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The Resistor–Capacitor Circuit

Welcome to the 27th article in the "Circuit Intuitions" column series. As the title suggests, each article provides insights and intuitions into circuit design and analysis. These articles are aimed at undergraduate students but may serve the interests of other readers. If you read this article, I would appreciate your comments and feedback as well as your requests and suggestions for future installments in this series. Please email me your comments at ali@ece.utoronto.ca.

Resistors and capacitors are perhaps the most common elements in all electrical circuits. Even if they are not explicitly shown on circuit schematics, they are present in the physical layout, for example, in the form of the unwanted (parasitic) resistance and capacitance of the wiring. In addition, resistors and capacitors appear in the models of most semiconductor devices, such as the output resistance of transistors and the parasitic capacitances of the p–n junctions of metal–oxide semiconductor transistors. Resistor–capacitor (RC) circuits are so fundamental to electrical engineering that their analysis is often taught during the first year of most undergraduate programs around the world. Past articles in this series [1]–[3] have discussed capacitor analogies. This article reviews the fundamental properties of RC circuits and examines some common assertions about their properties.

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Figure 1(a) presents a series RC circuit as a two-port network when the input voltage $v_i(t)$ is applied to the input port and the output voltage $v_o(t)$ is taken from the output port. The relationship between $v_i(t)$ and $v_o(t)$ can be written as

$$v_o(t+\Delta t) \approx v_o(t) + \frac{v_i(t) - v_o(t)}{RC}\Delta t.$$

This equation simply states that the output voltage increases during a short time interval Δt (defined as $\Delta t \ll RC$) by the charge that is accumulated on the capacitor through this period divided by C. It is common to write this equation as a first-order differential equation:

$$RC\frac{dv_o(t)}{dt} + v_o(t) = v_i(t).$$

This equation states that the input voltage (the right-hand side) is equal to the voltage across the resistor (the first term) plus the voltage across the capacitor (the second term). In the frequency domain, the series RC circuit simply divides the input signal between the impedances of R and C, that is, between R and 1/sC, where $s = j\omega$ for real frequencies. The ratio of the output voltage to the input voltage as a function of the frequency is known as the *transfer function* and can be written as

$$\frac{V_o}{V_i}(s) = \frac{1}{1+sRC} = \frac{1}{1+s/\omega_p},$$

where ω_p, known as the 3-dB frequency of the RC circuit, is the angular frequency at which the impedance magnitude of the capacitor $(1/\omega_p C)$ is equal to R. Although the equations in the time and frequency domains look different, both represent the same behavior, as shown by the block diagram in Figure 1(b). This diagram illustrates how the voltage difference between the input and the output is first multiplied by 1/R to produce a current and how this current is integrated by the capacitor (1/sC) to produce the output voltage.

Figure 2(a) and (b) gives the magnitude (in decibels) and the phase (in degrees) plots of the transfer function of the RC circuit as functions of ω. The magnitude and phase plots

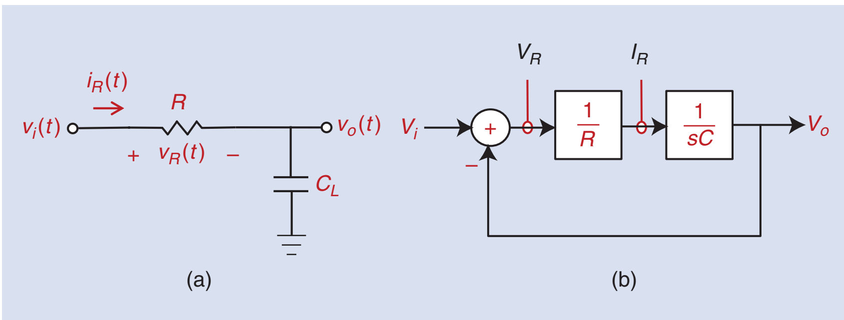

FIGURE 1: (a) A series RC circuit integrates the current that flows through the resistor to produce an output voltage. (b) A block diagram representing the operation principle of an RC circuit.

signify the filtering properties of the RC circuit, that is, how the RC circuit discriminates among input signals of different frequencies or how it filters the input signal according to its frequency content. To see the filtering behavior of the RC circuit, we distinguish three frequency regions in this plot: 1) the low-frequency region, where $\omega \ll \omega_p$; 2) the high-frequency region, where $\omega \gg \omega_p$; and 3) frequencies around ω_p. In each region, the circuit exhibits a different behavior. At low frequencies, by our definition, $R \ll 1/\omega C$, and hence all the input voltage appears across the capacitor. In other words, the circuit voltage gain is one. Also, in this region, the phase difference between the output and the input is close to zero, as detailed in Figure 2(b). In the time domain, one can argue that, since the input voltage changes slowly (i.e., it does not change much in one time-constant RC), the capacitor voltage is able to track the input voltage and not fall behind. That is, $v_o(t) \cong v_i(t)$, and the current flowing through the resistor and the capacitor $i_R(t) = i_C(t) = (v_i(t) - v_o(t)/R$ is close to zero.

At high frequencies, where $\omega \gg \omega_p$, we can simplify the transfer function equation to $\omega_p/j\omega$. The reader may recognize this as the transfer function of an integrator, where the gain magnitude is inversely proportional to ω and the phase shift from the input to the output is 90°. Simply put, in region 2, the transfer function of the RC circuit is close to that of an integrator. Let us now see this in the time domain. At high frequencies, the input voltage changes so quickly that the capacitor cannot keep up and hence maintains a voltage close to zero. As a result, we can assume all the input voltage appears across the resistor, creating a current proportional to the input voltage. This current is then fed directly to the capacitor, where it is integrated, producing a voltage proportional to the integral of the input voltage, with a scaling factor of $1/RC$. Note that there is no contradiction in the output voltage being close to zero and its value (albeit close to zero) being proportional to the integral of the input voltage. For example, an input signal $V_i \sin(\omega t)$ with $\omega \gg \omega_p$ results in an output voltage that is $(\omega_p/\omega) V_i \sin(\omega t - \pi/2)$, which is both close to zero (because $\omega_p/\omega \ll 1$) and proportional to the integral of the input voltage.

So far, we have said the RC circuit tracks the input voltage in region 1 and integrates it in region 2. How about in region 3? It turns out that in this region, the circuit neither tracks nor integrates the input but, rather, does something in between. At ω_p, for example, the circuit attenuates the signal amplitude by $1/\sqrt{2}$ (which is less than the value of one that is required for tracking) and shifts its phase by 45° (which is less than the 90° required by an integrator). One can also say that, in this region, the circuit partially tracks and integrates the input.

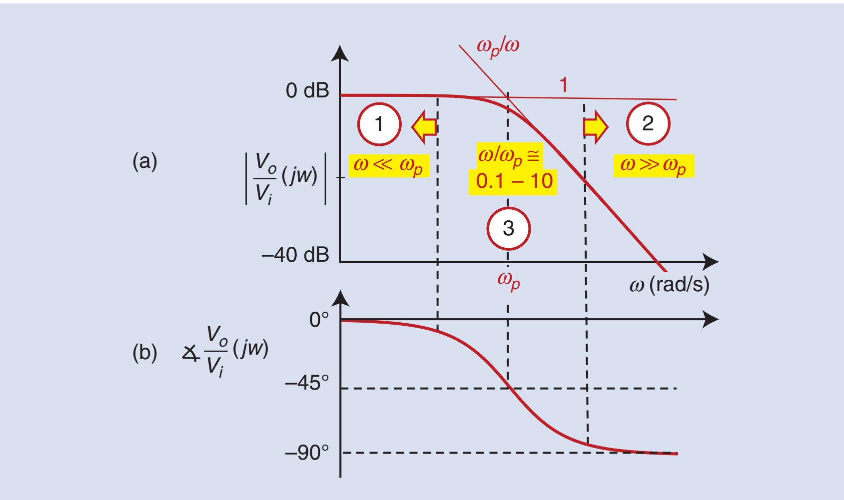

FIGURE 2: The (a) magnitude and (b) phase plots of the transfer function of a series RC circuit. The output tracks the input signal for frequencies far below ω_p but integrates the input signal for frequencies far above ω_p. For frequencies between the two extremes, the output neither tracks nor integrates the input signal.

With this basic understanding, let us now examine the following three statements about RC circuits and determine if each is true or false:

1) The RC circuit produces the time average of its input signal at the output.
2) The RC circuit takes the running average (also called the *moving window average*) of the input signal, where the time window is related to the RC time constant.
3) The RC circuit is continuously interpolating between the input and output voltages as time progresses.

Readers are encouraged to reflect on these assertions before proceeding further.

Statement 1 is correct only for dc inputs. As discussed earlier, a dc input will appear intact at the output, and, since the average of a dc voltage is equal to the dc voltage itself, this statement is true for dc inputs. However, any input signal that is changing with time will result in an output signal that is also changing with time, making statement 1 invalid, as the average of a signal must be constant by definition. Statement 2 is also correct only in special cases. It is certainly correct if the input is a pure dc voltage, as argued in relation to statement 1. In addition, this statement is correct for low-frequency signals, i.e., when $\omega \ll \omega_p$, because, similar to dc, the running average of a low-frequency signal is identical to the signal itself. And, since the output is equal to the input at low frequencies, this statement is true.

How about a signal with content only at very low and very high frequencies (i.e., with content in regions 1 and 2 in Figure 2 but not in region 3)?

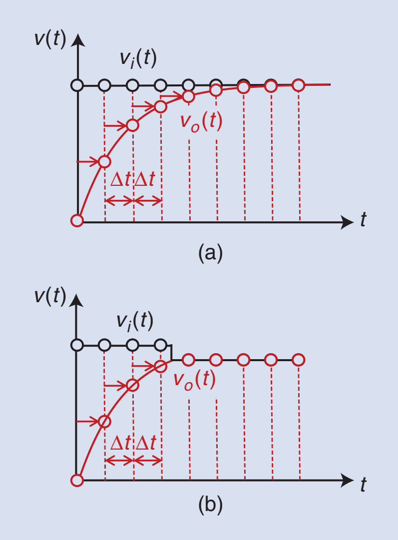

FIGURE 3: (a) The series RC circuit continuously interpolates between the input and output signals at time *t* to produce an output signal at time $t + \Delta t$. (b) The output voltage freezes when the input voltage steps down to meet it.

Interestingly, the low-frequency content simply appears at the output (without attenuation), while the high-frequency signals will be heavily attenuated. If the low-frequency content and high-frequency content have similar amplitudes, one will remain intact while the other is reduced to near zero. Therefore, the output appears to be the moving average of the input. In contrast, if the high-frequency content is much larger in magnitude than the low-frequency content amplitude, or if there is frequency content in region 3 of Figure 2, the output may no longer resemble the running average of the input.

Unlike statements 1 and 2, statement 3 is always correct. Let us see why. We can rewrite the very first equation in this article as

$$v_o(t+\Delta t) \approx \frac{\Delta t}{RC} v_i(t) + \left(1 - \frac{\Delta t}{RC}\right) v_o(t).$$

This equation tells us that the output voltage at any time $t+\Delta t$ is a weighted average or interpolation of the input voltage and the output voltage at time t. Since Δt is very small ($\Delta t \ll RC$), the input voltage has a much smaller weight compared to the output voltage. As a result, the output voltage cannot change quickly. In fact, if the input voltage is persistent (such as a unit step function), then it takes about five time constants (5RC) for the output voltage to approach 99% of the input voltage. This is demonstrated in Figure 3(a). If the input voltage changes quickly to a value far from the output voltage at that time, the capacitor maintains its output and changes only gradually to accommodate and track the change in the input. On the other hand, as in Figure 3(b), if the input voltage is suddenly brought to the level of the output voltage, then there will be no current flowing from the input to the output, prohibiting the output voltage from moving.

In summary, the series RC circuit has a low-pass filtering behavior, where it enables input signals at low frequencies ($\omega \ll \omega_p$) to pass to the output while it attenuates and integrates the input signals at high frequencies ($\omega \gg \omega_p$). In the time domain, the RC circuit continuously interpolates between the input and output voltage to produce a new output voltage. The larger the time constant RC is, the smaller the weight of the input is in this interpolation, taking longer for the output to change.

References

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[2] A. Sheikholeslami, "Circuit intuitions: A capacitor analogy, part 2," *IEEE Solid-State Circuits Mag.*, vol. 8, no. 4, pp. 8–9, Fall 2016. doi: 10.1109/MSSC.2016.2603221.

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ERRATA

The Fall 2020 issue of *IEEE Solid-State Circuits Magazine* included incorrect biographical information for two authors.

In [1], the final bullet entry in the third column should read as follows:

- Maneesha Yellepeddi, analog engineering manager in the Programmable Solutions Group at Intel, is an alumnus of the 2020 Rising Stars workshop.

In [2], the correct email address for Wing-Kong Ng should be billyng046@gmail.com. We apologize for any confusion.

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[2] W.-K. Ng, W.-T. Tam, A. H.-C. Ko, W.-S. Tam, and C.-W. Kok, "A review of wafer packing and new results," *IEEE Solid-State Circuits Mag.*, vol. 12, no. 4, Fall 2020, pp. 101—108. doi: 10.1109/MSSC.2020.3021845.