Where Is Mathematics?

BY ROBERT SINCLAIR
Mathematical Biology Unit
Okinawa Institute of Science and Technology Graduate University, Japan

As a mathematician, I read the Point of View article "The reasonable ineffectiveness of mathematics" [1] with great interest. While I do agree with much of what is written there, I wish to correct two impressions the article may have left in some readers' minds; first, that the belief that mathematics has its own existence somehow implies "the notion that mathematical forms underpin the physical universe."

For the record, let me state that I do believe mathematics has its own existence, but I do not see any compelling reason why mathematical forms should underpin the physical universe. Put simply, this is what I mean by "own existence." To be more specific, although I understand that there is a school of thought, inspired by Plato's allegory of the cave (Book 7 of *Republic* [2]), that the physical universe is but a shadow of a mathematical world, I do not feel bound to define my own position as either for or against it. I would not even know where to start: *Which mathematics is the right one?* Do we have different physical universes for different types of mathematics (with or without the Axiom of Choice, for example, [3])? Let me be clear that, in asking "which mathematics," I am not asking which branch of mathematics. I am actually referring to the fact that we have several different versions of mathematics as a whole, each one based upon different logical foundations [4]–[6]. Are they really so different? Yes, and the nonstandard versions are occasionally finding application in engineering contexts [7]. One can get a taste of what I mean by considering the fact that most working scientists and engineers are justifiably content to use only floating-point arithmetic in their calculations, and yet floating-point number representations are fundamentally different from the real numbers of "standard mathematics," starting with the fact that floating point (I am thinking of the IEEE 754 standard [8]) often has separate encodings for $+0$ and -0, and for good reasons [9]. Since one could, in principle, develop a type of mathematics upon the foundation of floating-point arithmetic [10], do I have to ask whether I should be looking for both $+0$ and -0 in the physical world? I think not. Furthermore, I should clearly state that even the most commonly accepted version of mathematics involves paradoxes, even of a sort that would allow one to cut up the sun and gently put the pieces back together to make something the size of a pea [3], [11]. I do not believe that such mathematics is the literal foundation of the physical world I live in, but neither do I reject it as mathematics.

Most pure mathematicians would point to proof as the defining characteristic of mathematics, and would do so without reference to anything physical. I agree, but I prefer to think of mathematics in terms of Cantor's comment that can be translated as

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"the essence of mathematics lies in its freedom" [12], and, for me, that also means freedom from having to define mathematics and the physical universe in terms of each other. Instead, I like to think of mathematics and engineering or basic science as equal partners.

The second impression I would like to correct is that mathematics may be purely a product or invention of the human mind.

There are two reasons why I cannot subscribe to such a point of view. First, we already live in an age where computers not only generate new mathematics, but also, in some cases, judge the mathematical work of humans. Second, *Homo sapiens* is not the only species to have evolved and used mathematical descriptions of the world.

Mathematics owes a great deal to the community of electrical engineers and computer scientists, since we would otherwise not have had reliable computing machines. While I have often heard fellow mathematicians state that computers could never complete a proof which involves an infinite number of cases, the real issue is whether the number of steps in a proof, and the resources required to process and store them, is finite. Using symbolic methods, even a computer can prove, in a finite number of elementary steps, that $2 + x^2$ is greater than unity for all real x. There is no need to evaluate $2 + x^2$ for specific values of x at all. At a less trivial level, one can ask whether a computer has ever found a mathematical proof that humans had failed to. The earliest examples of this that I am aware of date back to the last century (I consider a result relating to the Robbins conjecture to be a particularly nice one [13]). It would be tempting to retreat to the position that humans remain the final arbiters or custodians of mathematical truth, but there is already one journal I know of, *Formalized Mathematics*, published by Walter de Gruyter, Berlin, Germany, which subjects submitted articles to automated proof checking [14] before any peer review can begin.

The question, whether mathematics could ever have been the product of any but a human mind, depends in part on our definition of what it means to understand mathematics. It is no use shouting mathematical questions at an unsuspecting slime mould and then celebrating its failure to reply in English. Such mistaken logic has all too often been used in attempts to demonstrate the superiority of one group of humans over another. Instead, it has been suggested that repeated correct usage is an effective indicator of understanding [15]. It may come as a surprise to some that bees have evolved a polar representation of relative position that they not only use, but use to communicate relative positional information from one bee to another [16]. It is the act of abstract communication which I believe makes this example more relevant here than the otherwise impressive computational/cryptographic potential of some genetic systems [17] or the bowerbirds' use of perspective [18], etc. There is ample evidence that animals have and use abstract problem-solving abilities [19], including game theory as a real battle of wits, with the outcome determining life or death [20]. Furthermore, purely genetic explanations appear to be inadequate to explain the observed transmission of tool use in dolphins [21], meaning that we must take seriously the possibility that animals may be developing and communicating complex skills in ways which are difficult to distinguish from ways that humans develop and communicate mathematics. Even the frequently expressed thought, that "pure mathematics is play, and only humans do that, since animals only do what they need to survive," crumbles when confronted with observations of dolphins' bubble ring play [22].

All this leads me to the issue of the "lesser success" of mathematics in "describing biological systems" [1]. In my honest opinion, it is too early to make strong statements concerning the relationship between mathematics and biology. To write that biology is somehow "harder to model" than other natural sciences is to miss an important point: The relationship between mathematics and biology already goes deeper than modeling. Regarding the "idea of information and its physical embodiment in DNA sequences," the "fundamental theory was formulated by Turing in his notion of a universal Turing machine and deployed by von Neumann in his theory of self-reproducing machines" [23]. Also, as any engineer knows, biological systems seem to have much to teach us about the principles of control theory. This is one of many areas in which I expect new mathematics to eventually emerge, once again a reason to see the relationship between mathematics and biology as more than just modeling.

We can, of course, also expect to see other types of new engineering principles emerge from careful studies of living organisms. I do not wish to go into a list of specific examples here, but do want to make one general comment at a rather abstract level: Evolution teaches us that optimality is not always the strategy that wins in the long run [24]. There are many reasons for this. One is that optimality often implies a loss of flexibility in construction, and both parasites and predators usually target any aspect of an organism that is unvarying. Another is that optimality in a given context often translates into weakness in another, and *vice versa*. Thus, a small fraction of bacteria may occasionally, without any apparent reason, go into a "dormant" state, only to "wake up" again later. The switching is determined by a reversible stochastic mechanism which operates in all normal cells [25]. The growth of the colony is slightly reduced as a result, but it is these few cells which can survive when antibiotics are administered, and their seemingly ridiculous behavior is of great long-term benefit for their lineage.

Given the valid points raised by Abbott and others, it is useful to reexamine what it is that makes mathematics valuable. I wish to propose that it is precisely its "otherness" which makes mathematics, as a branch of philosophy, valuable to engineering and science, but only where there is real engagement from both sides (I have put science and engineering on the same "side" here, since they both deal more or less directly with the world we live in, whereas mathematics, in my opinion, does not). I am painfully aware of the failures of parts of the mathematical community to engage, and see this as one of the major social challenges to be overcome. I am one of the fortunate few who has had the experience of having done pure mathematical work with no application in mind, but then subsequently to see it being applied [26]. I wish more mathematicians could have such experiences. This does not mean that I propose any sort of dilution of mathematics itself. The power of mathematical approaches tends to be in their generality. The finite element method is fortunately not tied to a narrow area of application, for example, but is useful to many. This is because it has not been developed with any overly specific or exclusive set of problems in mind, and general mathematical principles were allowed to shape its foundations [27]. I suggest that we can learn much from this example, in the sense that a combination of a genuine need for a practical method and a genuine interest in basic mathematical issues, even when from a point of view that is completely divorced from applications, is a win–win situation. The only requirement is a willingness to communicate across traditional boundaries.

Engineering and the basic natural sciences have always been sources of inspiration for mathematicians. It is no accident that we have seen, time and again, new mathematics (and statistics) emerge exactly at the interface between mathematics and a practical need for answers. Abbott's Point of View [1] was an exciting read for me because it reminded me that engineers do care about the nature of mathematics, and this can only benefit us all. ■

REFERENCES

[1] D. Abbott, "The reasonable ineffectiveness of mathematics [Point of view]," *Proc. IEEE*, vol. 101, no. 10, pp. 2147–2153, Oct. 2013.

[2] Plato, *Republic*. Oxford, U.K.: Oxford Univ. Press, 2008.

[3] C. K. Gordon, "The third great crisis in mathematics," *IEEE Spectrum*, vol. 5, no. 5, pp. 47–52, May 1968.

[4] G. Priest, "Mathematical pluralism," *Logic J. IGPL*, vol. 21, pp. 4–13, Feb. 2013.

[5] C. W. Henson, "Foundations of Nonstandard Analysis," in *Nonstandard Analysis*, L. Arkeryd, N. Cutland, and C. W. Henson, Eds. Amsterdam, The Netherlands: Springer-Verlag, 1997, vol. 493, pp. 1–49.

[6] E. B. Davies, "Pluralism in mathematics," *Philos. Trans. R. Soc., Math. Phys. Eng. Sci.*, vol. 363, pp. 2449–2458, Oct. 15, 2005.

[7] L. Ferrucci, D. Mandrioli, A. Morzenti, and M. Rossi, "Modular automated verification of flexible manufacturing systems with metric temporal logic and non-standard analysis," in *Formal Methods for Industrial Critical Systems*, M. L. Stoelinga and R. Pinger, Eds. Berlin, Germany: Springer-Verlag, 2012, vol. 7437, pp. 162–176.

[8] *IEEE Standard for Floating-Point Arithmetic*, IEEE Std. 754-2008, 2008, pp. 1–70.

[9] W. Kahan, "Branch cuts for complex elementary functions, or much ado about nothing's sign bit," in *The State of the Art in Numerical Analysis*, A. Iserles and M. J. D. Powell, Eds. Oxford, U.K.: Oxford Univ. Press, 1987, pp. 165–211.

[10] W. S. Brown, "A simple but realistic model of floating-point computation," *ACM Trans. Math. Softw.*, vol. 7, pp. 445–480, 1981.

[11] L. M. Wapner, *The Pea and the Sun: A Mathematical Paradox*. Boca Raton, FL, USA: A K Peters/CRC Press, 2007.

[12] G. Cantor, "Ueber unendliche, lineare Punktmannichfaltigkeiten," *Mathematische Annalen*, vol. 21, pp. 545–591, Dec. 1883.

[13] W. McCune, "Solution of the Robbins problem," *J. Autom. Reason.*, vol. 19, pp. 263–276, Dec. 1997.

[14] A. Grabowski, A. Kornilowicz, and A. Naumowicz, "Mizar in a nutshell," *J. Formalized Reason.*, vol. 3, no. 2, pp. 153–245, 2010.

[15] L. Wittgenstein, *Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge, 1939*. Chicago, IL, USA: Univ. Chicago Press, 1989.

[16] F. C. Dyer, "The biology of the dance language," *Annu. Rev. Entomol.*, vol. 47, pp. 917–949, 2002.

[17] L. F. Landweber and L. Kari, "The evolution of cellular computing: Nature's solution to a computational problem," *Biosystems*, vol. 52, pp. 3–13, Oct. 1999.

[18] J. A. Endler, L. C. Endler, and N. R. Doerr, "Great bowerbirds create theaters with forced perspective when seen by their audience," *Current Biol.*, vol. 20, pp. 1679–1684, Sep. 28, 2010.

[19] A. H. Taylor, D. Elliffe, G. R. Hunt, and R. D. Gray, "Complex cognition and behavioural innovation in New Caledonian crows," *Proc. R. Soc. B, Biol. Sci.*, vol. 277, pp. 2637–2643, Sep. 7, 2010.

[20] R. R. Jackson and R. S. Wilcox, "Spider-eating spiders," *Amer. Sci.* vol. 86, pp. 350–357, Jul.–Aug. 1998.

[21] M. Krutzen, J. Mann, M. R. Heithaus, R. C. Connor, L. Bejder, and W. B. Sherwin, "Cultural transmission of tool use in bottlenose dolphins," *Proc. Nat. Acad. Sci. USA*, vol. 102, pp. 8939–8943, Jun. 21, 2005.

[22] B. McCowan, L. Marino, E. Vance, L. Walke, and D. Reiss, "Bubble ring play of bottlenose dolphins (Tursiops truncatus): Implications for cognition," *J. Comparat. Psychol.*, vol. 114, pp. 98–106, Mar. 2000.

[23] S. Brenner, "The revolution in the life sciences," *Science*, vol. 338, pp. 1427–1428, Dec. 14, 2012.

[24] K. Parvinen and U. Dieckmann, "Self-extinction through optimizing selection," *J. Theor. Biol.*, vol. 333, pp. 1–9, 2013.

[25] E. Maisonneuve, M. Castro-Camargo, and K. Gerdes, "(P)ppGpp controls bacterial persistence by stochastic induction of toxin-antitoxin activity," *Cell*, vol. 154, pp. 1140–1150, 2013.

[26] J. Osborne, G. Hicks, and R. Fuentes, "Global analysis of the double-gimbal mechanism—Dynamics and control on the torus," *IEEE Control Syst. Mag.*, vol. 28, no. 4, pp. 44–64, Aug. 2008.

[27] G. Strang, "Piecewise polynomials and the finite-element method," *Bull. Amer. Math. Soc.*, vol. 79, pp. 1128–1137, 1973.