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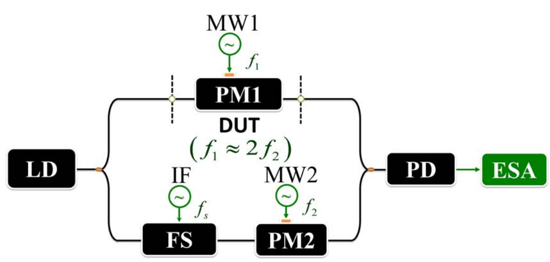

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Calibration-Free Electrical Spectrum Analysis for Microwave Characterization of Optical Phase Modulators Using Frequency-Shifted Heterodyning

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Abstract: A novel calibration-free electrical spectrum analysis method for microwave characterization of electrooptic phase modulators is proposed and experimentally demonstrated based on frequency-shifted optical heterodyning. The method achieves the electrical domain measurement of the modulation efficiency of phase modulators without the need for correcting the responsivity fluctuation in the photodetection. Moreover, it extends double the measuring frequency range through setting a specific frequency relationship between the driving microwave signals. Modulation depth and half-wave voltage of phase modulators are experimentally extracted from the heterodyning spectrum of two phase-modulated signals with and without frequency shifting, and the measured results are compared to those obtained with the traditional optical spectrum analysis method to check the consistency. The proposed method provides calibration-free and accurate measurement for high-speed optical phase modulators with the high-resolution electrical spectrum analysis.

Index Terms: Electro-optical modulation, microwave photonics signal processing, heterodyning, fiber optics systems.

1. Introduction

Optical phase modulation, where the instantaneous optical phase is varied in proportion to the electrical driving signal, is considered to be promising an alternative to intensity modulation due to intrinsically bias-drifting-free and linear modulation [1]–[3], and has attracted significant applications in the photonic microwave signal processing, such as photonic microwave filtering [4], microwave frequency measurement [5], [6], and photonic microwave generation [7]. In the phase-modulation-based microwave photonic system, the modulation efficiency of optical phase modulators, including modulation depth and half-wave voltage, will influence on the overall performance especially for broadband microwave operation, which should be critically characterized for the purpose of system evaluation and optimization.

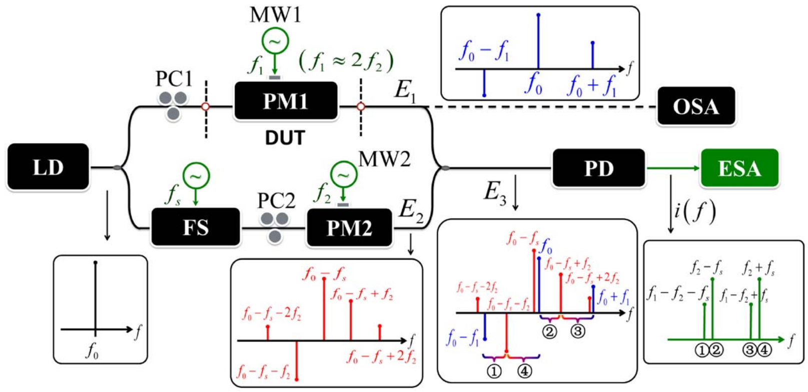

Fig. 1. Schematic diagram of the proposed electrical spectrum analysis method based on frequency-shifting optical heterodyning. LD, laser diode. PC, polarization controller. PM1, phase modulator under test. PM2, secondary phase modulator. FS, frequency shifter. MW, microwave source. DUT, device under test. PD, photodiode. ESA, electrical spectrum analyzer. OSA, optical spectrum analyzer.

Optical domain and electrical domain measurements are two main methods for microwave characterization of phase modulators. The optical spectrum analysis is very simple and direct, which is to measure the ratio of the sidebands with respect to the optical carrier in the optical spectrum of the phase modulated signal [8], [9]; however, the best resolving frequency is about 1.25 GHz (0.01 nm at 1550 nm), which is limited by the commercial grating-based optical spectrum analyzer (OSA) [10]. The resolution could be improved by using a high-resolution Brillouin-based OSA; however, the measurement will be affected by the line-width of laser source, since the optical spectrum after modulation is contributed by both the laser source and the phase modulator.

The electrical domain measurements often require the phase-to-intensity conversion with a Mach–Zehnder or Sagnac interferometer [11], [12], or frequency-to-intensity conversion with an optical discriminator or filter [13]–[15], since the phase modulation inherently changes the optical phase and keeps constant optical envelop. In this case, the phase modulated signal is eventually converted into electrical after square-law photo-detection for high resolution electrical domain measurement. Nevertheless, the phase-to-intensity conversion requires small-signal driving conditions and the frequency-to-intensity conversion should be operated at certain wavelengths in order to reduce the nonlinearity of interferometers and filters. Moreover, the major difficulty in these methods is the required extra calibration for correcting the responsivity fluctuation in the photo-detection especially when high frequency operation is involved [16], [17]. So, the methods that are capable of characterizing phase modulators with high resolution electrical domain measurement, and at the same time avoiding the calibration for the uneven responses of photodiodes are of great interest.

Recently, we demonstrated for the first time a proof-of-concept self-calibrating measurement of a high-speed electro-optic phase modulator based on the closely spaced two-tone modulation [18], which utilizes the measurement function of $J_1^2(m)/J_0^2(m)$ and requires four frequency components for each modulation frequency. In this paper, we propose an improved calibration-free measurement method for microwave characterization of optical phase modulators. As is shown in Fig. 1, the method consists of an acoustooptic frequency shifter and two phase modulators located in a Mach-Zehnder interferometer architecture. The two phase modulators are separately driven by different microwave signals with about half frequency relationship. The optical carriers of the two phase modulators are originated from the same optical source but detuned with respect to each other through acoustooptic frequency shifting. The phase-modulated signals after the two phase modulators are combined at the end of the interferometer and converted into electrical after photo-detection. The modulation efficiency of the phase modulator under test is extracted from the beat note of the detuned optical carriers and the modulation sidebands. The uneven responses of the photodiode (PD) are cancelled out through keeping the half frequency relationship of the driving microwave signals, and thus a calibration-free electrical domain measurement can be achieved.

The improved method has the following advantages: 1) The required four frequency components for each measurement in the former work are reduced to two frequency components in this work. 2) The measuring frequency range is doubled with the same PD through the half frequency relationship of the driving microwave signals. 3) The microwave driving levels are enhanced by 6 $\sim$ 7 dB due to without the microwave power combiner, which helps to improve the signal-to-noise ratio in the measurement. In this paper, theoretical description and experimental demonstration are presented to elaborate our method. The frequency-dependent modulation depths and half-wave voltages are experimentally measured for a commercial phase modulator, and the results are compared to those obtained using the conventional method to check the consistency and accuracy.

2. Operation Principle

Fig. 1 shows the schematic diagram of the proposed electrical spectrum analysis method. It consists of two electric-optic phase modulators (PM1 and PM2) and an acoustooptic frequency shifter (FS) located in a Mach-Zehnder interferometer. PM1 is the phase modulator under test located in the upper arm of the interferometer, and PM2 is cascaded with a FS and placed in the lower arm of the Mach-Zehnder interferometer. A continuous wave (CW) light from a laser diode (LD) is sent to PM1 and modulated by a sinusoidal microwave signal $v_1(t) = V_1 \sin 2\pi f_1 t$. The optical field after PM1 can be written as [3], [9]

$$E_1 = A_1 \exp j[2\pi f_0 t + m_1 \sin 2\pi f_1 t] = A_1 \sum_{n=-\infty}^{+\infty} J_n(m_1) \exp[j2\pi(f_0 + nf_1)t] \tag{1}$$

where A_1 is the amplitude of the optical carrier, f_0 is the frequency of the optical carrier and m_1 is the modulation depth of PM1. From Eq. (1), phase modulation generates multiple sidebands at both sides of the optical carrier with frequency space f_1 and relative amplitude $J_n(m_1)$. The modulation depth of phase modulator is related to its half-wave voltage by

$$m_i = \frac{\pi V_i}{V_{\pi i}}, \quad (i = 1, 2) \tag{2}$$

where V_i is the electrical amplitude of the driving microwave signal and $V_{\pi i}$ is the half-wave voltage of the phase modulator. The same optical carrier at f_0 is sent to PM2 after frequency shifting and modulated by another sinusoidal microwave signal $v_2(t) = V_2 \sin 2\pi f_2 t$. The optical field after PM2 can be expressed by

$$E_2 = A_2 \exp j[2\pi(f_0 - f_s)t + m_2 \sin 2\pi f_2 t] \tag{3}$$

where A_2 is the amplitude of the frequency shifted optical carrier, f_s is the detuned frequency resulted by the frequency shifting and m_2 is the modulation depth of PM2. The modulation depth m_2 of PM2 is defined analogously to Eq. (2). The output optical signals from PM1 and PM2 are combined at the end of the interferometer, and the combined optical field can be expressed by

$$E_3 = E_1 + \gamma \exp(j\phi) E_2 \tag{4}$$

where γ represents the relative amplitude and ϕ represents the phase difference between the two interferometer arms. The combined optical signal is sent to a PD for photo-detection to generate an instantaneous photocurrent given by

$$\begin{aligned} i(t) &= \Re E_3 E_3^* = \Re |E_1 + \gamma \exp(j\phi) E_2|^2 \\ &= \Re\left[A_1^2 + \gamma^2 A_2^2 + 2\gamma A_1 A_2 \cos(m_1 \sin 2\pi f_1 t - m_2 \sin 2\pi f_2 t + 2\pi f_s t + \phi)\right] \end{aligned} \tag{5}$$

with $\Re$ the responsivity of the PD. With the help of the Jacobi-Anger expansion [19], (5) can be reorganized as

$$i(t) = \Re\left\{A_1^2 + \gamma^2 A_2^2 + 2\gamma A_1 A_2 \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} J_p(m_1) J_q(m_2) \cos\left[2\pi(pf_1 + qf_2 + f_s)t + \phi\right]\right\}. \tag{6}$$

From (6), it is easy to quantify the amplitude of the following frequency components:

$$i(f_1 - f_2 \pm f_s) = 2\gamma A_1 A_2 J_1(m_1) J_1(m_2) \cdot \Re(f_1 - f_2 \pm f_s) \tag{7a}$$

$$i(f_2 \pm f_s) = 2\gamma A_1 A_2 J_0(m_1) J_1(m_2) \cdot \Re(f_2 \pm f_s). \tag{7b}$$

To obtain a calibration-free measurement, the microwave signal frequency of PM1 is set close to twice of that of PM2, i.e. $f_1 \approx 2f_2$, so that $\Re(f_1 - f_2 \pm f_s) \approx \Re(f_2 \pm f_s)$ is satisfied. In this case, we define the heterodyning ratio as the relative amplitude of the frequency component at $f_2 \pm f_s$ with respect to that at $f_1 - f_2 \pm f_s$, given by

$$H(m_1) = \frac{i(f_1 - f_2 \pm f_s)}{i(f_2 \pm f_s)} = \frac{J_1(m_1)}{J_0(m_1)}. \tag{8}$$

In (8), the heterodyning ratio in the electrical spectrum corresponds to the amplitude ratio of the first-order sideband with respect to the optical carrier of PM1 in the optical spectrum, that is, our method provides an equivalent electrical measurement of the phase-modulated sideband and at the same time eliminates the responsivity of the photo-detection by carefully choosing the frequency relationship of the driving microwave signals. One can also see from (8) that the desired heterodyning signals are close to the frequency of f_2 ($\approx f_1/2$), indicating that our method reduces half bandwidth requirement and extends double frequency range for the PD and ESA. Besides, our measurement is insensitive to the power unbalance and phase difference of the interferometer, since the heterodyning ratio is independent on the relative amplitude γ and phase difference ϕ. It is worthy noticing that the Mach–Zehnder interferometer is operated at heterodyning mode instead of interference mode and it does not introduce any nonlinear transfer function, which is totally different from the traditional interferometer method. Our method can also be easily extended to measuring PM2 with the same setup simply by swapping the position of PM1 and PM2.

Compared to our previous method, the improved method includes the following changes: 1) The former two-tone modulation of one phase modulator is changed to the separate modulation of two modulators. 2) The closed two-tone modulation signal with very closed frequencies ($f_1 \approx f_2$) is replaced by two microwave signals with about half frequency relationship ($f_1 \approx 2^* f_2$). 3) The measurement function is changed to be $J_1(\mathrm{m})/J_0(\mathrm{m})$ instead of the former $J_1^2(\mathrm{m})/J_0^2(\mathrm{m})$. 4) The power unbalance and phase difference of the Mach–Zehnder interferometer are taken into account in this work while they were not in our former work.

3. Experimental Demonstration

We experimentally characterize a commercial phase modulator based on the setup shown in Fig. 1. A narrow line-width DFB LD with a central wavelength of 1550.12 nm is used as the optical carrier, which is divided into two arms of the Mach–Zehnder interferometer by an optical coupler. The optical carrier in the upper arm is sent to the phase modulator under test (PM1, COVEGA 10027) via a polarization controller (PC1), and the optical carrier in the lower arm is first frequency-shifted by 70 MHz with an acoustooptic modulator (CETC F-YSG70) and then sent to the second phase modulator (PM2, COVEGA 10053) via a second polarization controller (PC2). PM1 is driven by a microwave source up to 40 GHz (MW1, R&S SMB100A), while PM2 is driven by another microwave source up to 20 GHz (MW2, HP86320A). The signal frequency of MW1 is set to be twice and less 20 MHz of that of MW2, i.e., $f_1 = 2f_2 - 0.02$ (GHz). The optical signals after PM1 and PM2 are combined and detected by a high-speed PD (HP 11982A)

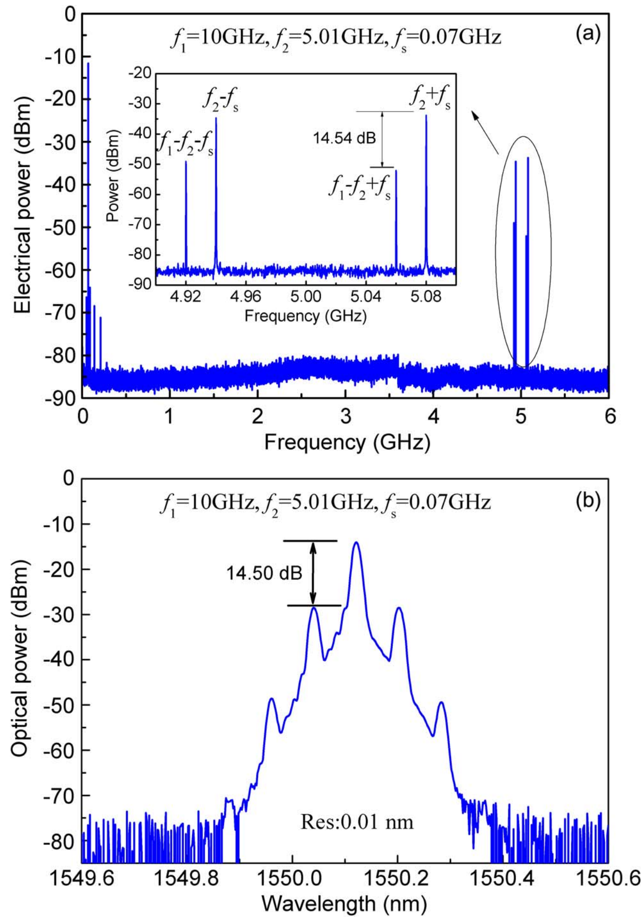

Fig. 2. (a) Measured electrical spectrum of the heterodyning signal and (b) the corresponding optical spectrum of PM1 in the cases of $f_1 = 10$ GHz, $f_2 = 5.01$ GHz, and $f_s = 70$ MHz.

and an electrical spectrum analyzer (ESA, R&S FSU50) for measuring the heterodyning ratio. The output of PM1 is partially monitored with an OSA (YOKOGAWA AQ6370C) for the optical spectrum measurement. In the experiment, all the optical components are pigtailed with angle polished (APC) finishes in order to minimize the residual reflection as much as possible. The residual reflection can be further reduced with optical isolators if necessary.

Fig. 2(a) shows the electrical spectrum of the heterodyning signal, of which the microwave signal frequencies are set as $f_1 = 10$ GHz, $f_2 = 5.01$ GHz, and $f_s = 70$ MHz, respectively. The electrical spectrum holds extremely narrow spectrum lines due to the inherent coherence of the optical heterodyning signals originating from the same optical carrier. From the electrical spectrum, the frequency components of 5.06 GHz $(f_1 - f_2 + f_s)$ and 5.08 GHz $(f_2 + f_s)$, are measured to be -49.12 dBm and -34.58 dBm, respectively, from which the heterodyning ratio is determined to be -14.54 dB. Therefore, the modulation depth of PM1 at 10 GHz is solved to be 0.3686 (rad) based on (8). Fig. 2(b) illustrates the optical spectrum of the phase modulated signal after PM1under the same driving condition, where the ratio of -14.50 dB of the first-order sideband with respect to the optical carrier corresponds to a modulation depth of 0.3702 (rad).

Fig. 3(a) and (b) show the measured heterodyning ratios and modulation depths at different modulation frequencies, respectively, where the results obtained with the optical spectrum analysis method are also included for comparison. The minimum starting frequency of about 4 GHz in the optical spectrum measurement is limited by the 0.01 nm resolution of the OSA [8], [9], [12], [15], since the best resolving wavelength of 0.01 nm can be achieved only when the two closed wavelengths are with the same intensity according to the Rayleigh Criterion. However, in

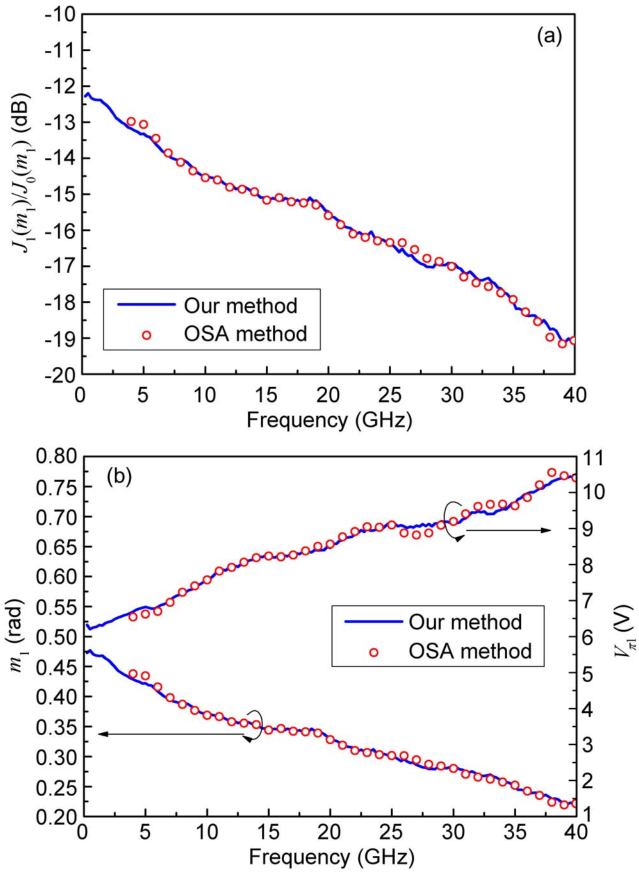

Fig. 3. Measured (a) heterodyning ratios and (b) modulation efficiency as a function of modulation frequency, where the open circles represent the results measured with OSA method.

our case, the optical carrier and the first-order sideband have a relative amplitude of $J_0(\mathrm{m})$ and $J_1(\mathrm{m})$, respectively, and the best resolution cannot be guaranteed, which explains why the optical spectrum measurement cannot start from $\sim$1.25 GHz (0.01 nm). Nevertheless, the good agreement between the measurable results indicates the equivalence between the electrical spectrum and the optical spectrum measurement, and our method does eliminate the frequency responses of the PD, since the optical spectrum measurement does not include any photodetection.

It should be noted that the PD (HP11982) only has a bandwidth of 15 GHz, and it is still qualified for the measurement up to 40 GHz because the desired heterodyning signals are about half frequency of the microwave signal applied on the phase modulator under test. Besides, the microwave signal of PM1 in our measurement is swept from 0.25 GHz up to 40 GHz with a frequency step of 0.25 GHz, which can be further reduced as required for higher resolution measurement with an ESA.

For the accuracy, we investigate the error dependence of modulation depth on the uncertainty of heterodyning ratio by the error transfer factor

$$F = \frac{\delta m_1/m_1}{\delta H/H} = \frac{1}{m_1\left[\frac{J_1(m_1)}{J_0(m_1)} - \frac{J_0(m_1)-J_2(m_1)}{2J_1(m_1)}\right]} \tag{9}$$

which is deduced from the total derivative of (8). For $F < 1$, the uncertainty of the modulation depth is less than that of the heterodyning ratio, whereas for $F > 1$, the uncertainty of the modulation depth is larger than that of the heterodyning ratio. As is shown in Fig. 4, the error transfer

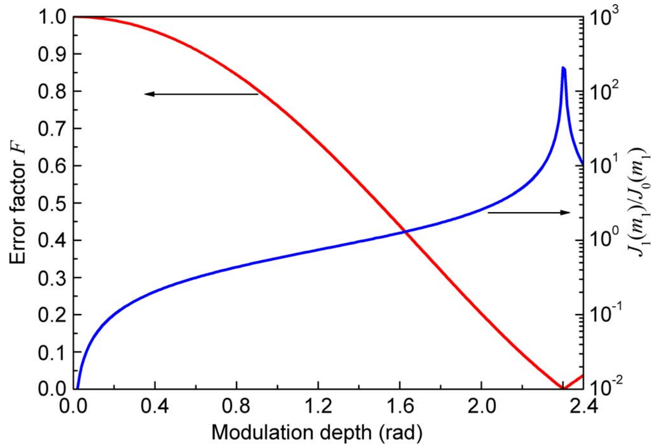

Fig. 4. Calculated heterodyning ratio and its error transfer factor as a function of the modulation depth.

factor is less than 1 within the whole range of modulation depth, indicating that the uncertainty of the extracted modulation depth will be less than that of the measured heterodyning ratio. In our experiment, the main error sources come from the response fluctuation of PD and the uncertainty of ESA, since our method is based on the approximation of $\Re(f_1 - f_2 + f_s) \approx \Re(f_2 + f_s)$. According to the PD's specification, the response difference is less than 0.2 dB within the frequency difference of 20 MHz. According to the specification of ESA, the measurement uncertainty is within 0. 1 dB. Therefore, a relative error of less than 0.3 dB will be transferred to the measured heterodyning ratio in the worst case, which means an uncertainty of less than 3.51% $(= (10.^\wedge(0.3/20) - 1)^* 100\%)$ delivered to the extracted modulation depths.

Furthermore, the half-wave voltage of PM1 is measured as a function of modulation frequency up to 40 GHz with the help of (2). The measured half-wave voltages with our method and with the optical spectrum measurement are also illustrated in Fig. 3(b) for comparison. The error dependence of the half-wave voltage on the uncertainty of the electrical amplitude and the modulation depth can be similarly written as

$$\frac{\delta V_{\pi 1}}{V_{\pi 1}} = -\frac{\delta m_1}{m_1} + \frac{\delta V_1}{V_1}. \tag{10}$$

In our measurement, the measured electrical amplitude has an uncertainty of less than 0.05 dB $(1.16\% = (10.^\wedge(0.05/10) - 1)^* 100\%)$ according to the specification of ESA. Therefore, the total relative error of less than 4.67% is transferred to the measured half-wave voltage in the worst case.

4. Discussion and Conclusion

During the experiment, our measurements show stable and repeatable results because both phase modulated signals originate from the same source, leading to extremely narrow heterodyning spectra lines. Compared to the OSA method, the proposed electrical spectrum analysis method achieves very high frequency resolution measurement, and at the same time avoids the line-width influence of laser source due to the nature of self-referenced heterodyning. Different from the traditional electrical domain methods, our method realizes a calibration-free measurement for phase modulators through setting a specific frequency relationship between the two driving microwave signals. In contrast to our previous work, the improved method extends double frequency range for the PD and ESA, because the desired heterodyning signal is about half frequency of the driving microwave signal of the phase modulator under test. Our measurement is robust to path impairments, thermal fluctuations and mechanical vibrations, since the measurement function is insensitive to the power and phase difference between the two interferometer arms. Besides, our scheme is established without any small-signal assumption, and it is applicable for different driving levels and operating wavelengths as long as the required frequency relationship is satisfied.

In summary, we have proposed and demonstrated a calibration-free measurement for microwave characterization of phase modulators based on frequency-shifted optical heterodyning. The modulation efficiency at microwave frequencies was evaluated by measuring the heterodyning spectrum between the two phase-modulated signals with and without frequency-shifting. Our method realizes a calibration-free and accurate measurement for optical phase modulators with the high resolution electrical spectrum analysis, eliminating the need for correcting the responsivity fluctuation in the photo-detection.

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