetilde{Y}^* + \sum_{t=1}^{l} L_{2k-1}^{(t)}| \leqslant |X| < \frac{2}{15}n$, then $|I_2| < \frac{5}{15}n + \frac{2}{15}n = \frac{7}{15}n$.

If $|I_2| < \frac{4}{15}n$, then Step 2.1 is activated. If Step 2.1

succeeds to increase the size of I_2 such that $|I_2| \geqslant \frac{4}{15}n$, then $\frac{4}{15}n \leqslant |I_2| < \frac{5}{15}n$ holds after Step 2.1. Since $|I_2| \geqslant \frac{4}{15}n$, Step 2.2 runs the "Otherwise" branch by putting $\widetilde{Y}^*$ into I_2. With $|\widetilde{Y}^*| < \frac{1}{15}n$, we can conclude that $|I_2| < \frac{5}{15}n + |\widetilde{Y}^*| < \frac{6}{15}n$ (see Fig. 8).

Now we consider $|I_3|$ for the two aforementioned conditions. I_1 and I_2 are proved to contain at least $\frac{5}{15}n$ and $\frac{4}{15}n$ nodes, respectively. Thus, the size of I_3 is at most $\frac{6}{15}n$ and $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\} \leqslant \frac{7}{5}OPT(T_r)$ holds.

If Step 2.1 fails to add sufficient nodes, such that $|I_2| \geqslant \frac{4}{15}n$, then Step 2.2 runs the "If" branch by resetting I_2. Before Step 2.2, I_2 contains all the triangles except Y^* and the children of $\widetilde{Y}^*$. Since $|Y^*| < \frac{6}{15}n$, $|I_2| < \frac{4}{15}n$, and $|Y| > \frac{13}{15}n$, then the size of children nodes of $\widetilde{Y}^*$ exceeds $\frac{3}{15}n$. Recall $|Y \setminus \sum_{t=1}^{k} L_{2t}| < \frac{4}{15}n$, the size of the children of $\widetilde{Y}^*$ is less than $\frac{4}{15}n$. After Step 2.2, with $|\sum_{t=1}^{l} L_{2k-1}^{(t)}| < \frac{2}{15}n$, then $|I_2| < \frac{4}{15}n + \frac{2}{15}n = \frac{6}{15}n$, and I_2 includes at least $\frac{3}{15}n$ triangles. Notably, I_1 and I_2 contain at least $\frac{5}{15}n$ and $\frac{3}{15}n$ triangles, respectively. $|I_3| \leqslant \frac{7}{15}n$. Thus, $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\} \leqslant \frac{7}{5}OPT(T_r)$ holds (see Fig. 9). ∎

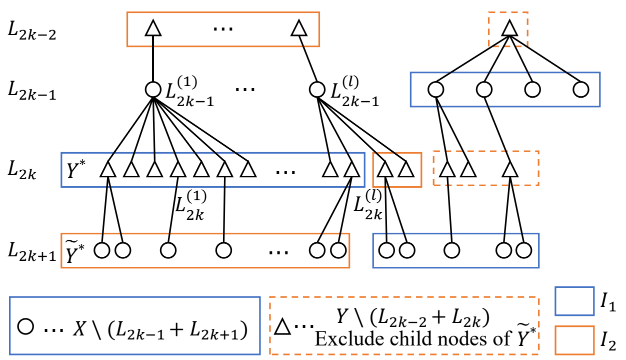


Fig. 8 Overview of Partition Approach 11 (○ and △ represent single vertices in *X* and *Y*, respectively).

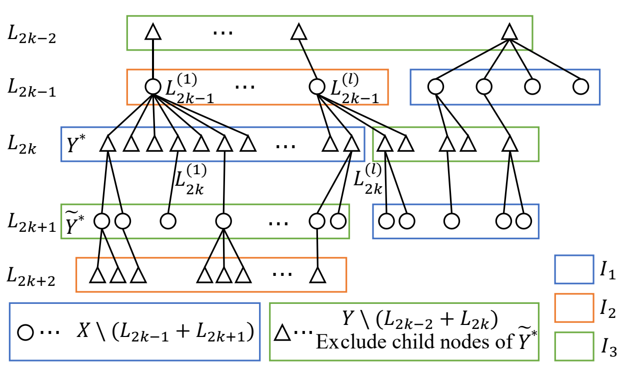


Fig. 9 "If" branch within Step 2.2 of Partition Approach 14 (○ and △ represent single vertices in *X* and *Y*, respectively).

Subcase 16 $|L_{2k}^{(1)}| < \frac{5}{15}n$ and $\frac{4}{15}n < |L_{2k}| \leqslant \frac{5}{15}n$.

Partition Approach 15

Step 1: Let $I_1 = L_{2k} + X \setminus \{L_{2k-1} + L_{2k+1}\}$.

Step 2: Let $I_2 = \sum_{t=1}^{k-1} L_{2t} + L_{2k+1}$.

Step 3: Let $I_3 = Y \setminus \sum_{t=1}^{k} L_{2t} + L_{2k-1}$.

Step 4: Output $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\}$.

Lemma 11 Partition Approach 15 can produce a feasible partition solution, such that $Alg(T_r) \leqslant \frac{7}{5}OPT(T_r)$.

Proof Since the size of L_{2k} is at most $\frac{5}{15}n$, then $|X \setminus \{L_{2k-1} + L_{2k+1}\}| < |X| < \frac{2}{15}n$ and $|I_1| < \frac{7}{15}n$. Recall that $|\sum_{t=1}^{k-1} L_{2t}| < \frac{5}{15}n$ and $|L_{2k+1}| < |X| < \frac{2}{15}n$. Thus, $|I_2|$ is less than $\frac{7}{15}n$. Similarly, from $|Y \setminus \sum_{t=1}^{k} L_{2t}| < \frac{4}{15}n$ and $|L_{2k-1}| < \frac{2}{15}n$, we can obtain $|I_3| < \frac{6}{15}n$. Therefore, $Alg(T_r) = \max\{|I_1|, |I_2|, |I_3|\} \leqslant \frac{7}{5}OPT(T_r)$ holds (see Fig. 10). ∎

4 Conclusion

We consider the partitioning problem of a given graph into three independent sets of minimizing the maximum one. We first provide a simple $\frac{3}{2}$-approximation algorithm for any 2-colorable graph and then design an improved $\frac{7}{5}$-approximation algorithm for a tree. One of the possible and significant directions for future research lies in the computational complexity of the partitioning problem of a tree into three independent sets of minimizing the maximum one.

Acknowledgment

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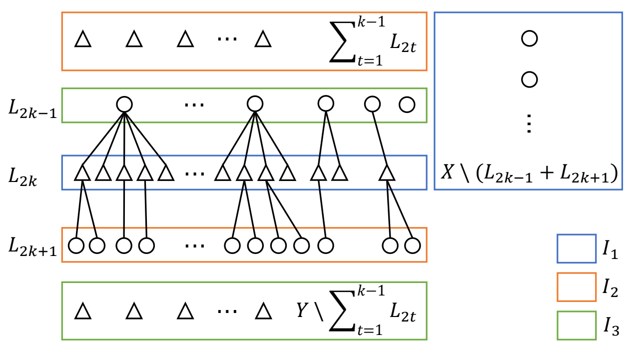

Fig. 10 Overview of Partition Approach 15 (○ and △ represent single vertices in *X* and *Y*, respectively).

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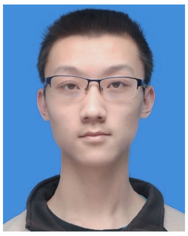

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