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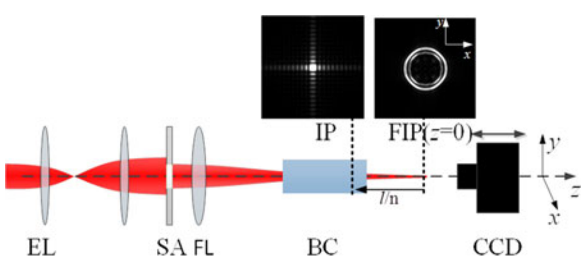

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Method Based on Fast Fourier Transform for Calculating Conical Refraction of Beams With Noncircular Symmetry

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Abstract: Conical refraction of optical beams with circular symmetry is often analyzed using Belsky–Khapalyuk–Berry (BKB) theory; however, for beams with noncircular symmetry, it is difficult to obtain analytical expressions for far-field diffraction patterns. We propose a method, based on fast Fourier transform (FFT), for calculating conical refraction of beams with noncircular symmetry and verify it experimentally using a quasi-plane wave passing through a square aperture and focusing lens. Excellent agreement between theoretical and experimental results has been achieved.

Index Terms: Diffractive optics, optical and other properties.

1. Introduction

Conical refraction (also referred to as conical diffraction) was first predicted in 1832 by Hamilton [1]. It occurs when the light beam propagates along one of optic axes of biaxial crystal and spreads out into a hollow skewed cone inside the materials. In 1978, Belsky and Khapalyuk proposed an exact paraxial theory of conical refraction [2], [3]. Then, Berry reformulated and expanded this theory [4],which is generally called Belsky–Khapalyuk–Berry (BKB) solution [5], [6]. For simplicity and convenience, the incident beam is assumed to be uniformly polarized and circularly symmetric [2], [4], and in this case the solution can be expressed in a form of integrals of Bessel function.

Based on BKB theory, experimental and theoretical studies of conical refraction have been carried out for cylindrically symmetric beams, such as Gaussian beams [7]–[9], top-hat input beams [10], and Laguerre-Gauss input beams [11], [12]. In addition, the case of the incident beam out of the optical axis has also been studied [13]–[15]. For non-cylindrically symmetric and non-homogeneously polarized incident beams, Turpin and co-workers have given a formula by separately considering Fourier transform of incident beams in x and y directions [15]. In the above research, the analytical expressions of Fourier transforms for incident beams with symmetry are required for calculation of the conical refraction. However, there are no analytical expressions for the most practical incident beams, such as the beam with non-circular symmetry. The calculation of the conical refraction of these incident beams becomes a problem. Furthermore, the time overhead of diffraction integrals of Bessel function is large, especially in the case of non-circularly symmetric diffraction patterns, where the calculations in every point of field are required. For example, the incident beam is directed out of an optical axis.

In order to solve this problem and to improve the calculation efficiency, we propose a method based on fast Fourier transform (FFT) [16] to calculate the conical refraction of beams with non-circular symmetry. It is well known that with the development of computer techniques the FFT has become a popular method to calculate the diffraction patterns. When the calculations meet Nyquist sampling theorem [17], the numerical results from discrete Fourier transform (DFT) is similar to that made by the analytical approaches. Conical refraction is a refractive effect that can be described in terms of diffractive optics and that one can use FFT by selecting the appropriate transform function, thus it can also be expressed as a convolution form and calculated by the Fourier transform and inverse Fourier transform. The only difference between conical refraction formulae and classical diffraction formulae [18] is the transform function. Our goal here is to verify the feasibility of the application of FFT for calculating the conical refraction of complicated incident beams including elliptical input beams, axicon input beams [19], and partially blocking beams [20]. This method that we demonstrated is also the computationally efficient, which is useful for beam shaping and optical tweezers. Furthermore, a non-circular symmetry and rectangle diffracted incident beam was used in this study. The conical refraction phenomenon exhibits interesting intensity profiles beyond the crystal with a rich evolution structure.

The theory used here is based on Belsky and Khapalyuk [2], [3], but we modified it and calculate the conical refraction of non-circular symmetry beam based on FFT. The transverse component of the electric field of a beam can be described as an angular spectrum of plane waves

$$U(x, y, z) = \int_{-\infty}^{+\infty} \int A(f_x, f_y, 0) \exp\left[-i2\pi\left(xf_x + yf_y + z\sqrt{n^2 - f_x^2 - f_y^2}\right)\right] df_x df_y \tag{1}$$

where $A(f_x, f_y)$ is the angular spectrum of the field $U(x, y, 0)$, and integration variables f_x and f_y are spatial frequencies. Assuming the three principal refractive indices of biaxial crystal $n_1 < n_2 < n_3$, the semi angle of the internal conical refraction A_C can be expressed as

$$A_C = \frac{1}{2}\arctan\left[n_2^2\sqrt{(n_1^{-2} - n_2^{-2})(n_2^{-2} - n_3^{-2})}\right]. \tag{2}$$

For common biaxial materials A_C is very small, and then the paraxial approximation can be used [2], [4]. The radius of the ring of conical refraction scales with the length of material l and can be described as

$$R_0 = \frac{l}{2}\tan(2A_C) \approx A_C l. \tag{3}$$

For different angular spectra, the refractive indices are different and can be written as [21]

$$n_f = n_2\left[1 + \frac{A(-\lambda f_x) \pm \sqrt{f_x^2 + f_y^2}}{n_2}\right]. \tag{4}$$

Here, the n_f is an approximate solution, but it is suitable for most situations [21]. Considering the double refraction (anisotropy) of crystal, the electrical field vector of the beam in the principle plane of the crystal (p-polarization) and the electrical field vector that is perpendicular to the principle plane of the crystal (s-polarization) should be handled separately. Then, the total fields can be expressed as the superposition of four parts as (5), shown below. The superscripts and subscripts represent the polarization directions before and after the propagation in the crystal, respectively

$$U_{x,y}^{s,p}(x, y) = C\int_{-\infty}^{+\infty} \int A(f_x, f_y, 0) H_{x,y}^{s,p}(f_x, f_y) \exp[-ik(x\lambda f_x + y\lambda f_y)] df_x df_y \tag{5}$$

where

$$H_x^p = H_B \cdot \left[\cos\left(kR_0\sqrt{(\lambda f_x)^2 + (\lambda f_y)^2}\right) + i\sin(kR_0\sqrt{(\lambda f_x)^2 + (\lambda f_y)^2}\right) \frac{f_x}{\sqrt{f_x^2 + f_y^2}} \right]$$

$$H_y^p = H_B \cdot \left[i\sin(kR_0\sqrt{(\lambda f_x)^2 + (\lambda f_y)^2}) \frac{f_y}{\sqrt{f_x^2 + f_y^2}} \right]$$

$$H_x^s = H_B \cdot \left[i\sin\left(kR_0\sqrt{(\lambda f_x)^2 + (\lambda f_y)^2}\right) \frac{f_y}{\sqrt{f_x^2 + f_y^2}} \right]$$

$$H_y^s = H_B \cdot \left[\cos\left(kR_0\sqrt{(\lambda f_x)^2 + (\lambda f_y)^2}\right) - i\sin(kR_0\sqrt{(\lambda f_x)^2 + (\lambda f_y)^2} \right] \tag{6}$$

where H_B, which is transfer function of angular spectrum, can be expressed as

$$H_B\left(f_x, f_y\right) = \exp\left[ikd\sqrt{1 - (\lambda f_x)^2 - (\lambda f_y)^2}\right]. \tag{7}$$

Variable d in (7) is the effective distance along the axial

$$d = z_t + l(1 - 1/n_2). \tag{8}$$

Here, z_t is the distance between rear surface of crystal and observe plane. Using Fourier transform, (5) can be modified into

$$U_{x,y}^{s,p}(x, y, z) = F^{-1}\left\{F\{U_0(x_0, y_0, 0)\}H_{x,y}^{s,p}(f_x, f_y)\right\}. \tag{9}$$

Finally, the intensity of diffraction pattern can be expressed as

$$I\,(x, y) = \left|U_x^s(x, y) + U_x^p(x, y)\right|^2 + \left|U_y^s(x, y) + U_y^p(x, y)\right|^2. \tag{10}$$

Comparing to the transfer function of the effective distance in free space in (7), the effect of crystal on H_B can be considered to be introduced an additional length of $-l/n_2$. It means the incident plane can be selected freely and the position of crystal does not depend on the conical refraction pattern. Then the variables of interest are the length of crystal l and the length between the focusing lens (FL) and the observe plane z.

In most cases, except biaxial crystals there will be a free space between FL and the observe plane. We assume the angular spectrum of incident beam at the observe plane is $A\,(f_x, f_y, z)$, and thus

$$A\,(f_x, f_y, z) = A\,(f_x, f_y, 0)\exp\left[-ikz\sqrt{1 - (\lambda f_x)^2 - (\lambda f_y)^2}\right]. \tag{11}$$

Then, (9) can be modified as

$$U_{x,y}^{s,p}(x, y, z) = F^{-1}\left\{A\,(f_x, f_y, z)H_{x,y}^{s,p}(f_x, f_y)\right\} \tag{12}$$

where transfer function of H_B can be expressed as

$$H_B\left(f_x, f_y\right) = \exp\left[-ikl\sqrt{1 - (\lambda f_x)^2 - (\lambda f_y)^2}/n_2\right]. \tag{13}$$

Using Collins formula [22], it is convenient to calculate the angular spectrum of observe plane $A\,(f_x, f_y, z)$ and then to calculate the diffraction pattern using (10) and (12).

In addition, as for the field inside the crystal, the R_0 should be replaced by $R_0(l)$, which is the radius of ring of propagation length inside the crystal. The transfer function H_B can be replaced by the transfer function of Fresnel diffraction, but this change generates little difference in the calculation.

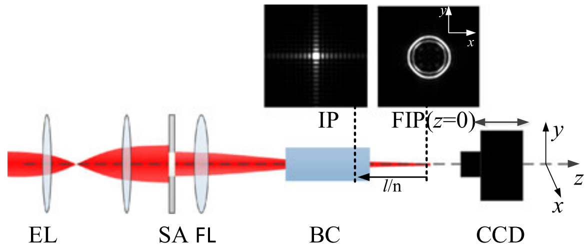

Fig. 1. Experimental setup. A laser beam collimated and expanded by a pair of lenses (EL) passes through a square aperture (SA) (l = 1.8 mm) and then is focused by the focusing lens (FL) (focal length 75 mm) and enters a biaxial crystal (BC) of KTP of length 10 mm. The emerging light is captured by CCD. The figure also plots the beam profiles at incident plane (IP) and focal image plane (FIP).

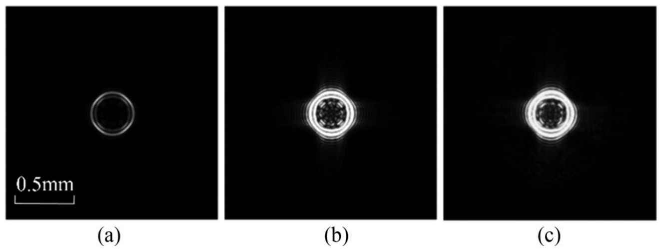

Fig. 2. Contrast of calculating diffraction pattern at $z = 0$ mm. (a) Normalized light intensity, (b) saturated light intensity, and (c) experimental result

2. Experimental Details

In order to verify our method of using FFT for calculating the diffraction pattern, we have done experiment with a beam diffracted by square aperture, which is a simple case of incident beams without circular symmetry, and the density is further calculated using (11) and (12). Fig. 1 shows the experimental setup. Input beam from a He-Ne laser was collimated and expanded by a pair of lenses. The beam radius was about 3.5 mm after the expansion and the beam was considered as a plane wave. A square aperture with a length of 1.8 mm was placed before the focusing lens with a focal length of 75 mm. The beam then propagated along the optic axis of a KTP biaxial crystal with a length of 10.0 mm and a cross section of 5.0 mm × 5.0 mm. The three principal refractive indices for the KTP crystals at 632.8 nm are $n_1 = 1.7636$, $n_2 = 1.7733$, $n_3 = 1.8636$. The surfaces of the crystal were cut to be perpendicular to one of its optic axes. A CCD fixed on the displacement platform was used to record the diffraction pattern. Although a square aperture was used as an example in the experiment to verify the theory, it can be replaced by arbitrary apertures.

The experimental parameters of interest in this study are the longitudinal shift from FL to CCD, the length of crystal, the focal length of FL and the length of square aperture. Since the position of the crystal does not affect the pattern, the incident plane is selected freely. Conventionally, the focal plane of the FL is considered as the incident plane which corresponding to the focal image plane (FIP).

Before the comparison, the intensities of calculating diffraction patterns are processed to get better contrast of patterns. It is noted that some diffraction fringes cannot be observed due to its low intensity. To see fringe detailes clearly we saturated the pattern, which is a kind of image enhancement. The intensity of Fig. 2(a) is normalized, intensity of Fig. 2(b) is 20 times higher than that of Fig. 2(a), and Fig. 2(c) is the corresponding experimental result. Therefore, the simulation results in Fig. 3 are saturated appropriately to see more detail of patterns.

3. Results

Due to incident beam without circular symmetry, the intensity distribution varies in different directions. This makes it difficult to compare the experimental conical refraction patterns with that of the

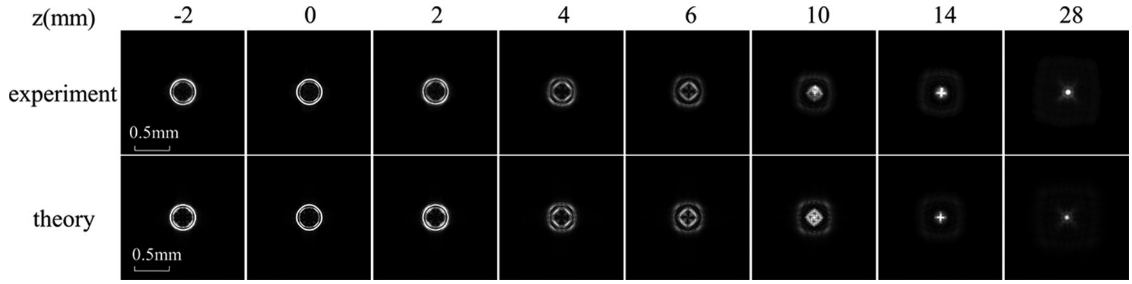

Fig. 3. Experimental and theoretical results of diffraction pattern at different distances.

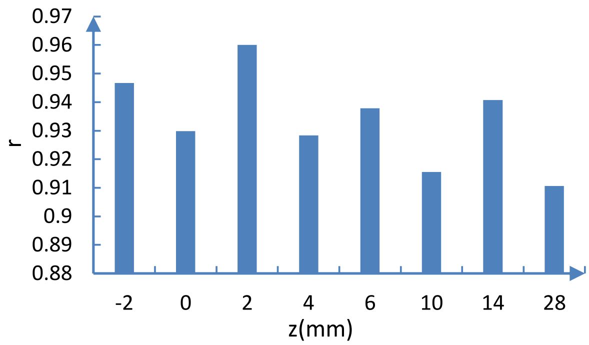

Fig. 4. Correlation coefficient for the theoretical and experimental patterns.

Gauss beam or the top-hat beam. However, the key features of conical refraction are comparable. The hollow double-ring is clearest at FIP, as shown in Fig. 3 ($z = 0$). The pattern seems an octagon with the highest intensity at diagonal line and has diffraction fringes inside and outside the rings. In Fig. 3, with the light propagation along the axial, the inner ring turns into a spot gradually with the outer ring expanding. This evolution is a typical feature of classical conical refraction. However, it is apparent that the diffraction feature of the square aperture becomes clearer with the light propagation. The axial spike [4] can be seen in Fig. 3 ($z = 28$ mm), with a spot surrounded by enormous fringes. Patterns at $z = -2$ mm and $z = 2$ mm are almost the same, which shows the longitudinal symmetry of diffraction patterns on FIP. This feature is also found in the reported work on conical refraction [8], [23]. The essential reason of this symmetry is that the angular spectra before and after the focal point of FL are identical.

Fig. 4 presents the correlation coefficient of our theoretical and experimental results. The correlation coefficient r is defined as

$$r = \frac{\sum_m \sum_n (A_{mn} - \bar{A})(B_{mn} - \bar{B})}{\left[\left(\sum_m \sum_n (A_{mn} - \bar{A})^2\right)\left(\sum_m \sum_n (B_{mn} - \bar{B})^2\right)\right]^{1/2}} \tag{14}$$

where A and B are intensity matrices of experiment and theory, and subscripts m and n are x and y coordinates.

From Fig. 4, the average correlation coefficient is larger than 0.93. The correlation coefficients are less than 1 due to the offset between the theoretical and experimental patterns. However, these correlation coefficients uniformly larger than 0.934 show excellent agreement between theoretical and experimental patterns, demonstrating the feasibility of our method using FFT. Since the proposed FFT method uses the Collins formula to calculate the spectra of incident beams, more complicated incident beam can be calculated using this method. Thus, our method is general.

In addition, for a 1024×1024 pattern of the Gauss input beam, $1024 \times 1024 \times 2$ operations are required in the classic integral method and at least 512×2 operations required considering the circular symmetry, while only one operation of FFT and four operations of IFFT (inverse fast Fourier transform) are consumed to obtain the1024×1024 pattern in our method. The total time consumption for our method and the classic integral method are about 0.1 s and 1.6 s measured by using the built-in timer in Matlab, respectively. It shows the high efficiency of our method.

4. Conclusion

This work presents a phenomenon of conical refraction and provides an efficient and convenient method to calculate the conical refraction pattern using FFT. The conical refraction of an incident beam without circular symmetry is calculated, of which the analytical expression using convolution is difficult to obtain. Observations of a beam diffracted by a square aperture propagating through a conical refraction crystal at different distances have been made. The diffraction pattern shows the features of classical conical refraction, such as Raman spot and hollow double-ring, while it also shows the features of incident beam, such as the shape of aperture and the diffraction fringes. The experimental results demonstrate the effectiveness of our method based on FFT for calculating the conical refraction. This approach allows the calculation for the conical refraction of complicated incident beams and is significant to further research and related applications.

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