# Radiation Pressure Field Reconstruction for Ultrasound Midair Haptics by Greedy Algorithm With Brute-Force Search

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***Abstract*—Non-contact tactile presentation using ultrasound phased arrays is becoming a powerful method for providing haptic feedback on bare skin without restricting the user's movement. In such ultrasonic mid-air haptics, it is often necessary to generate multiple ultrasonic foci simultaneously, which requires solving the inverse problem of amplitudes and phases of the transducers in a phased array. Conventionally, matrix calculation methods have been used to solve this inverse problem. However, a matrix calculation requires a non-negligible amount of time when the number of control points and the number of transducers in the array are large. In this article, we propose a simple method based on a greedy algorithm and brute-force search to solve the field reconstruction problem. The proposed method directly optimizes the desired field without matrix calculation or target field phase optimization. The empirical results indicate that the proposed method can reproduce the target sound with an accuracy of more than 80%.**

***Index Terms*—mid-air haptics, phased array, sound field reconstruction.**

## I. Introduction

Airborne ultrasound phased arrays [1]–[3] have been a typical non-contact stimulator in haptics [4]. Acoustic nonlinear effects can provide haptic feedback in non-contact user interfaces as well as AR/VR applications and improve usability hygienically as there is no physical contact. For scientific studies on haptics, this non-contact method can create stable forces on the skin with good reproducibility and the spatial and temporal pattern can be freely designed. In such ultrasonic mid-air haptics, it is often necessary to solve the inverse problem to obtain transducers' inputs from the desired radiation pressure pattern, typically with multiple foci. For example, Long *et al.* proposed a system that presents volumetric tactile sensations to the palm in a non-contact manner by generating multiple foci [5]. Matsubayashi *et al.* also proposed a midair haptic system where a user can pinch and handle a virtual object [6], [7]. In these systems, it is needed to calculate the optimal phase and amplitude of each transducer in the array, desirably in real time.

In the aforementioned applications, the nonlinear term is crucial, but it is quit smaller than the linear term. Therefore, we assume that the acoustic pressure is linearly related to the amplitude and frequency of the vibration at transducer, which yields a satisfactory approximation. In particular, in a narrowband and single-frequency case, the complex acoustic amplitude $\boldsymbol{p}$ sampled at discrete points in the space is written as $\boldsymbol{p} = G\boldsymbol{q}$, where $\boldsymbol{q}$ represents the amplitudes and phases of the transducers in the phased array.

A simple approach is to obtain $\boldsymbol{q} = G^{-1}\boldsymbol{p}$, where $G^{-1}$ is the inverse or pseudo inverse matrix of $G$. However, this approach often gives a useless solution for the aforementioned applications. In haptic applications, to produce strong stimulation, it is desired to collect the power of many transducers close to the maximum of each transducer. Therefore, a solution where only a limited number of transducers work with unrealistic power while most transducers are resting is unsuitable. This implies that the constraint on the amplitude of each transducer is practically crucial in the inverse problem.

In this paper, we propose a method for finding the solution under this constraint. Our proposed method is based on a simple greedy algorithm, which is easy to implement. We compare the proposed method with other methods and show experimentally that the proposed method reproduces the sound field with an accuracy of more than 80% and has a short computation time, especially for a sound field with many foci. Such performance is sufficient for applications, for example, providing tactile feedback to an aerial touch panel. On the other hand, the proposed method is not suitable for applications that require high accuracy, for example, to reproduce the tactile sensation of real objects.

The problem discussed in this paper is the reproduction of a desired radiation pressure distribution on a two-dimensional surface by a phased array. A more complicated problem is examined in applications other than haptics, for example, in the levitation applications [8]. By designing the spatial derivative of the acoustic field as well as the pressure pattern on the object surface, levitation of an object larger than the

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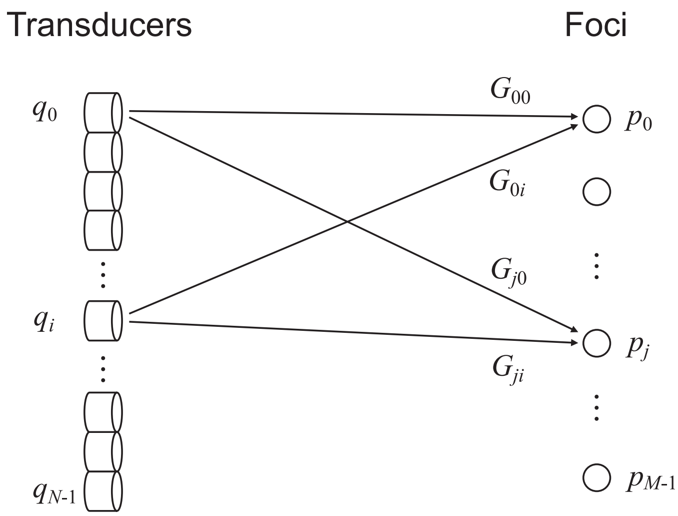



Fig. 1. Schematic diagram of the the system in this paper.

wavelength has been achieved [9]. Although such a problem is interesting, we have excluded it in this study.

In the following section, we formulate the problem and clarify its novelty and relationship with previous studies.

## II. Problem Formulation and Related Works

Principally, the crucial design freedom for finding a good solution under such a hard amplitude constraint is the arbitrariness of the phase of the target acoustic field. The previous effective methods [5], [10], [11] used this arbitrariness with or without emphasizing it.

The following is the formulation of the problem discussed in this paper. For simplicity of calculation, we model the acoustic pressure as a complex value $a\mathrm{e}^{-\mathrm{j}(\omega t-\phi)}$, where $a$ and $\phi$ are the amplitude and phase, $\omega$ is angular frequency, and $t$ is a time. The actually observed sound pressure is the real part of this, and its absolute value is proportional to the effective sound pressure. Because we use only a single frequency, we ignore the ultrasonic vibration term $\mathrm{e}^{-\mathrm{j}\omega t}$ in the following. Let $\boldsymbol{p}$ be the complex amplitude of the acoustic pressure sampled at discrete points, and $\boldsymbol{q}$ be the vibration velocities of the transducer in the phased array. They can be combined as

$$\boldsymbol{p} = G\boldsymbol{q}, \tag{1}$$

$$\boldsymbol{p} = [p_0, \ldots, p_{M-1}]^{\mathsf{T}} \in \mathbb{C}^M, \tag{2}$$

$$\boldsymbol{q} = [q_0, \ldots, q_{N-1}]^{\mathsf{T}} \in \mathbb{C}^N, q_i = a_i \mathrm{e}^{\mathrm{j}\phi_i}, \tag{3}$$

$$G \in \mathbb{C}^{M\times N}, \tag{4}$$

where $a_i$ and $\phi_i$ are the amplitude and phase of the $i$-th transducer, respectively, $M$ is a number of control points, $N$ is a number of transducers, and $G$ is the propagation matrix. Fig. 1 shows a schematic picture of this model.

To simplify the problem, we assume that each transducer of the phased array is a point source in which the surface velocity is controlled electrically. If we represent the surface velocity with the equivalent acoustic pressure to the velocity and there are no obstacles between the target surface and the phased array, the element of $G$ can be expressed as

$$G_{ji} = \frac{1}{\|\boldsymbol{r}_j - \boldsymbol{r}_i\|}\mathrm{e}^{\mathrm{j}k\|\boldsymbol{r}_j-\boldsymbol{r}_i\|}, \tag{5}$$

where $\boldsymbol{r}_j$ and $\boldsymbol{r}_i$ are the positions of the $j$-th control point and $i$-th transducer, respectively, and $k$ is a wavenumber.

The problem here is an inverse problem of estimating $\boldsymbol{q}$ from a given $\boldsymbol{p}$, minimizing the sum of the squares of the residuals, that is,

$$\min\|\boldsymbol{p} - G\boldsymbol{q}\|^2. \tag{6}$$

We only need the surface radiation pattern; thus, the phase of the acoustic pressure can be arbitrary. In haptic applications, the amplitudes of the control points are necessary, whereas the phase differences among the control points are not of interest because humans cannot perceive ultrasonic vibrations. Therefore, the problem we should actually solve is to obtain $\boldsymbol{q}$ as

$$\min \sum_{j}^{M-1} \Big| |p_j| - |(G\boldsymbol{q})_j| \Big|^2. \tag{7}$$

It should be noted that the radiation pressure depends on the vibration direction as well as the pressure amplitude; however, we neglect the term of the vibration direction [9].

In practical applications, the number of foci $M$ is smaller than the number of transducers $N$. That is, the inverse problem is underdetermined, and there may be multiple solutions. Therefore, it is necessary to choose one solution in some way.

Methods that optimize the phases of $\boldsymbol{p}$, thereby utilizing the phase arbitrariness, have been proposed to solve this problem. Long *et al.* proposed a method that optimized the phase of $\boldsymbol{p}$ through eigenvalue decomposition (EVD) and then solved eq. (6) using Tikhonov regularization [5]. Inoue *et al.* also introduced a method that relaxed the phase optimization problem to semi-definite programming (SDP) and used the block coordinate descent (BCD) [10]. In the following, these two methods are referred to as 'EVD' and 'SDP+BCD,' respectively. In these two methods, the solution with small transducer output is selected by Tikhonov regularization. Another solution for eq. (6) is to take the amplitudes of $\boldsymbol{p}$ and $\boldsymbol{q}$ as input parameters simultaneously. The Levenberg-Marquardt (LM) method is known to be an efficient algorithm for solving the nonlinear least-squares problem [12], [13]. There are some example of multi foci tactile presentation using the LM method [14], [15]. The solution in the LM method converges to a single solution close to the initial value by iteration.

A similar problem has been studied in optics, and various solutions are known. A well-known algorithm is an iterative method that repeats the propagation and back-propagation while applying the amplitude constraint to the optical element and the observed wavefront. This algorithm is called the Gerchberg-Saxton (GS) method [16] or the Iterative Fourier transform algorithm [17], [18]. Marzo *et al.* proposed a method to generate multiple foci with an ultrasound phased array based on this GS method [19]. Later, Plasencia *et al.*

**Algorithm 1.** Greedy Pressure Field Reconstruction

**Input:** $|p_0|, \ldots, |p_{M-1}|, p_0 = \cdots = p_{N-1} = 0$
**Output:** $\boldsymbol{q}$
**for** $i = 0, \ldots, N-1$ **do**
find $q_i$ s.t.

$$\min E(q_0, \ldots, q_i) = \min \sum_{j}^{M-1} \left| |p_j| - |\mathcal{P}(\boldsymbol{r}_j; q_0, \ldots, q_i)| \right|^2$$

**end for**

proposed a faster algorithm, GS-PAT, by omitting some of the computations in this method [11]. GS-PAT optimizes the target sound field phase by alternately and repeatedly projecting between the sound field space that can be represented as the output of the phased array and the target sound field space. GS-PAT is expected to select a single target sound field $\boldsymbol{p}$ close to the initial value by iteration. And then, a single driving vector $\boldsymbol{q}$ can be obtained by back-propagation.

By contrast, we propose a scheme that minimizes eq. (7) directly through a greedy algorithm. Furthermore, we discretize the amplitude and phase of the transducers with low resolution and apply the brute-force search method to find the solution. The computational time of the proposed method is linear with respect to number of control points $M$ and number of transducers $N$ and is predictable.

## III. Proposed Method

Our greedy algorithm is expressed as follows:
where $\mathcal{P}(\boldsymbol{r}_j; q_0, \ldots, q_i)$ is sound pressure at $\boldsymbol{r_j}$ generated by the $0, 1, \ldots, i$-th transducers.

First, we use only one transducer in the algorithm. Then, we drive this transducer and find the $q_0$ that minimizes objective function $E(q_0)$. Subsequently, we keep the $q_0$ fixed, add a new transducer and find the optimal $q_1$ for $E(q_0, q_1)$. We repeat this process for all the transducers.

In this study, to perform Algorithm 1, we applied an exhaustive search (brute-force search) for the digitized amplitudes and phases of the added transducer while calculating $E$ and determined the optimum amplitude and phase. Here the algorithm is expressed as follows:
It should be noted that the solution of the method satisfies the maximum amplitude constraint because we search the optimum parameters within the limit of each transducer.

The time required for Algorithm 2 is obviously $O(NKLR)$, where $R$ is the time taken to observe $E_i^{kl}$. In the case of a field where the superposition principle works, e.g., a sound field, the calculation of $E_i^{kl}$ can be performed in $O(M)$. Here, $\mathcal{P}$ is expressed as

$$\mathcal{P}(\boldsymbol{r}_j; q_0, \ldots, q_{n-1}) = \sum_{i=0}^{n-1} \frac{1}{\|\boldsymbol{r}_j - \boldsymbol{r}_i\|} \mathrm{e}^{\mathrm{j}k\|\boldsymbol{r}-\boldsymbol{r}_i\|} q_i, \tag{8}$$

**Algorithm 2.** Greedy Pressure Field Reconstruction with Brute-force Search

**Input:**$|p|, \ldots, |p_{M-1}|, p_0 = \cdots = p_{N-1} = 0$, amplitude limit $A$
**Output:**$\boldsymbol{q} = \{q_i \mid |q_i| \le A\}$
discretize $a_i$ as $\{a_i^k \mid |a_i^k| \le A, k = 0, 1, \ldots, K-1\}$
discretize $\phi_i$ as $\{\phi_i^l \mid \phi_i^l \in [0, 2\pi), l = 0, 1, \ldots, L-1\}$
**for** $i = 0, \ldots, N-1$ **do**
**for** $k = 0, \ldots, K-1$ **do**
**for** $l = 0, \ldots, L-1$ **do**
obtain $E_i^{kl} = E(q_0, \ldots, q_{i-1}, a_i^k \mathrm{e}^{\mathrm{j}\phi_i^l})$
**end for**
**end for**
$q_i \leftarrow a_i^{k^*} \mathrm{e}^{\mathrm{j}\phi_i^{l^*}}$ s.t. $E_i^{k^* l^*} = \min\{E_i^{kl}\}$
**end for**

that means we can rewrite $\mathcal{P}(\boldsymbol{r}; q_0, \ldots, q_{n-1})$ as follows:

$$\begin{aligned} &\mathcal{P}(\boldsymbol{r}; q_0, \ldots, q_{n-1}) \\ &\quad = \mathcal{P}(\boldsymbol{r}; q_0, \ldots, q_{n-2}) + \frac{1}{\|\boldsymbol{r} - \boldsymbol{r}_{n-1}\|} \mathrm{e}^{\mathrm{j}k\|\boldsymbol{r}-\boldsymbol{r}_{n-1}\|} q_{n-1}. \end{aligned} \tag{9}$$

Therefore, by caching $\mathcal{P}$, we can calculate $E_i^{kl}$ in $O(M)$. Therefore, Algorithm 2 can be performed in $O(KLMN)$. For simplicity, we use simple spherical waves in this study but note that the essence of the above discussion does not change even if we consider other factors, such as directivity and attenuation terms.

Although the advantage of linear computation time, the proposed method does not have any mathematical guarantees. Due to determining the phase and amplitude one by one, a single solution can be obtained. However, the solution depends on the order of selecting the transducers to determine the phase and amplitude, and there is no good strategy for how to determine this order. This sometimes lead to problematic solutions in certain situations. We discuss this limitation in Section V.

## IV. Evaluation

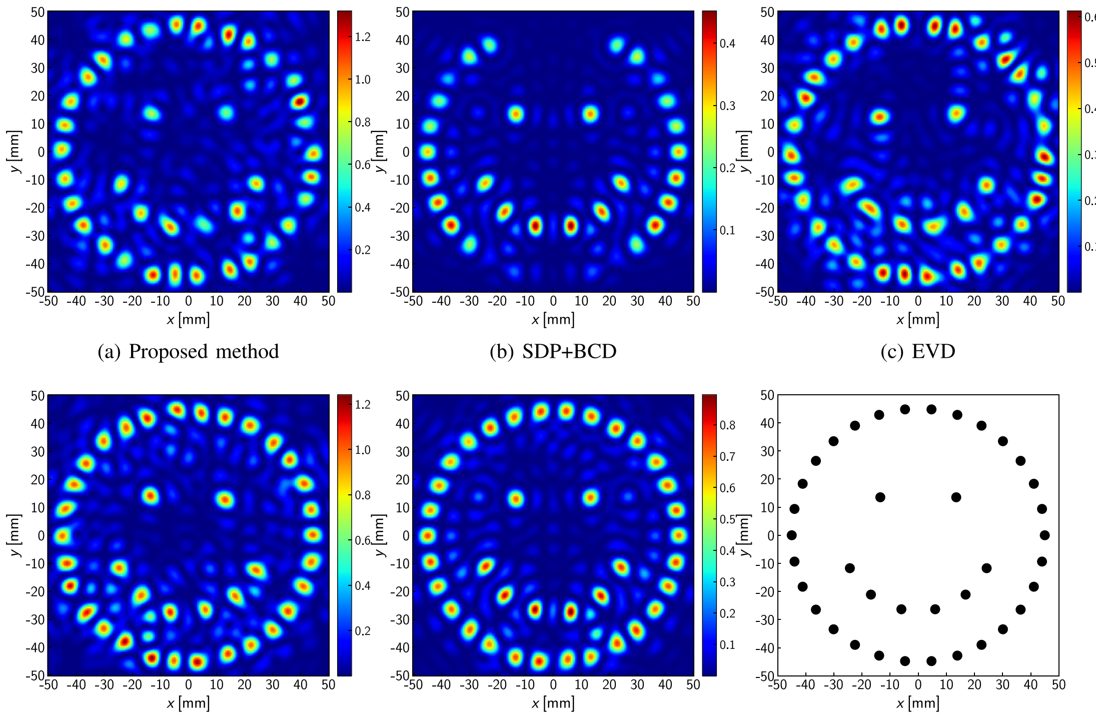
### A. Sound Field Reconstruction

*Experiments*. First, we evaluated the performance of a sound field reconstruction. Fig. 2(a) shows an example of the results obtained using the proposed method.

We used a phased array consisting of $36 \times 36$ transducers arranged in a grid pattern with 10 mm spacing; the wavelength of the ultrasound was set to 8.5 mm. This arrangement and wavelength followed the phased-array setup actually used. The amplitudes of the transducers were nondimensionalized, and the maximum was normalized to 1. In this study, the phase and amplitude were divided equally as follows:

$$a_i^k = \frac{k+1}{K}, \tag{10}$$

$$\phi_i^l = 2\pi \frac{l}{L}. \tag{11}$$




(a) Proposed method (b) SDP+BCD (c) EVD

(d) LM method (e) GS-PAT (f) Target pattern

Fig. 2. Results of the sound field reconstruction using different methods. It should be noted that the squares of the amplitude are plotted to enhance visibility after optimizing the amplitudes. (a) Proposed method with $K = 1, L = 16$, (b) SDP+BCD method [10], (c) EVD method [5], (d) LM method, (e) GS-PAT [11], and (f) target pattern.

In Fig. 2, the parameters of the proposed method were set to $K = 1$ and $L = 16$. Then, we determined the phases in random order. There were 38 control points, with $|p_j| = 1$ placed in the $xy$-plane at $z = 150\,\mathrm{mm}$, where the $xy$-plane was parallel to the array surface, and the origin was placed at the center of the array.

In Fig. 2(b)–(e), the results of the SDP+BCD method [10], the EVD method [5], LM method, and GS-PAT [11] are also depicted for comparison. The SDP+BCD method parameters were similar to those in the original study [10], and the regularization parameter, $\gamma$, for the EVD method was set to $\gamma = 1$. The implementation and parameters of the LM method followed the lecture notes [20], and the initial values were set randomly. In the LM method, we optimized only the phase of the control points and transducers and set the amplitude of the transducers to 1 because the LM method did not converge when the amplitudes were variable. The number of iterations in GS-PAT was set to 100 as in the original paper [11].

The results presented in Fig. 2 are an example and do not provide an evaluation of the general performance of the compared methods. To evaluate the general performance, we evaluated the accuracy using random control points. Following [11], the accuracy was defined as

$$\text{accuracy} = \frac{\sum_j |(\boldsymbol{G}\boldsymbol{q})_j|}{\sum_j |p_j|}. \tag{12}$$

Fig. 3 shows the accuracy over the number of control points. The number of transducers was $20 \times 20$ to reduce the computation time, and the other configuration was similar to the one in Fig. 2. The control points were randomly located in the $xy$-plane $(x, y) \in [-100\,\mathrm{mm}, 100\,\mathrm{mm}] \times [-100\,\mathrm{mm}, 100\,\mathrm{mm}]$ at $z = 150\,\mathrm{mm}$. To balance the energy of the sound field with the energy that can be transmitted from the array, the amplitude of the control points was set to $p_1/\sqrt{M}$. Here, $M$ was the number of control points, and $p_1$ was the sound pressure at a single focus at $(x, y, z) = (0, 0150)\,\mathrm{mm}$ when the phased array generated it with full power. We repeated the trial 1000 times for each $M$. We evaluated two patterns of the parameters of the proposed method: ($K = 1, L = 16$) and ($K = 16, L = 256$).

Additionally, for tactile-display applications, it is necessary to suppress the amplitude variation among the control points as well as their intensities. Therefore, we depict the standard deviation among the normalized amplitudes of the control points in Fig. 4. Here, the standard deviation $\sigma$ was defined as follows:

$$\begin{gathered}\sigma = \sqrt{\frac{1}{N}\sum_{j=0}^{M-1}(x_j - \mu)^2}, \\ x_j = \frac{|(\boldsymbol{G}\boldsymbol{q})_j|}{\max_j|(\boldsymbol{G}\boldsymbol{q})_j|}, \mu = \frac{1}{M}\sum_{j=0}^{M-1} x_j.\end{gathered} \tag{13}$$

*Discussion*. As shown in Fig. 2, the proposed method could reconstruct a certain sound field to some extent; however, the control point amplitudes were varied. The amplitudes at the control points obtained using the SDP+BCD and EVD methods were weaker than the target value; particularly, the SDP

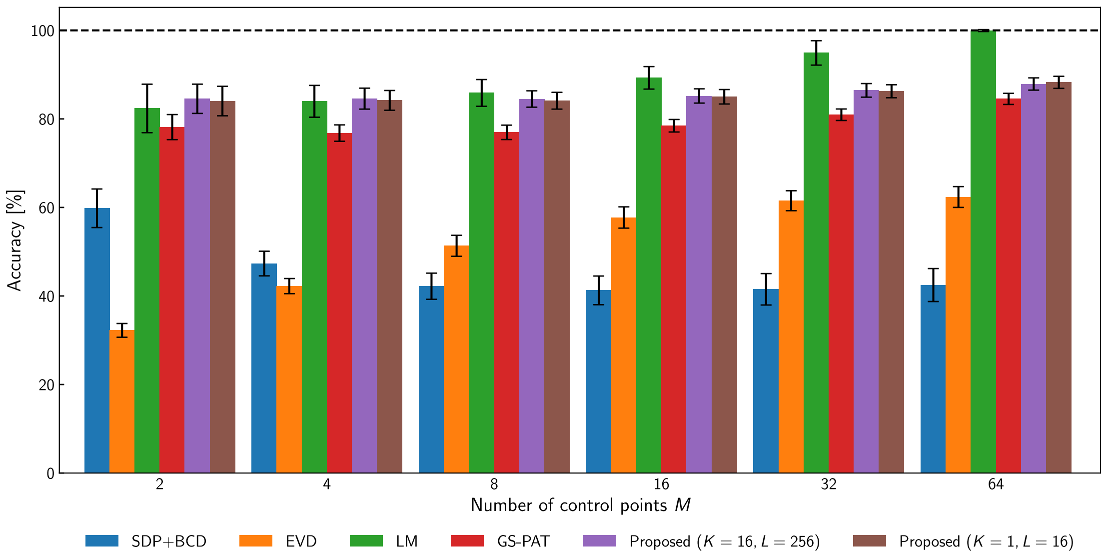



Fig. 3. Accuracy of multiple foci reconstruction.

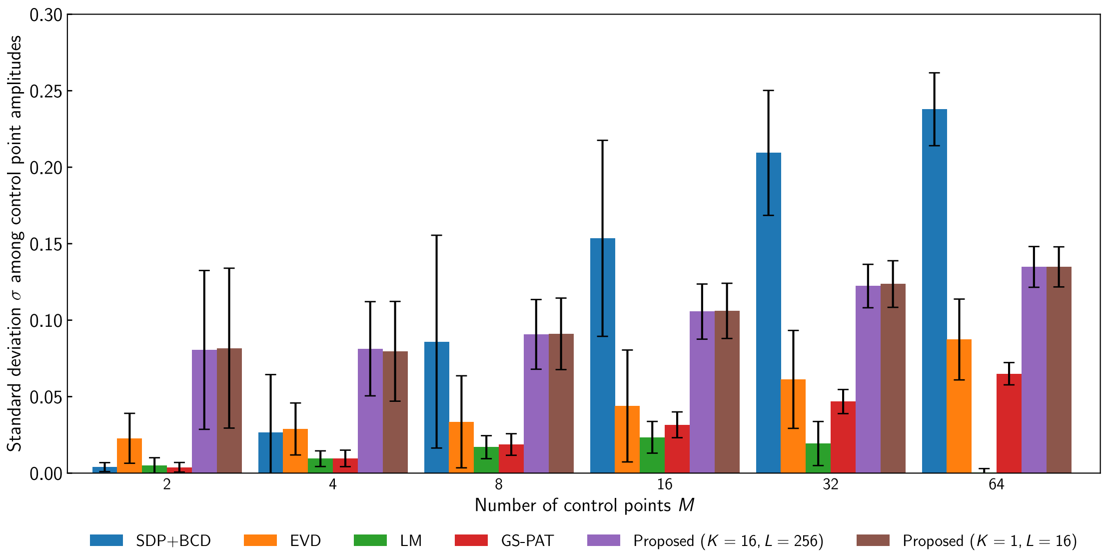



Fig. 4. Standard deviation $\sigma$ among the amplitudes of foci.

+BCD method lost some control points. The LM method reconstructed the sound field best among the four methods, and its amplitude variation was small. GS-PAT also showed good results with low amplitude variation, but the sound pressure were slightly lower than that of the LM method.

As shown in Fig. 3 and 4, the same trend can be observed for randomized control points. The LM method has the highest accuracy, followed by the proposed method. The variation of the amplitudes of the control points also exhibited an almost similar tendency as the relative error of the amplitudes. The LM method had the least variation on average, and that of the proposed method is relatively large, regardless of the control point number.

The reproducibility of the SDP+BCD and EVD methods appeared to be lower than that of the proposed method as well as the LM method. This is because these methods implicitly consider the suppression of amplitudes other than those at the control points through regularization. It must be noted that the sound field outside the control points is out of consideration in the evaluation of Fig. 3 and 4.

Further, the performances of the two patterns of the proposed method, $(K = 1, L = 16)$ and $(K = 16, L = 256)$, were almost similar; the difference in their accuracy was less than $1\%$. These results indicate that the number of phase divisions is sufficient at 16, and the amplitude freedom is unnecessary.

### B. Computation Time

*Experiment*. Second, we measured the computation time of the proposed method using a Laptop PC (CPU: Intel Core i7-8550 U). Fig. 5 presents the results over the number of control points when the number of transducers is fixed at $20 \times 20 =$

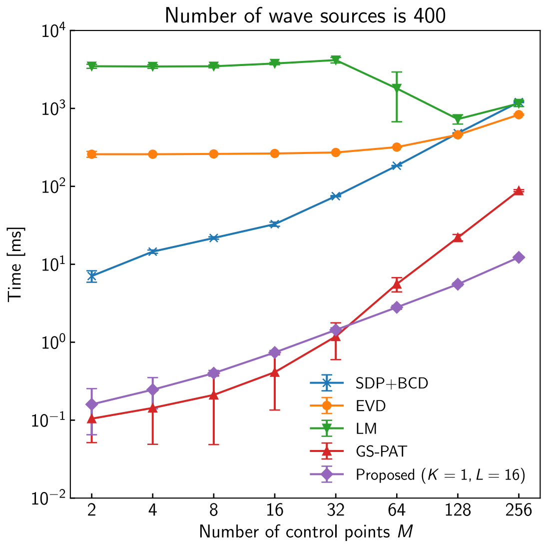



Fig. 5. Relative computation performance for a number of control points randomly placed using a fixed number of transducers ($N = 400$).

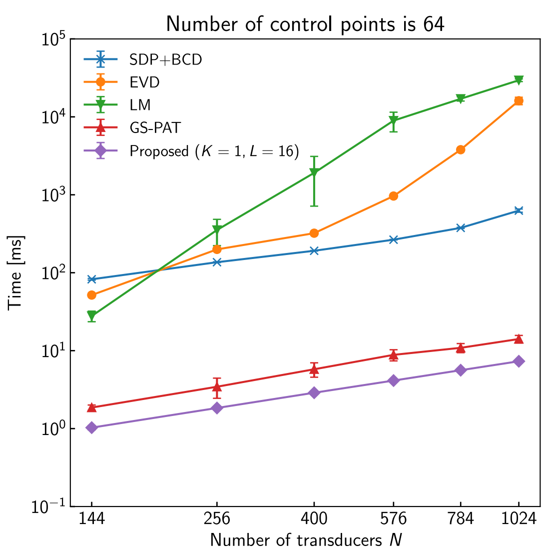
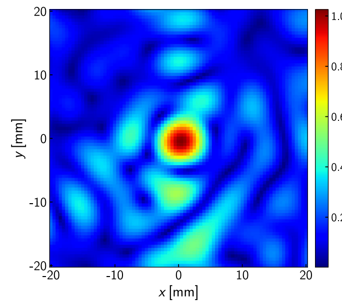


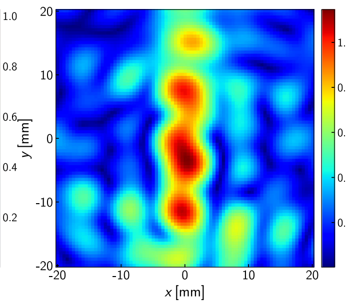

Fig. 6. Relative computation performance for a number of transducers to produce randomly placed control points ($M = 64$).

400. Additionally, Fig. 6 presents the results over the number of transducers when the number of control points is fixed at 64. It should be noted that these are log-log graphs. The computation times of the SDP+BCD, EVD, LM, and GS-PAT methods are also presented for comparison. LAPACK and OpenBLAS are used to compute the matrix calculations in these methods. The parameters used are similar to those in Section IV-A.

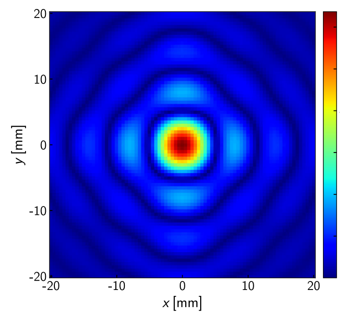
*Discussion*. As expected, the computational time of the proposed method is linear for number of transducers $N$ and control points $M$. We expect that other methods will be proportional to $M^3$ or $N^3$; however, this is not merely true owing to complicated factors involved in the practice. The computation time of GS-PAT and the proposed method is more than one order of magnitude smaller than the others.



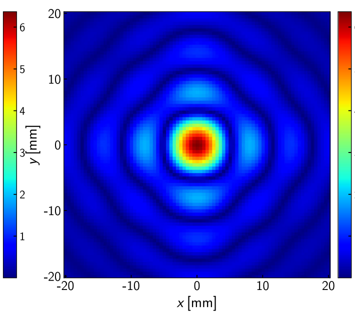

(a) $|p| = 1$ (ramdom)




(b) $|p| = 1$ (fixed)




(c) $|p| = 10$ (ramdom)




(d) $|p| = 10$ (fixed)

Fig. 7. Results of generating a single focus using the proposed method. When the target amplitude is small, artifacts or a laterally extended focus is generated.

Although taking more time than GS-PAT when the number of foci $M$ is small, the proposed method has a considerable advantage for large $M$.

Note that, in this experiment, to be fair, all the calculations were done on the CPU. GS-PAT can be easily parallelized, and the original paper [11] shows that it can be further accelerated by using GPUs. Matsubayashi *et al.* also claim that the LM method can be implemented on GPUs and used in real-time. On the other hand, the parallelization of the proposed method is difficult because the phase and amplitude of the transducers are determined one by one, which makes it unsuitable for GPU implementation.

## V. LIMITATION

We have presented good reconstruction results and a fast computation time that is linear with the number of transducers and control points in the previous section. However, the proposed method does not have any mathematical guarantee of optimality and has an obvious weak point.

In the cases examined in the previous sections, we assumed the pressure values at the control points, $|p_i|$, are balanced to the total phased array power in advance. Though the large $|p_i|$ do not affect optimization, when $|p_i|$ are too small, even a single control point does not give the best results and may produce some artifacts.

For example, Fig. 7(a) presents the reconstruction results of a single control point of $|p| = 1$ where transducers are selected in random order, and Fig. 7(b) presents that where the transducers are selected in a fixed order, that is, column-first order from the corner. The other settings are the same used in Section IV-A. Both in Fig. 7(a) and (b), the amplitude at the control point is seen to be 1. However, in Fig. 7(b), the focus extends laterally. Fig. 7(a) seems better, but it still accompanies undesired artifacts. If the target amplitude of the control point is small, the transducers selected at the beginning achieve the target sound pressure supposed at the fucus. As a result, the transducers selected after that cannot contribute to forming the focus. This phenomenon was not seen in the SDP+BCD, EVD, and GS-PAT. The LM method showed the same problem because the amplitude was fixed in this paper.

Note that this problem does not occur for large amplitudes. Fig. 7(c) and (d) shows the results for the case where $|p| = 10$ with a random and fixed order, respectively. In this case, the best results are obtained regardless of the order. Thus, a practical solution to this problem is to adjust the amplitude of the control points so that the total power around the control points is greater than or equal to the total output power of the transducers. Then, after the optimization, the output power of the transducers should be reduced uniformly if necessary.

## VI. Conclusion

In this study, we developed a method using a greedy algorithm and brute-force search to determine the amplitude and phase of transducers in a phased array such that the given sound field is reconstructed. The proposed method has a linear computation time for the number of control points $M$ and transducers $N$, which indicates that it can be computed very quickly, even for large $M$ and $N$.

The empirical results demonstrated that the proposed method could reconstruct the given sound pressures at control points with an accuracy of at least 80%. We also confirmed that the LM method accurately reconstructed the sound field, although it requires massive computation time, and GS-PAT [11] produced stable control points with sufficient speed.

Our method used only a simple greedy algorithm and the brute-force search and did not require complex matrix calculation. Therefore, this method can be easily implemented in embedded systems. Moreover, although we have only performed an optimization of the amplitude of a two-dimensional sound field in this study, our method is expected to be suitable for three-dimensional sound fields or more general fields. The method can also be applied for the optimization of a sound field with obstacles and has the potential to optimize secondary fields, such as acoustic streaming, directly. Furthermore, the proposed method does not need to compute the field; it can be adapted without simulations by replacing the $\mathcal{P}$ in this study with measurement values obtained using a microphone or camera, for example.

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