# Design and performance analysis of multiple-relay cooperative MIMO networks

Donatella Darsena, Giacinto Gelli, and Francesco Verde

***Abstract:*** **In this paper, we propose a closed-form (i.e., without the need of any iterative procedure) joint optimization framework of the source precoder, the amplify-and-forward relaying matrices, and the destination equalizer for a cooperative multiple-input multiple-output (MIMO) wireless network. We study in depth such a design and carry out its performance analysis in terms of average symbol error probability (ASEP), which allows one to calculate the diversity order of the cooperative system. The ASEP is also evaluated via Monte Carlo simulations and compared with recent competitive alternatives. Results show that the proposed design performs better than or comparably to recently proposed iterative approaches, with lower computational requirements.**

***Index Terms:*** **Amplify-and-forward relays, closed-form designs, minimum-mean-square-error criterion, multiple-input multiple-output (MIMO) systems, performance analysis.**

## I. INTRODUCTION

COMPARED to direct source-to-destination transmissions, cooperative multiple-input multiple-output (MIMO) communications can assure significant performance gains over some distance ranges [1], [2], even considering the additional energy overhead required for enabling cooperation. Such an overhead is mainly due to the need to acquire channel state information (CSI) at the source and the relays. In this paper, *full short-term* CSI is hereinafter assumed to be available at both the source and the relay, i.e., they have knowledge of the instantaneous values of the source-to-relay and relay-to-destination channels.[1]

Several papers (see [3]–[9]) dealt with optimization of cooperative MIMO networks encompassing multiple amplify-and-forward (AF) relays, with different power constraints (see Table 1 for a concise taxonomy). However, due to the non-convex nature of the considered optimization problems, closed-form (i.e., without the need of any iterative procedure) optimal solutions have been found only for simple cases (e.g., power-constrained relays equipped with a single antenna or, alternatively, multi-antenna relays without power constraints [3]) or only for a subset of the parameters to be optimized while the remaining ones are kept fixed [6]–[9]. To jointly design the source precoder, the AF relaying matrices, and the destination equalizer, several solutions have been developed [4]–[6], [8], [9], which basically rely on iterative schemes. These algorithms require a proper initialization to find a local optimum and their rate of convergence is sometimes not even simple to predict. Moreover, the lack of closed-form designs does not allow one to develop insightful theoretical analyses, aimed at directly linking the network performance to the main system parameters. For instance, evaluation of the diversity order achieved by iterative designs in the asymptotic signal-to-noise ratio (SNR) regime is a challenging task.

Adopting the minimum-mean-square-error (MMSE) criterion with power constraints at the source [10] and at destination [3], we propose a joint optimization framework of the source precoder, the AF relaying matrices, and the destination equalizer.[2] In particular, rather than attempting to iteratively search for locally optimal solutions as in [4]–[6], [8], [9], we propose to relax the original constrained MMSE nonconvex optimization problem to derive a closed-form solution.

Specifically, capitalizing on our preliminary results in [11], three additional contributions are reported in this paper. First, we develop an in-depth study of the proposed design by showing that it entails a reduced complexity, compared to recent iterative designs, e.g., [4]–[6]. Second, capitalizing on our closed-form design, we carry out the performance analysis of the proposed cooperative system in terms of average symbol error probability (ASEP), by calculating the corresponding asymptotic diversity order. Third, Monte Carlo numerical comparisons with existing iterative approaches are developed, which show that the proposed closed-form design is competitive with the iterative approach of [4] and even outperforms those of [5], [6].

The paper is organized as follows. The model of the cooperative MIMO network with multiple AF relays is introduced in Section II. Development and performance analysis of the proposed closed-form design are carried out in Section III. Monte Carlo numerical simulations are presented in Section V. Finally, conclusions are drawn in Section VI.

## II. NETWORK MODEL AND BASIC ASSUMPTIONS

In the considered MIMO network (see Fig. 1), the transmission between a source and a destination is assisted by $N_C$ half-duplex relays. The numbers of antennas at the source, relays, and destination are $N_S$, $N_R$, and $N_D$, respectively. We assume that there is no direct link between the source and the destination, due to high path loss values or obstructions. As in [4], [6], [9], and [11], the received signal at the destination can be

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[1]Extension of the proposed approach to the case of long-term CSI (i.e., only the second-order statistics of the relevant channel coefficients are known) is an interesting issue, but it is outside the scope of this paper.

[2]This paper is an extended version of [11].

Table 1. A taxonomy of cooperative MIMO relaying techniques (see legend below).

| Ref. | Scenario | Number of antennas | Source precod. | Optim. criterion | Dest. equalizer | Power constraints | Solution type |
|---|---|---|---|---|---|---|---|
| [3] | 1S-MR-1D | Multiple | No | MMSE w/target SNR | MMSE | NOC | Closed |
| | | | | MMSE | MMSE | RPC | Closed |
| | | | | Max dest. SNR | ZF | NOC | Closed |
| | | | | Max out. SNR | ZF | RPC | Closed |
| | | | | Max rate | Arbitrary | RPC | Iterative |
| [4] | 1S-MR-1D | Multiple | Yes | MMSE | MMSE, MMSE-DFE | SPC, RPC w/rescaling | Iterative |
| [5] | MS-MR-MD | Multiple | Yes | Max sum-rate | MMSE | SPC, S-TPC, I-TPC | Iterative |
| [6] | 1S-MR-1D | Multiple | No | MMSE | MMSE | Weighted S-TPC, I-TPC | Iterative |
| [7] | MS-MR-MD | Single | No | MMSE, ZF | MMSE, ZF | S-TPC, RPC, NOC | Closed |
| [8] | 1S-MR-MD | Multiple | Yes | Worst-stream SINR, source-relay power | Arbitrary | SPC, S-TPC | Iterative |
| [9] | MS-MR-MD | Multiple | Yes | MMSE | MMSE | SPC, S-TPC | Iterative |
| Proposed | 1S-MR-1D | Multiple | Yes | MMSE | MMSE | SPC, RPC w/rescaling | Closed |

1S/MS = single/multiple sources, 1R/MR = single/multiple relays, 1D/MD = single/multiple destinations, NOC = no constraints, RPC = RX power at the destination, SPC = source power constraint, S-TPC = sum of the TX powers of the relays, I-TPC = individual TX power of the relay.

expressed as

$$\mathbf{r} = \mathbf{C}\,\mathbf{s} + \mathbf{v}, \tag{1}$$

where $\mathbf{C} \triangleq \mathbf{G}\,\mathbf{F}\,\mathbf{H}\,\mathbf{F}_0 \in \mathbb{C}^{N_D \times N_B}$ is the *dual-hop* matrix, with matrices $\mathbf{H} \triangleq [\mathbf{H}_1^{\mathrm{T}}, \mathbf{H}_2^{\mathrm{T}}, \cdots, \mathbf{H}_{N_C}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{C}^{(N_C N_R) \times N_S}$ and $\mathbf{G} \triangleq [\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_{N_c}] \in \mathbb{C}^{N_D \times (N_C N_R)}$ collecting the *first-(backward)* and *second-hop (forward)* MIMO channel coefficients of all the relays, respectively, the diagonal blocks of $\mathbf{F} \triangleq \mathrm{diag}(\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_{N_C}) \in \mathbb{C}^{(N_C N_R) \times (N_C N_R)}$ denoting the corresponding *relaying* matrices, and $\mathbf{F}_0 \in \mathbb{C}^{N_S \times N_B}$ representing a *source precoding* matrix, $\mathbf{s} \in \mathbb{C}^{N_B}$ is the symbol block transmitted by the source, and $\mathbf{v} \triangleq \mathbf{G}\,\mathbf{F}\,\mathbf{w} + \mathbf{n}$ is the *equivalent* noise vector, with $\mathbf{w} \in \mathbb{C}^{N_C N_R}$ and $\mathbf{n} \in \mathbb{C}^{N_D}$ gathering the noise samples at all the relays and at the destination, respectively.

The block $\mathbf{s}$ is modeled as a zero-mean circularly symmetric complex random vector, with covariance matrix $\mathbb{E}[\mathbf{s}\,\mathbf{s}^{\mathrm{H}}] = \mathbf{I}_{N_B}$. The entries of $\mathbf{H}$ and $\mathbf{G}$ are assumed to be zero-mean unit-variance circularly symmetric complex Gaussian (CSCG) random variables. Additionally, the noise vectors $\mathbf{w}$ and $\mathbf{n}$ are modeled as mutually independent zero-mean CSCG random vectors statistically independent of $(\mathbf{s}, \mathbf{H}, \mathbf{G})$, with covariance matrix $\mathbb{E}[\mathbf{w}\,\mathbf{w}^{\mathrm{H}}] = \sigma_w^2\,\mathbf{I}_{N_C N_R}$ and $\mathbb{E}[\mathbf{n}\,\mathbf{n}^{\mathrm{H}}] = \mathbf{I}_{N_D}$, respectively. The vector $\mathbf{r}$ is subject to linear equalization at the destination through the *equalizing* matrix $\mathbf{D} \in \mathbb{C}^{N_B \times N_D}$, hence yielding an estimate $\hat{\mathbf{s}} \triangleq \mathbf{D}\,\mathbf{r}$ of the source block $\mathbf{s}$, whose entries are then subject to minimum-distance hard quantization.

Hereinafter, we assume that the dual-hop channel matrix $\mathbf{C}$

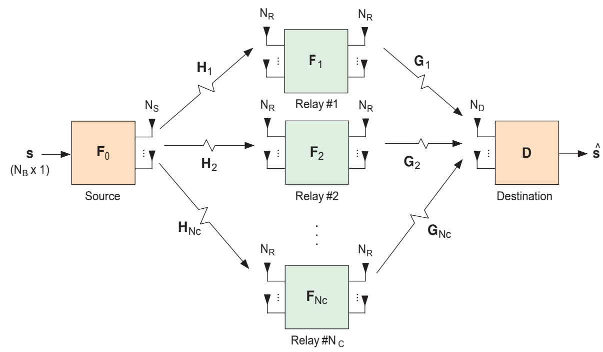


Fig. 1. Model of the considered cooperative MIMO network.

and the conditional covariance matrix

$$\mathbf{K}_{\mathbf{vv}} \triangleq \mathbb{E}[\mathbf{v}\,\mathbf{v}^{\mathrm{H}} \,|\, \mathbf{G}] = \sigma_w^2\,\mathbf{G}\,\mathbf{F}\,\mathbf{F}^{\mathrm{H}}\;\mathbf{G}^{\mathrm{H}} + \mathbf{I}_{N_D} \tag{2}$$

of $\mathbf{v}$, given $\mathbf{G}$, have been previously acquired at the destination during a training session. We assume that $\mathbf{H}$ is known at the source and, as in [3], [4], [5]–[9], it is also known at every relay node, whereas the $i$th second-hop channel matrix $\mathbf{G}_i$ is known only to the $i$th relay, for $i \in \{1, 2, \cdots, N_C\}$. Even though these assumptions require, in addition to a training phase, feedback links when the channel reciprocity property cannot be invoked, this case has been widely recognized in many works as a useful benchmark.

## III. THE PROPOSED CLOSED-FORM DESIGN

As a global performance of the cooperative network, we consider the trace of conditional mean square error (MSE) matrix, given the matrices $\mathbf{H}$ and $\mathbf{G}$, which is defined as $\mathbf{E}(\mathbf{F}_0, \mathbf{F}, \mathbf{D}) \triangleq \mathbb{E}[(\hat{\mathbf{s}} - \mathbf{s})\,(\hat{\mathbf{s}} - \mathbf{s})^{\mathrm{H}} \,|\, \mathbf{H}, \mathbf{G}]$. It is well-known (see, e.g., [10]) that, for fixed matrices $\mathbf{F}_0$ and $\mathbf{F}$, the matrix $\mathbf{D}$ minimizing $\mathrm{MSE}(\mathbf{F}_0, \mathbf{F}, \mathbf{D}) \triangleq \mathrm{tr}[\mathbf{E}(\mathbf{F}_0, \mathbf{F}, \mathbf{D})]$ is the Wiener filter $\mathbf{D}_{\mathrm{mmse}} = \mathbf{C}^{\mathrm{H}}(\mathbf{C}\mathbf{C}^{\mathrm{H}} + \mathbf{K}_{\mathbf{vv}})^{-1}$. Substitution of $\mathbf{D}_{\mathrm{mmse}}$ into $\mathrm{MSE}(\mathbf{F}_0, \mathbf{F}, \mathbf{D})$ yields

$$\mathbf{E}(\mathbf{F}_0, \mathbf{F}) \triangleq \mathbf{E}(\mathbf{F}_0, \mathbf{F}, \mathbf{D}_{\mathrm{mmse}}) = (\mathbf{I}_{N_B} + \mathbf{C}^{\mathrm{H}}\,\mathbf{K}_{\mathbf{vv}}^{-1}\,\mathbf{C})^{-1} \,. \quad (3)$$

Further minimization of $\mathrm{MSE}(\mathbf{F}_0, \mathbf{F}) \triangleq \mathrm{tr}[\mathbf{E}(\mathbf{F}_0, \mathbf{F})]$ with respect to $\mathbf{F}_0$ and $\mathbf{F}$ must be carried out under appropriate power constraints involving the matrices $\mathbf{F}_0$ and $\mathbf{F}$.

In order to limit the average transmit power of the source, we impose the customary constraint $\mathrm{tr}(\mathbf{F}_0\,\mathbf{F}_0^{\mathrm{H}}) \le \mathcal{P}_S$, with $\mathcal{P}_S > 0$. Regarding the precoding matrices of the relays, as in [3] and [4], we also impose a constraint on the average power received at the destination, i.e., $\mathrm{tr}(\mathbf{G}\,\mathbf{F}\,\mathbf{K}_{\mathbf{zz}}\,\mathbf{F}^{\mathrm{H}}\,\mathbf{G}^{\mathrm{H}}) \le \mathcal{P}_D$, where $\mathbf{K}_{\mathbf{zz}} \triangleq \mathbb{E}[\mathbf{z}\,\mathbf{z}^{\mathrm{H}} \,|\, \mathbf{H}] = \mathbf{H}\,\mathbf{F}_0\,\mathbf{F}_0^{\mathrm{H}}\,\mathbf{H}^{\mathrm{H}} + \sigma_w^2\,\mathbf{I}_{N_C N_R}$ is the conditional covariance matrix of the vector $\mathbf{z} \in \mathbb{C}^{N_C N_R}$ collecting the signals transmitted by all the relays, given $\mathbf{H}$, with $\mathcal{P}_D > 0$. Such a constraint is appropriate, e.g., for resource-constrained nodes that are mainly designed to solve a single network-wide signal processing task [1]. So doing, one has the problem

$$\min_{\mathbf{F}_0 \mathbf{F}} \mathrm{MSE}(\mathbf{F}_0, \mathbf{F}) \quad \text{s.t.} \quad \begin{cases} \mathrm{tr}(\mathbf{F}_0\,\mathbf{F}_0^{\mathrm{H}}) \le \mathcal{P}_S \,; \\ \mathrm{tr}(\mathbf{G}\,\mathbf{F}\,\mathbf{K}_{\mathbf{zz}}\,\mathbf{F}^{\mathrm{H}}\,\mathbf{G}^{\mathrm{H}}) \le \mathcal{P}_D \,. \end{cases} \quad (4)$$

Compared with a point-to-point MIMO system [10], the objective function $\mathrm{MSE}(\mathbf{F}_0, \mathbf{F})$ depends not only on the source precoder $\mathbf{F}_0$, but also on the relaying matrices contained in $\mathbf{F}$. Moreover, the additional constraint on the received power at the destination is imposed, which is a function of both $\mathbf{F}_0$ and $\mathbf{F}$, too. Problem (4) is nonconvex and, for such a reason, the matrices $\mathbf{F}_0$ and $\mathbf{F}$ are often designed by resorting to iterative algorithms (see, e.g., [4]). However, such algorithms rarely compute a global minimum of (4): Usually, they compute a local minumum or, at least, a Karush-Kuhn-Tucker (KKT) point.[3]

A different approach is pursued herein: To derive closed-form suboptimal expressions for $\mathbf{F}_0$ and $\mathbf{F}$, a simplification of the cost function in (4) and a modified version of the power constraint at the destination are developed. Specifically, in this section, we present the proposed design and discuss its computational complexity requirement. In Subsection IV, we derive an upper bound on the ASEP of the cooperative system designed with our proposed solution and its achievable diversity order.

A key result to simplify (4) is the following one.

**Lemma 1:** The mean-square error matrix $\mathrm{MSE}(\mathbf{F}_0, \mathbf{F})$ can be lower bounded as follows

$$\begin{aligned} \mathrm{MSE}(\mathbf{F}_0, \mathbf{F}) &= \mathrm{tr}\left[\left(\mathbf{I}_{N_B} + \mathbf{C}^{\mathrm{H}}\,\mathbf{K}_{\mathbf{vv}}^{-1}\,\mathbf{C}\right)^{-1}\right] \\ &\ge \mathrm{MSE}_{\mathrm{min}}(\mathbf{F}_0, \mathbf{F}) \triangleq \mathrm{tr}\left[\left(\mathbf{I}_{N_B} + \mathbf{C}^{\mathrm{H}}\,\mathbf{C}\right)^{-1}\right] , \end{aligned} \quad (5)$$

where the equality holds if $\mathbf{K}_{\mathbf{vv}} = \mathbf{I}_{N_D}$.

*Proof:* See Appendix I. □

According to (2), the equality in (5) holds as $\sigma_w^2 \to 0$, i.e., when the relays are noiseless. For noisy relays, the lower bound in (5) can be regarded as an approximation, provided that $\sigma_w^2 \ll \min(1, \mu_{\mathrm{min}})$, where $\mu_{\mathrm{min}}$ is the smallest eigenvalue of $\mathbf{G}\,\mathbf{F}\,\mathbf{F}^{\mathrm{H}}\,\mathbf{G}^{\mathrm{H}}$. Capitalizing on Lemma 1, we relax the original constrained optimization problem (4) by minimizing the cost function $\mathrm{MSE}_{\mathrm{min}}(\mathbf{F}_0, \mathbf{F})$, under a modified version of the power constraint at the destination. More specifically, by exploiting the additive property of the trace operator [12], and applying three times the matrix trace inequality for positive semidefinite matrices [13], the average power received at the destination in Phase II can be upper-bounded as

$$\begin{aligned} &\mathrm{tr}(\mathbf{G}\,\mathbf{F}\,\mathbf{K}_{\mathbf{zz}}\,\mathbf{F}^{\mathrm{H}}\,\mathbf{G}^{\mathrm{H}}) \\ &\le \mathrm{tr}(\mathbf{G}\,\mathbf{F}\,\mathbf{F}^{\mathrm{H}}\,\mathbf{G}^{\mathrm{H}}) \cdot \left[\mathrm{tr}(\mathbf{H}\,\mathbf{H}^{\mathrm{H}})\,\mathrm{tr}(\mathbf{F}_0\,\mathbf{F}_0^{\mathrm{H}}) + \sigma_w^2\,N_C\,N_R\right] \\ &\le \mathrm{tr}(\mathbf{G}\,\mathbf{F}\,\mathbf{F}^{\mathrm{H}}\,\mathbf{G}^{\mathrm{H}}) \,\left[\mathcal{P}_S\,\mathrm{tr}(\mathbf{H}^{\mathrm{H}}\,\mathbf{H}) + \sigma_w^2\,N_C\,N_R\right] , \end{aligned} \quad (6)$$

where the source constraint $\mathrm{tr}(\mathbf{F}_0\,\mathbf{F}_0^{\mathrm{H}}) \le \mathcal{P}_S$ has also been used. It is noteworthy that, for fixed $N_S$, by the law of large numbers $\mathbf{H}^{\mathrm{H}}\,\mathbf{H}/(N_C\,N_R) \to \mathbf{I}_{N_S}$ almost surely as $N_C\,N_R$ gets large. Hence, in the large $N_C\,N_R$ limit, one gets $\mathrm{tr}(\mathbf{G}\,\mathbf{F}\,\mathbf{K}_{\mathbf{zz}}\,\mathbf{F}^{\mathrm{H}}\,\mathbf{G}^{\mathrm{H}}) \le N_C\,N_R\,\mathrm{tr}(\mathbf{G}\,\mathbf{F}\,\mathbf{F}^{\mathrm{H}}\,\mathbf{G}^{\mathrm{H}})\,\left(N_S\,\mathcal{P}_S + \sigma_w^2\right)$. Thus, to limit the power received at the destination, we impose that $\mathrm{tr}(\mathbf{G}\,\mathbf{F}\,\mathbf{F}^{\mathrm{H}}\,\mathbf{G}^{\mathrm{H}}) \le \widetilde{\mathcal{P}}_D$, with $\widetilde{\mathcal{P}}_D > 0$. Such a constraint allows one to simplify the derivation of the source precoder and relaying matrices, thus implying

$$\mathrm{tr}(\mathbf{G}\,\mathbf{F}\,\mathbf{K}_{\mathbf{zz}}\,\mathbf{F}^{\mathrm{H}}\,\mathbf{G}^{\mathrm{H}}) \le N_C\,N_R\,\widetilde{\mathcal{P}}_D\,\left(N_S\,\mathcal{P}_S + \sigma_w^2\right) \,. \quad (7)$$

Therefore, by introducing the matrix $\mathbf{B} \triangleq \mathbf{G}\,\mathbf{F} \in \mathbb{C}^{N_D \times (N_C N_R)}$, one can formulate the relaxed optimization problem as

$$\min_{\mathbf{F}_0, \mathbf{B}} \mathrm{tr}\left[\left(\mathbf{I}_{N_B} + \mathbf{F}_0^{\mathrm{H}}\,\mathbf{H}^{\mathrm{H}}\,\mathbf{B}^{\mathrm{H}}\,\mathbf{B}\,\mathbf{H}\,\mathbf{F}_0\right)^{-1}\right] \quad \text{s.t.} \begin{cases} \mathrm{tr}(\mathbf{F}_0\mathbf{F}_0^{\mathrm{H}}) \le \mathcal{P}_S \,; \\ \mathrm{tr}(\mathbf{B}\mathbf{B}^{\mathrm{H}}) \le \widetilde{\mathcal{P}}_D \,. \end{cases} \quad (8)$$

The following theorem characterizes the solution of the matrix-valued optimization problem (8).

**Theorem 1:** Assume that: **a1)** The precoding matrix $\mathbf{F}_0 \in \mathbb{C}^{N_S \times N_B}$ is full-column rank, i.e., $\mathrm{rank}(\mathbf{F}_0) = N_B \le N_S$; **a2)** the matrix $\mathbf{B}\,\mathbf{H} \in \mathbb{C}^{N_D \times N_S}$ is full-column rank, i.e., $\mathrm{rank}(\mathbf{B}\,\mathbf{H}) = N_S \le N_D$. Moreover, let $\mathbf{H} = \mathbf{U}_{\mathrm{h}}\,\mathbf{\Lambda}_{\mathrm{h}}\,\mathbf{V}_{\mathrm{h}}^{\mathrm{H}}$ denote the singular value decomposition (SVD) of $\mathbf{H}$, where $\mathbf{U}_{\mathrm{h}} \in \mathbb{C}^{(N_C N_R) \times (N_C N_R)}$ and $\mathbf{V}_{\mathrm{h}} \in \mathbb{C}^{N_S \times N_S}$ are unitary matrices, and $\mathbf{\Lambda}_{\mathrm{h}} \in \mathbb{R}^{(N_C N_R) \times N_S}$ gathers the corresponding singular

[3] In the absence of convexity, a KKT point of (4) can be a global minimum, a local minimum, a saddlepoint, or even a maximum.

values arranged in increasing order. Then, the solution of (8) has the following form:

$$\mathbf{F}_0 = \mathbf{V}_{\text{h,right}} \, [\mathbf{O}_{L_\text{h} \times (N_B - L_\text{h})}, \mathbf{\Omega}]; \tag{9}$$

$$\mathbf{B} = \mathbf{Q} \, \mathbf{\Delta} \, \mathbf{U}^{\text{H}}_{\text{h,right}}, \tag{10}$$

where $\mathbf{V}_{\text{h,right}} \in \mathbb{C}^{N_S \times L_\text{h}}$ contains the $L_\text{h}$ rightmost columns of $\mathbf{V}_\text{h}$, $\mathbf{U}_{\text{h,right}} \in \mathbb{C}^{(N_C N_R) \times r_\text{h}}$ collects the $r_\text{h}$ rightmost columns of $\mathbf{U}_\text{h}$, with $r_\text{h} \triangleq \text{rank}(\mathbf{H})$ and $L_\text{h} \triangleq \min(N_B, r_\text{h})$, $\mathbf{\Omega} \in \mathbb{R}^{L_\text{h} \times L_\text{h}}$ and $\mathbf{\Delta} \in \mathbb{R}^{r_\text{h} \times r_\text{h}}$ are diagonal matrices whose nonzero *real* entries will be specified soon after, and $\mathbf{Q} \in \mathbb{C}^{N_D \times r_\text{h}}$ is an arbitrary semi-unitary matrix, i.e., $\mathbf{Q}^\text{H} \, \mathbf{Q} = \mathbf{I}_{r_\text{h}}$.

*Proof:* See Appendix II. □

Under **a1)** and **a2)**, the dual-hop channel matrix $\mathbf{C} = \mathbf{B} \, \mathbf{H} \, \mathbf{F}_0$ is full-column rank, i.e., $\text{rank}(\mathbf{C}) = N_B \leq N_D$: this ensures perfect recovery of the source symbol vector $\mathbf{s}$ at the destination in the absence of noise by means of linear equalizers. Although Theorem 1 holds for any value of $N_B$, we will assume herein that $N_B = r_\text{h}$, which allows the source to transmit as many symbols as possible with an acceptable performance in practice. In this case, one has $L_\text{h} = r_\text{h} = N_B$. Relying on Theorem 1, the matrix optimization problem (8) can be written in the scalar form

$$\min_{\substack{\{z_\ell\}_{\ell=1}^{r_\text{h}} \\ \{w_\ell\}_{\ell=1}^{r_\text{h}}}} f(\{z_\ell\}, \{w_\ell\}) \quad \text{s.t.} \quad \begin{cases} \sum_{\ell=1}^{r_\text{h}} z_\ell \leq \mathcal{P}_S, & z_\ell > 0\,; \\ \sum_{\ell=1}^{r_\text{h}} w_\ell \leq \widetilde{\mathcal{P}}_D, & w_\ell > 0\,, \end{cases} \tag{11}$$

where $f(\{z_\ell\}_{\ell=1}^{r_\text{h}}, \{w_\ell\}_{\ell=1}^{r_\text{h}}) \triangleq \sum_{\ell=1}^{r_\text{h}} [1 + \lambda_\ell^2(\mathbf{H}) \, z_\ell \, w_\ell]^{-1}$, with $z_\ell$ and $w_\ell$ representing the $\ell$th *square* diagonal entry of $\mathbf{\Omega}$ and $\mathbf{\Delta}$, respectively, whereas $\lambda_\ell(\mathbf{H})$ denotes the $\ell$th nonzero singular value of $\mathbf{H}$, for $\ell \in \{1, 2, \cdots, r_\text{h}\}$.

It is shown that the objective function $f(\{z_\ell\}_{\ell=1}^{r_\text{h}}, \{w_\ell\}_{\ell=1}^{r_\text{h}})$ is convex if $z_\ell \, w_\ell \geq [3 \, \lambda_\ell^2(\mathbf{H})]^{-1}$, for $\ell \in \{1, 2, \cdots, r_\text{h}\}$. Since $\mathbf{H}$ is composed of independent identically distributed CSCG random variables with zero mean and unit variance, its smallest singular value $\lambda_{\text{min}}(\mathbf{H})$ converges [14] almost surely to $N_C \, N_R \, (1 - \sqrt{\alpha})$ as $N_C \, N_R$ gets large, with $N_S/(N_C \, N_R) \to \alpha \in (0, 1)$. Therefore, in the large $N_C \, N_R$ limit with $\alpha \in (0, 1)$, one has $\lambda_{\text{min}}(\mathbf{H}) \gg 1$ and, thus, condition $z_\ell \, w_\ell \geq [3 \, \lambda_\ell^2(\mathbf{H})]^{-1}$ boils down to $z_\ell > 0$ and $w_\ell > 0$, for each $\ell \in \{1, 2, \cdots, r_\text{h}\}$.

The convex optimization problem (11) can be efficiently solved using the interior-point methods [15], which usually have local quadratic convergence rates. However, one can approximately calculate the solution of (11) in closed-form. Let $\mathcal{L} : \mathbb{R}^{2(r_\text{h}+1)} \to \mathbb{R}$ define the Lagrangian associated with the problem (11), it results

$$\mathcal{L}(\{z_\ell\}_{\ell=1}^{r_\text{h}}, \{w_\ell\}_{\ell=1}^{r_\text{h}}, \mu_1, \mu_2) = \sum_{\ell=1}^{r_\text{h}} \left[1 + \lambda_\ell^2(\mathbf{H}) \, z_\ell \, w_\ell\right]^{-1} + \mu_1 \left( \sum_{\ell=1}^{r_\text{h}} z_\ell - \mathcal{P}_S \right) + \mu_2 \left( \sum_{\ell=1}^{r_\text{h}} w_\ell - \widetilde{\mathcal{P}}_D \right), \tag{12}$$

where $\mu_1$ and $\mu_2$ are the Lagrange multipliers. By equating to zero the first-order partial derivatives of (12) with respect to $z_\ell$ and $w_\ell$, for $\ell \in \{1, 2, \cdots, r_\text{h}\}$, after straightforward calculations, one obtains that the nonnegative solution for $z_\ell$ (i.e., the $\ell$th square diagonal entry of $\mathbf{\Omega}$) is given by

$$z_\ell = \frac{-\mu_1 + \lambda_\ell(\mathbf{H}) \, \sqrt{\mu_1 \, w_\ell}}{\mu_1 \, \lambda_\ell^2(\mathbf{H}) \, w_\ell}, \tag{13}$$

whereas $w_\ell$ (i.e., the $\ell$th square diagonal entry of $\mathbf{\Delta}$) can be determined by solving the following quartic equation

$$\mu_2 \, \lambda_\ell^2(\mathbf{H}) \, \delta_\ell^4 - \sqrt{\mu_1} \, \lambda_\ell(\mathbf{H}) \, \delta_\ell + \mu_1 = 0 \tag{14}$$

with $\delta_\ell \triangleq \sqrt{w_\ell}$. Euler showed [16] that the solution of (14) can be brought back to finding the roots of a cubic resolvent, for which there exist explicit expressions given by Cardano's formula [17] (see also [18]). After tedious but straightforward calculations, it can be verified that, when $\lambda_{\text{min}}(\mathbf{H}) \gg 1$, the value of $w_\ell$ satisfying (14) is approximately given by

$$w_{\text{opt},\ell} \approx \left[ \frac{\mu_2^2}{\mu_1} \, \lambda_\ell^2(\mathbf{H}) \right]^{-1/3} \tag{15}$$

for $\ell \in \{1, 2, \cdots, r_\text{h}\}$, which, substituted back into (13), leads to the solution $z_{\text{opt},\ell}$, with the constants $\mu_1$ and $\mu_2$ chosen to satisfy the constraints $\sum_{\ell=1}^{r_\text{h}} z_\ell \leq \mathcal{P}_S$ and $\sum_{\ell=1}^{r_\text{h}} w_\ell \leq \widetilde{\mathcal{P}}_D$, respectively. The diagonal entries of $\mathbf{\Omega}$ and $\mathbf{\Delta}$ turn out to be the square roots of $\{z_{\text{opt},\ell}\}_{\ell=1}^{r_\text{h}}$ and $\{w_{\text{opt},\ell}\}_{\ell=1}^{r_\text{h}}$, respectively.

To calculate the relaying matrices, let us partition solution (10) as $\mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_{N_C}]$, with $\mathbf{B}_i \in \mathbb{C}^{N_D \times N_R}$, and assume that $\mathbf{G}_i$ is full-row rank, i.e., $\text{rank}(\mathbf{G}_i) = N_D \leq N_R$. Therefore, the $i$th relay can construct its relaying matrix by solving the matrix equation [19] $\mathbf{G}_i \, \mathbf{F}_i = \mathbf{B}_i$, whose minimum-norm solution [20] is given by $\mathbf{F}_i = \mathbf{G}_i^\dagger \, \mathbf{B}_i$, where the superscript † denotes the Moore-Penrose inverse.

A final comment on complexity is now in order. It is apparent that, unlike previous MIMO designs of multiple-relay cooperative networks, the proposed solution is obtained without the need of any iterative procedure. For instance, after recasting the AF MIMO relay system into a set of parallel single-input single-output (SISO) channels, the optimal power allocation over the parallel SISO channels is numerically found in [4] by using the alternative iterative technique described in [21]. Even though the alternating algorithm exhibits a reduced computational complexity compared with other methods, e.g., dual decomposition or high-dense grid search, it is more complex than calculating the solution of (14) through the formulas of Euler and Cardano or its approximated version (15) directly; moreover, it requires proper initialization to monotonically converge to a local optimum. Additionally, as shown in the subsequent section, the existence of a closed-form design such as (15) results into a relatively simple performance analysis.

## IV. PERFORMANCE ANALYSIS

When the equalizing matrix is equal to the Wiener filter, the signal-to-interference-plus-noise ratio (SINR) on the $n$th symbol stream $\text{SINR}_n(\mathbf{F}_0, \mathbf{F})$ is related to the corresponding $\text{MSE}_n(\mathbf{F}_0, \mathbf{F}) \triangleq \{\mathbf{E}(\mathbf{F}_0, \mathbf{F})\}_{nn}$ as follows (see, e.g., [22])

$$\text{SINR}_n(\mathbf{F}_0, \mathbf{F}) = \frac{1}{\{(\mathbf{I}_{N_B} + \mathbf{C}^\text{H} \, \mathbf{K}_{\mathbf{vv}}^{-1} \, \mathbf{C})^{-1}\}_{nn}} - 1 \tag{16}$$

for $n \in \{1, 2, \cdots, N_B\}$. An overall performance measure of the cooperative system is the ASEP $P(e)$, which is defined as $P(e) \triangleq \sum_{n=1}^{N} P_n(e)/N$, where $P_n(e)$ is the ASEP in the maximum likelihood (ML) detection of the $n$th entry of $\mathbf{s}$. Let the entries of $\mathbf{s}$ be drawn from a $Q$-dimensional quadrature amplitude modulation (QAM) square constellation. In this case, after a change of variables from rectangular to polar coordinates in the integral defining the complementary error function, the ASEP $P_n(e)$ in the ML detection of the $n$th entry of $\mathbf{s}$ is (approximatively) given [23] by

$$P_n(e) = \frac{2\,b}{\pi} \int_0^{\pi/2} \mathbb{E}\left\{\exp\left[-\frac{u}{\sin^2\theta}\,\text{SINR}_n(\mathbf{F}_0, \mathbf{F})\right]\right\} \mathrm{d}\theta \tag{17}$$

for $n \in \{1, 2, \cdots, N\}$, where the parameters $b \triangleq 2\,(1 - 1/\sqrt{Q})$ and $u \triangleq 3/[2\,(Q-1)]$ depend on the symbol constellation, whereas the expectation is taken over the sample space of $(\mathbf{F}_0, \mathbf{F})$.

Let us assume that $N_B = N_S = N_D \triangleq N$ and $\mathbf{H}$ is full-column rank, i.e., rank$(\mathbf{H}) = N$; by substituting solutions (9) and (10) into (16) and remembering that $\mathbf{H} = \mathbf{U}_\text{h}\,\mathbf{\Lambda}_\text{h}\,\mathbf{V}_\text{h}^\text{H}$, after some calculations, (16) can be written as

$$\begin{aligned}\text{SINR}_n(\mathbf{\Omega}, \mathbf{\Delta}) &= \frac{1}{\{[\mathbf{I}_N + \mathbf{\Omega}^2\,\mathbf{\Lambda}_\text{h}^2\,\mathbf{\Delta}^2(\mathbf{\Delta}^2 + \mathbf{I}_N)^{-1}]^{-1}\}_{nn}} - 1 \\ &= \frac{z_n\,w_n\,\lambda_n^2(\mathbf{H})}{1 + w_n}\,.\end{aligned} \tag{18}$$

To simplify the analysis, we also assume that $\mathcal{P}_S = \widetilde{\mathcal{P}}_D = \mathcal{P}$: In this case, it can be seen from (13) and (15) that

$$z_{\text{opt},\ell} = w_{\text{opt},\ell} \approx \frac{\mathcal{P}}{\sum_{n=1}^{N} \lambda_n^{-\frac{2}{3}}(\mathbf{H})}\,\lambda_\ell^{-\frac{2}{3}}(\mathbf{H}), \tag{19}$$

whose corresponding value of (18) is denoted with $\text{SINR}_n$ hereinafter. If $\mathcal{P}$ is sufficiently high, then $w_{\text{opt},n} \gg 1$, hence yielding the approximation $\text{SINR}_n \approx w_{\text{opt},n}\,\lambda_n^2(\mathbf{H})$ in the high-SNR region. Consequently, accounting for (15), one has

$$\text{SINR}_n \approx \frac{\mathcal{P}}{\sum_{\ell=1}^{N} \lambda_\ell^{-\frac{2}{3}}(\mathbf{H})}\,\lambda_n^{\frac{4}{3}}(\mathbf{H}) \geq \text{SINR}_\text{lb} \triangleq \frac{\mathcal{P}}{N}\,\lambda_\text{min}^2(\mathbf{H}), \tag{20}$$

where we have also used $\sum_{\ell=1}^{N} \lambda_\ell^{-\frac{2}{3}}(\mathbf{H}) \leq N\,\lambda_\text{min}^{-\frac{2}{3}}(\mathbf{H})$. The equality in (20) holds when the singular values of $\mathbf{H}$ are all equal to $\lambda_\text{min}(\mathbf{H})$ and $\mathcal{P}/N \gg 1$. Inequality (20) naturally leads to the upper bound on the ASEP of the cooperative system

$$\begin{aligned}P(e) &\lessapprox \frac{2\,b}{\pi} \int_0^{\pi/2} \mathbb{E}\left\{\exp\left[-\frac{u}{\sin^2\theta}\,\frac{\mathcal{P}}{N}\,\lambda_\text{min}^2(\mathbf{H})\right]\right\} \mathrm{d}\theta \\ &= \frac{2\,b}{\pi} \int_0^{\pi/2} \int_0^{+\infty} \exp\left(-\frac{u}{\sin^2\theta}\,\frac{\mathcal{P}}{N}\,x\right) f_{\lambda_\text{min}^2(\mathbf{H})}(x)\,\mathrm{d}\theta\,\mathrm{d}x,\end{aligned} \tag{21}$$

where $f_{\lambda_\text{min}^2(\mathbf{H})}(x)$ is probability distribution function (pdf) of the (positive) random variable $\lambda_\text{min}^2(\mathbf{H})$. Since $\lambda_\text{min}^2(\mathbf{H})$ turns out to be the smallest eigenvalue of the central Wishart matrix $\mathbf{H}^\text{H}\mathbf{H}$, its pdf can be written [24] in the following way

$$f_{\lambda_\text{min}^2(\mathbf{H})}(x) = \bar{K} \sum_{n,m=1}^{N} (-1)^{n+m}\,x^{n+m-2+N_C\,N_R-N} \cdot e^{-x} \det[\mathbf{\Gamma}(x)] \tag{22}$$

for $x \geq 0$, where

$$\bar{K} \triangleq \left[\prod_{i=1}^{N}(N_C\,N_R - i)! \prod_{j=1}^{N}(N-j)!\right]^{-1}. \tag{23}$$

$\det[\mathbf{\Gamma}(x)]$ is the determinant of $\mathbf{\Gamma}(x)$, whose $(i,j)$th element is given by $\mathbf{\Gamma}_{i,j}(x) = \Gamma\left(\alpha_{i,j}^{n,m} + N_C\,N_R - N + 1, x\right)$, for $i, j \in \{1, 2, \cdots, N-1\}$, with $\Gamma(k, x) \triangleq \int_0^x u^{k-1}\,e^{-u}\,\mathrm{d}u$ denoting the incomplete Gamma function and

$$\alpha_{i,j}^{n,m} \triangleq \begin{cases} i+j-2 & \text{if } i < n \text{ and } j < m\,; \\ i+j & \text{if } i \geq n \text{ and } j \geq m\,; \\ i+j-1 & \text{otherwise}\,. \end{cases} \tag{24}$$

Substituting (22) into (21), one obtains an approximated upper bound on the ASEP, which can be numerically evaluated through, e.g., the Matlab function `integral2`. We will show in Section V that the accuracy of such an upper bound improves for increasing values of $N_C$.

At this point, we study the diversity order achieved with our proposed design. Since $\text{SINR}_\text{lb}$ is a linear transformation of $\lambda_\text{min}^2(\mathbf{H})$, one has that its pdf is given by

$$f_{\text{SINR}_\text{lb}}(y) = \frac{N}{\mathcal{P}}\,f_{\lambda_\text{min}^2(\mathbf{H})}\left(\frac{N}{\mathcal{P}}\,y\right), \tag{25}$$

for $y \geq 0$, where the summand with lower degree in (22) is that corresponding to $n = m = 1$. After some straightforward but tedious calculations, it is seen that all derivatives of $f_{\text{SINR}_\text{lb}}(y)$ up to order $N_C\,N_R - N - 1$ are null at zero, while the $(N_C\,N_R - N)$-th order derivative is greater than zero: according to [25], this implies that the system has a diversity order of $N_C\,N_R - N + 1$. Therefore, the proposed system can achieve a diversity order which increases linearly with the number $N_C$ of relays, as also confirmed by the numerical results reported in Section V.

## V. SIMULATION RESULTS

In this section, to assess the performance of the considered design, we present the results of Monte Carlo computer simulations, aimed at evaluating the ASEP of the cooperative system. We considered a MIMO system equipped with $N_B = N_S = N_R = N_D = N = 2$ antennas, transmitting QPSK symbols, i.e., $Q = 4$. We also assumed that $\mathcal{P}_S = \widetilde{\mathcal{P}}_D = \mathcal{P}$ and set $\sigma_w^2 = 1$. Consequently, the SNR was defined as $\text{SNR} \triangleq \mathcal{P}$, which measures the per-antenna link quality of both the first- and second-hop transmissions. The ASEP was evaluated by carrying out $10^3$ independent Monte Carlo trials, with each run using independent sets of channel realizations and noise, and an

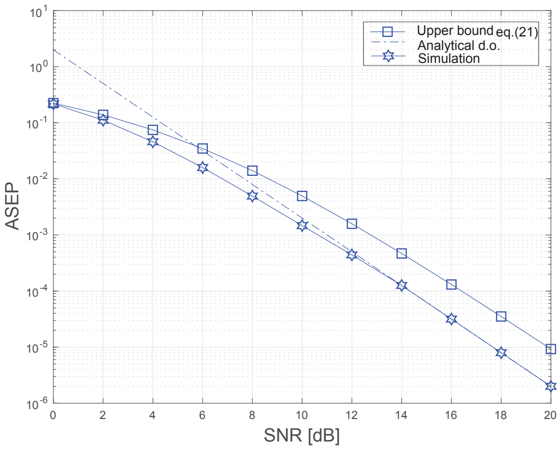


Fig. 2. ASEP versus SNR of the proposed design ($N_C = 2$).

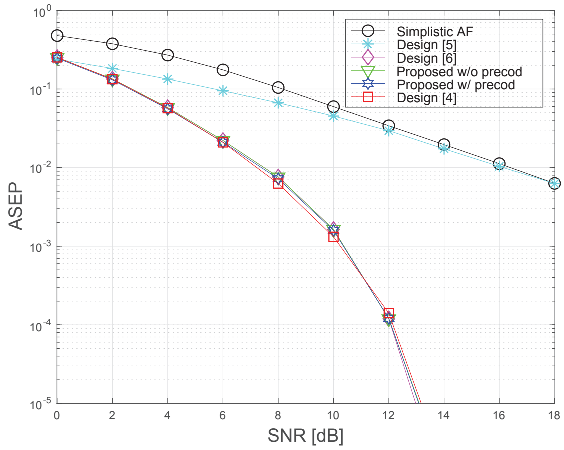


Fig. 4. ASEP versus SNR for different designs ($N_C = 2$).

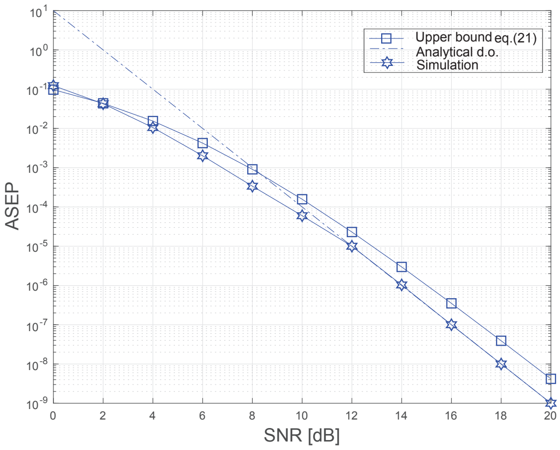


Fig. 3. ASEP versus SNR of the proposed design ($N_C = 3$).

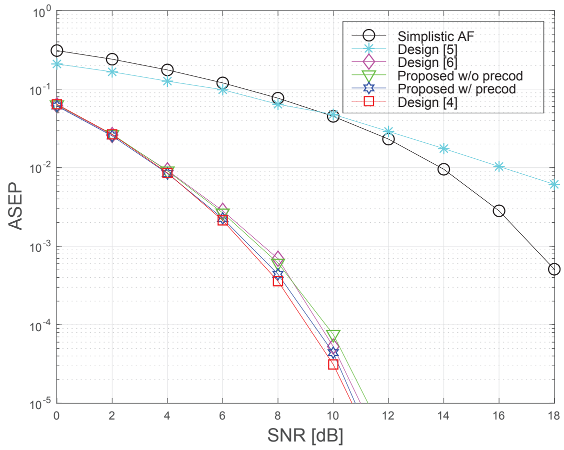


Fig. 5. ASEP versus SNR for different designs ($N_C = 3$).

independent record of $10^6$ source symbols.

To corroborate the performance analysis carried out in Section IV, we reported in Figs. 2 and 3 the ASEP of the proposed design evaluated by means of Monte Carlo simulation (referred to as "Simulation") for $N_C = 2$ and $N_C = 3$ relays, respectively, compared to the upper bound (21); we also depicted in the same figures, the line having slope equal to the theoretical diversity order $N_C N_R - N + 1$ (labeled as "Analytical d.o."). It is apparent from Figs. 2 and 3 that the accuracy of the upper bound improves for increasing values of $N_C$: Indeed, when $N_C$ gets larger and larger, the singular values of $\mathbf{H}$ approximately lie in a smaller and smaller range [14], i.e., they tend to be equal to each other. Specifically, for the case of three relays, the gap between the ASEP curve and the upper bound (21) is about one dB for moderate-to-high SNR values. According to our analysis, the diversity order is exactly equal to $N_C N_R - N + 1$.

In Figs. 4 and 5, we reported the ASEP for different designs when $N_C = 2$ and $N_C = 3$, respectively. In particular, we carried out a performance comparison among two different versions of our proposed design (labeled as "Proposed") and three iterative designs recently proposed in the literature. Specifically, in addition of the design developed in Section III, we implemented its modified version with no precoding at the source, i.e., $\mathbf{F}_0 = \sqrt{\mathcal{P}_S}\,\mathbf{I}_N$. As competitive alternatives, we reported the ASEP performance of the methods devised in [4]–[6], which all rely on iterative algorithms. It is worthwhile to note that all strategies under comparison require the same amount of CSI. Furthermore, since the aforementioned approaches impose different power constraints on the design of the relaying matrices, the corresponding solutions for $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_{N_C}$ are properly scaled such that to ensure that the global power expenditure in the network is the same for all methods. As a further reference, we additionally reported the ASEP of the "Simplistic AF" strategy [26], which is obtained by setting $\mathbf{F}_i \propto \mathbf{I}_{N_R}$, for each $i \in \{1, 2, \cdots, N_C\}$. Results of Figs. 4 and 5 show that the proposed closed-form design, based on the solution of the relaxed problem (8), outperforms both [5] and [6], by paying only a slight performance penalty with respect to [4]. However, such a penalty reduces as the number of relays $N_C$ increases. It is noteworthy that the iterative methods [4]–[6] involve a larger computational complexity, compared to our design.

## VI. CONCLUSIONS

We proposed a joint design of the source precoder, the relaying matrices, and the destination equalizer for multi-relay cooperative networks. Such a design is expressed in closed-form, thus avoiding the use of troublesome iterative procedures. Comparative numerical simulations showed that the proposed cooperative system performs better than or comparably to recently proposed designs, which rely on iterative algorithms and exhibit a larger computational complexity. Moreover, theoretical evaluation of the ASEP and the diversity order of the proposed approach were carried out, showing that the diversity order is a linearly increasing function of the number of relays.

## APPENDICES

## I. PROOF OF LEMMA 1

By virtue of the matrix inversion lemma [12], one has from (2) that

$$\mathbf{K}_{\mathbf{vv}}^{-1} = \left(\sigma_w^2\,\mathbf{G}\,\mathbf{F}\,\mathbf{F}^{\rm H}\,\mathbf{G}^{\rm H} + \mathbf{I}_{N_D}\right)^{-1} = \mathbf{I}_{N_D} - \sigma_w^2\,\mathbf{G}\,\mathbf{F}\,\left(\mathbf{I}_{N_C N_R} + \sigma_w^2\,\mathbf{F}^{\rm H}\,\mathbf{G}^{\rm H}\,\mathbf{G}\,\mathbf{F}\right)^{-1}\mathbf{F}^{\rm H}\,\mathbf{G}^{\rm H}\,. \tag{26}$$

By substituting (26) into the mean-square error $\mathrm{MSE}(\mathbf{F}_0, \mathbf{F})$, one obtains

$$\mathrm{MSE}(\mathbf{F}_0, \mathbf{F}) = \mathrm{tr}\left[\left(\mathbf{I}_{N_B} + \mathbf{C}^{\rm H}\,\mathbf{C} - \mathbf{D}\right)^{-1}\right], \tag{27}$$

with $\mathbf{D} \triangleq \sigma_w^2\,\mathbf{C}^{\rm H}\,\mathbf{G}\,\mathbf{F}\,\left(\mathbf{I}_{N_C N_R} + \sigma_w^2\,\mathbf{F}^{\rm H}\,\mathbf{G}^{\rm H}\,\mathbf{G}\,\mathbf{F}\right)^{-1}\mathbf{F}^{\rm H}\,\mathbf{G}^{\rm H}\,\mathbf{C}$. Let $\lambda_\ell(\mathbf{A})$, for $\ell \in \{1, 2, \cdots, m\}$, denote the eigenvalues of a generic matrix $\mathbf{A}$ having $\mathrm{rank}(\mathbf{A}) = m$, arranged in increasing order, i.e., $\lambda_1(\mathbf{A}) \geq \lambda_2(\mathbf{A}) \geq \cdots \geq \lambda_m(\mathbf{A})$. Relying on Weyl's inequality [12], it results

$$\lambda_\ell(\mathbf{I}_{N_B} + \mathbf{C}^{\rm H}\,\mathbf{C} - \mathbf{D}) \leq \lambda_\ell(\mathbf{I}_{N_B} + \mathbf{C}^{\rm H}\,\mathbf{C}) - \lambda_1(\mathbf{D}) \leq \lambda_\ell(\mathbf{I}_{N_B} + \mathbf{C}^{\rm H}\,\mathbf{C}), \tag{28}$$

for $\ell \in \{1, 2, \cdots, N_B\}$, where we have exploited the positive semidefiniteness property of the matrix $\mathbf{D}$ in the rightmost-hand side of (28). By taking into account (28), the MSE (27) can be lower-bounded as

$$\mathrm{MSE}(\mathbf{F}_0, \mathbf{F}) = \sum_{\ell=1}^{N_B} \frac{1}{\lambda_\ell(\mathbf{I}_{N_B} + \mathbf{C}^{\rm H}\,\mathbf{C} - \mathbf{D})} \geq \sum_{\ell=1}^{N_B} \frac{1}{\lambda_\ell(\mathbf{I}_{N_B} + \mathbf{C}^{\rm H}\,\mathbf{C})} = \mathrm{tr}\left[\left(\mathbf{I}_{N_B} + \mathbf{C}^{\rm H}\,\mathbf{C}\right)^{-1}\right]. \tag{29}$$

## II. PROOF OF THEOREM 1

We note that under **a1)** and **a2)**, one has $\mathrm{rank}(\mathbf{B}\,\mathbf{H}\,\mathbf{F}_0) = N_B \leq N_S$. Let $\mathbf{U}_{\rm a}\,\mathbf{\Lambda}_{\rm a}\,\mathbf{U}_{\rm a}^{\rm H}$ be the eigenvalue decomposition (EVD) of $\mathbf{A} \triangleq \mathbf{H}^{\rm H}\,\mathbf{B}^{\rm H}\,\mathbf{B}\,\mathbf{H} \in \mathbb{C}^{N_S \times N_S}$, where the diagonal matrix $\mathbf{\Lambda}_{\rm a} \in \mathbb{R}^{N_S \times N_S}$ and the unitary matrix $\mathbf{U}_{\rm a} \in \mathbb{C}^{N_S \times N_S}$ collect the eigenvalues, arranged in increasing order, and the eigenvectors of $\mathbf{A}$, respectively. The objective function in (8) is a Schur-concave function of the diagonal elements of $\left(\mathbf{I}_{N_B} + \mathbf{F}_0^{\rm H}\,\mathbf{A}\,\mathbf{F}_0\right)^{-1}$. In this case, it can be shown [10] that there is an optimal $\mathbf{F}_0$ such that $\mathbf{F}_0^{\rm H}\,\mathbf{A}\,\mathbf{F}_0$ is diagonal, whose diagonal elements are assumed to be arranged in increasing order, and such an optimal matrix, which also minimizes $\mathrm{tr}(\mathbf{F}_0\mathbf{F}_0^{\rm H})$, is given by

$$\mathbf{F}_0 = \mathbf{U}_{\text{a,right}}\,\widetilde{\mathbf{\Omega}}, \tag{30}$$

where $\mathbf{U}_{\text{a,right}} \in \mathbb{C}^{N_S \times N_B}$ contains the $N_B \leq N_S$ rightmost columns from $\mathbf{U}_{\rm a}$ and $\widetilde{\mathbf{\Omega}} \in \mathbb{R}^{N_B \times N_B}$ is a diagonal matrix. Let $\widetilde{\mathbf{Q}} \in \mathbb{C}^{N_D \times N_S}$ be an arbitrary semi-unitary matrix, i.e., $\widetilde{\mathbf{Q}}^{\rm H}\,\widetilde{\mathbf{Q}} = \mathbf{I}_{N_S}$, it follows from the EVD of the matrix $\mathbf{A}$ that $\mathbf{B}\,\mathbf{H} = \widetilde{\mathbf{Q}}\,\mathbf{\Lambda}_{\rm a}^{1/2}\,\mathbf{U}_{\rm a}^{\rm H}$. By substituting the SVD of $\mathbf{H}$ (defined in the theorem statement) in this equation, after some algebraic manipulations, one has that the minimum-norm solution [20] of the matrix equation $\mathbf{B}\,\mathbf{U}_{\rm h}\,\mathbf{\Lambda}_{\rm h} = \widetilde{\mathbf{Q}}\,\mathbf{\Lambda}_{\rm a}^{1/2}\,\mathbf{U}_{\rm a}^{\rm H}\,\mathbf{V}_{\rm h}$ is

$$\mathbf{B} = \widetilde{\mathbf{Q}}\,\mathbf{\Lambda}_{\rm a}^{1/2}\,\widetilde{\mathbf{U}}_{\rm a}\,[\mathbf{O}_{r_{\rm h} \times (N_C N_R - r_{\rm h})}, \mathbf{\Lambda}_{\text{h,right}}^{-1}]\,\mathbf{U}_{\rm h}^{\rm H}, \tag{31}$$

where $r_{\rm h}$, $\mathbf{V}_{\text{h,right}}$, and $\mathbf{U}_{\rm h}$ are defined in the theorem statement, whereas $\widetilde{\mathbf{U}}_{\rm a} \triangleq \mathbf{U}_{\rm a}^{\rm H}\,\mathbf{V}_{\text{h,right}} \in \mathbb{C}^{N_S \times r_{\rm h}}$ and the diagonal matrix $\mathbf{\Lambda}_{\text{h,right}} \in \mathbb{R}^{r_{\rm h} \times r_{\rm h}}$ gathers the $r_{\rm h}$ nonzero singular values of $\mathbf{H}$ in increasing order. The aim is now to further determine (31) by properly choosing $\widetilde{\mathbf{U}}_{\rm a}$ such that $\mathrm{tr}(\mathbf{B}\mathbf{B}^{\rm H}) = \mathrm{tr}[(\widetilde{\mathbf{U}}_{\rm a}\,\mathbf{\Lambda}_{\text{h,right}}^{-2}\,\widetilde{\mathbf{U}}_{\rm a}^{\rm H})\,\mathbf{\Lambda}_{\rm a}]$ has the smallest value.[4] By observing that $\widetilde{\mathbf{U}}_{\rm a}^{\rm H}\,\widetilde{\mathbf{U}}_{\rm a} = \mathbf{I}_{r_{\rm h}}$ and using a known trace inequality,[5] one has

$$\mathrm{tr}[(\widetilde{\mathbf{U}}_{\rm a}\,\mathbf{\Lambda}_{\text{h,right}}^{-2}\,\widetilde{\mathbf{U}}_{\rm a}^{\rm H})\,\mathbf{\Lambda}_{\rm a}] \geq \sum_{\ell=1}^{r_{\rm h}} \lambda_{\text{h},\ell}^{-2}\,\lambda_{\text{a},\ell}, \tag{32}$$

where $\lambda_{\text{h},\ell}$ and $\lambda_{\text{a},\ell}$ denote the $\ell$th diagonal entry of $\mathbf{\Lambda}_{\text{h,right}}$ and $\mathbf{\Lambda}_{\rm a}$, respectively. The equality in (32) holds when

$$\widetilde{\mathbf{U}}_{\rm a} = \mathbf{U}_{\rm a}^{\rm H}\,\mathbf{V}_{\text{h,right}} = [\mathbf{O}_{(N_S - r_{\rm h}) \times r_{\rm h}}^{\rm T}, \mathbf{I}_{r_{\rm h}}]^{\rm T}\,. \tag{33}$$

Substituting (33) in (31), after some algebraic manipulations, one obtains (10), with $\mathbf{\Delta} = \mathbf{\Lambda}_{\text{h,right}}^{-1}\,\mathbf{\Delta}_{\text{a,right}}^{1/2}$, where the diagonal matrix $\mathbf{\Delta}_{\text{a,right}} \in \mathbb{R}^{r_{\rm h} \times r_{\rm h}}$ collects the $r_{\rm h}$ largest eigenvalues of $\mathbf{A}$. Solution (9) comes from substituting in (30) the minimum-norm solution [20] of (33), i.e., $\mathbf{U}_{\rm a} = \mathbf{V}_{\text{h,right}}\,[\mathbf{O}_{r_{\rm h} \times (N_S - r_{\rm h})}, \mathbf{I}_{r_{\rm h}}]$, and, then, separately considering the cases $N_B \leq r_{\rm h}$ and $N_B > r_{\rm h}$, where the diagonal matrix $\mathbf{\Omega} \in \mathbb{R}^{L_{\rm h} \times L_{\rm h}}$ gathers the largest $L_{\rm h}$ diagonal entries of $\widetilde{\mathbf{\Omega}}$.

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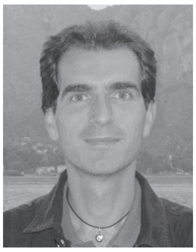

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