Spatially Asymptotic Behavior of Structured Covariance Matrix Estimation for Massive MIMO

Jinhui Chen, *Member, IEEE*, Zhan Xu, *Member, IEEE*, David Gesbert, *Fellow, IEEE*, Kai Yang, *Member, IEEE*, and Ruxin Zhi

***Abstract*— In this letter, the truncated redundancy averaging (TRA) method for structured covariance matrix estimation and its spatially asymptotic behavior for massive MIMO are studied. The TRA method can be applied to the antenna arrays exhibiting correlation redundancy, including linear and non-linear arrays. Resorting to Khinchin's statement on the law of large numbers for correlated random variables, it is derived that, for a uniform array, if its physical size is a strictly increasing linear or sub-linear function of the number of antenna elements, the convergence of the TRA estimate to the true covariance matrix occurs within one single channel realization. We also derive and demonstrate that lower spatial correlation leads to increased estimation performance.**

***Index Terms*— Asymptotic behavior, structured covariance matrix estimation, spatial correlation, massive MIMO.**

I. Introduction

SPATIAL covariance matrix estimation is instrumental to the functioning of a host of massive MIMO methods. Antenna correlation can be desirable as it provides *structure* in signal statistics which can, in turn, be used to mitigate multi-user interference or pilot contamination in massive MIMO systems [1], [2]. More generally, the spatial covariance matrix can be used in DoA estimation [3], channel estimation and feedback [4], [5], user scheduling [6], and precoding [7]. Hence, the efficiency of a spatial covariance matrix estimation method can play an important role in improving the performance of massive MIMO techniques.

Consider n sample vectors $\{\mathbf{x}_i\}_{i=1}^n$ from a distribution of p-dimensional random vector $\mathbf{x}$. The sample covariance matrix (SCM) method $\hat{\mathbf{R}} = \frac{1}{n}\Sigma_{i=1}^n \mathbf{x}_i\mathbf{x}_i^T$ is a classic estimation method. When $\mathbf{x}$ is zero-mean multivariate normal distributed (MVN) and no structure information on the true covariance matrix $\mathbf{R} = \mathbb{E}(\mathbf{x}\mathbf{x}^T)$ is known a priori, the SCM method results in the maximum likelihood (ML) estimate. When the n samples are independent or linearly correlated Gaussian vectors, $n = O(p)$ is sufficient to estimate the covariance matrix accurately [8], where $O(\cdot)$ is the big-O notation. When $\mathbf{x}$ is not MNV or the structure of the covariance matrix $\mathbf{R}$ is known a priori, the SCM estimate is no longer the ML estimate.

In the literature, covariance matrix estimators for random vector distributions other than MVN are investigated, including robust covariance estimators whose performances are robust to various random vector distributions, *e.g.,* in [9]–[11] and the references therein; the study of covariance matrix estimators for structured covariance matrices with structure information a priori mainly focuses on Toeplitz or block-Toeplitz for linear arrays, *e.g.,* in [12]–[15] and the references therein.

In practical MIMO systems, considering physical limitation and various coverage objectives, non-linear arrays such as planar arrays are widely deployed. The structures of their spatial covariance matrices might not be either Toeplitz or block Toeplitz. In the case of regular antenna arrays, an interesting property arises from the fact that the distance between a pair of antenna elements is subject to shift-invariance. This means that several distinct pairs of antenna elements will exhibit the same inter-element spacing. This in turn yields some redundancy in the correlation coefficients found across the covariance matrix, which we term as *correlation redundancy* below, bestowing some useful structure of the matrix itself. Redundancy averaging (RA) is an element-wise covariance matrix estimation approach to exploit the above correlation redundancy [12]. In the RA method, the samples of the correlation coefficients relating to the antenna pairs separated by the same distance are averaged across. The RA method can be applied to various array configurations, including linear and non-linear arrays, and is robust to various random vector distributions.

Intuitively, large antenna arrays in context of massive MIMO systems give the extra benefit that they can provide large correlation redundancy for enhancing covariance matrix estimation. This motivates us to investigate the spatially asymptotic behavior of the RA method with the number of antenna elements. Under the intuition that zeroing small correlation coefficients has small enough impact on the accuracy of a covariance matrix estimate, we modify the conventional RA method described in [12] into the truncated redundancy averaging (TRA) method. In the TRA method, if a pair of antenna elements are orthogonally cross-polarized or the distance between them is equal to or greater than δ_0,

Manuscript received February 28, 2021; revised April 17, 2021; accepted May 9, 2021. Date of publication May 19, 2021; date of current version August 12, 2021. This letter was supported in part by National Key Research and Development Project (No. 2020YFC1511701), in part by Key Laboratory of Modern Measurement & Control Technology (No. KF20201123201), in part by Beijing Excellent Talent Support Program (No. 2016000026833ZK08), in part by the Beijing Science and Technology Project (No. Z191100001419001), and in part by the French DGE-funded Beyond5G project. The associate editor coordinating the review of this letter and approving it for publication was Y. Liu. *(Corresponding author: Zhan Xu.)*

Jinhui Chen, Zhan Xu, and Ruxin Zhi are with the Key Laboratory of Modern Measurement and Control Technology, Beijing Information Science and Technology University, Beijing 100101, China, and also with the School of Information and Communication Engineering, Beijing Information Science and Technology University, Beijing 100101, China (e-mail: jinhui.chen@bistu.edu.cn; xuzhan@bistu.edu.cn; zhiruxin@bistu.edu.cn).

David Gesbert is with the Department of Communication Systems, Eurecom, 06410 Biot, France (e-mail: david.gesbert@eurecom.fr).

Kai Yang is with the School of Information and Electronics, Beijing Institute of Technology, Beijing 100081, China (e-mail: yangkai@bit.edu.cn).

Digital Object Identifier 10.1109/LCOMM.2021.3081733

e.g., $\delta_0 = 4\lambda$ with the wavelength of λ, their related correlation coefficient is zeroed in the covariance matrix estimate.

In this letter, our main contributions are as follows:

- We modify the conventional RA method into the TRA method for complexity reduction.
- Resorting to Khinchin's statement on the law of large numbers (LLN) for correlated random variables [16], we establish that, for a uniform array, if its physical size is a strictly increasing linear or sublinear function of the number of antenna elements, the convergence of the TRA estimate to the true covariance matrix occurs within one single channel realization;
- Based on the inequality in Khinchin's statement, we show that lower spatial correlation leads to increased estimation performance of the TRA method;
- Our theoretical results are demonstrated by numerical results with Jake's one-ring model [17].

The rest of this letter is organized as follows. In Section II, system assumptions for spatial covariance matrix estimation are presented. In Section III, the procedure of antenna pair grouping in the TRA method is described. In Section IV, the spatially asymptotic behavior of the TRA method is analyzed theoretically. In Section V, numerical results demonstrate the spatially asymptotic convergence of the TRA method and the impact of spatial correlation under certain conditions. Section VI contains our conclusion and discussion.

II. System Assumptions

Consider a MIMO system comprising a base station (BS) equipped with an M-antenna array and K mobile stations (MS) with N_k antennas each, $M > N_k$, $k = 1, \dots K$. Denote the uplink channel matrix at the BS for MS k by a $M \times N_k$ matrix $\mathbf{H}_k$, $\mathbf{H}_k = [\mathbf{h}_{1,k}, \dots, \mathbf{h}_{N_k,k}]$ where $\mathbf{h}_{n,k}$'s are M-length column vectors, $n = 1, \dots, N_k$. Assume $\mathbf{h}_{n,k}$'s are identically distributed zero-mean random vectors. The uplink receiving covariance matrix at the BS for MS k is

$$\tilde{\mathbf{R}}_k = \mathbb{E}\left(\mathbf{H}_k \mathbf{H}_k^\dagger\right) = \sum_{n=1}^{N_k} \mathbb{E}\left(\mathbf{h}_{n,k}\mathbf{h}_{n,k}^\dagger\right) = N_k \mathbb{E}\left(\mathbf{h}_{1,k}\mathbf{h}_{1,k}^\dagger\right), \tag{1}$$

where $\dagger$ denotes the Hermitian transpose. Eq.(1) shows that it can be sufficient to estimate the uplink receiving covariance matrix $\tilde{\mathbf{R}}_k$ at the BS for MS k with the channel coefficients relating to one of the multiple antennas at MS k; that is, for acquiring the estimate of $\tilde{\mathbf{R}}_k$, it is required to at least estimate the covariance matrix $\mathbf{R}_k = \mathbb{E}\left(\mathbf{h}_{1,k}\mathbf{h}_{1,k}^\dagger\right)$. In the rest of this letter, for simplicity, $\mathbf{h}_{1,k}$ is denoted by $\mathbf{h}_k$.

III. Truncated Redundancy Averaging

In the proposed TRA method for the spatial covariance matrix estimation at the BS, considering MS k in isolation, far-spaced or orthogonally cross-polarized antenna pairs are not taken much attention as their related small correlation coefficients are treated as zeros, and the rest antenna pairs will be grouped into Q antenna pair groups, $\mathcal{P}_1, \dots, \mathcal{P}_Q$, relating to Q distinct non-zero correlation coefficients $\rho_{1,k}, \dots, \rho_{Q,k}$.

The antenna pair grouping procedure can be processed as follows:

1) When a pair of antenna elements are orthogonally cross-polarized, or the distance between them is equal to or greater than δ_0, *e.g.*, $\delta_0 = 4\lambda$ with λ the wavelength, they are approximated as uncorrelated, and the related correlation coefficient is treated as zero. Note that empirically, when the distance between a pair of antenna elements is equal to or greater than 4λ, they are approximated as uncorrelated.
2) Compare the inter-element distances for the rest antenna pairs and figure out the number of antenna pair groups Q which denotes the number of distinct values of distances $d_1, d_2, \dots, d_Q$ with $d_1 < d_2 < \dots < d_Q < \delta_0$.
3) The antenna pair group $\mathcal{P}_q$ is
$$\mathcal{P}_q = \{(i,j) | d_{ij} = d_q\}, q = 1, \dots, Q, \tag{2}$$
where (i,j) denotes the antenna pair composed of antenna elements i and j and d_{ij} is the distance between antenna elements i and j.

After antenna pair grouping, the Q distinct values of non-zero correlation coefficients $\rho_{1,k}, \dots, \rho_{Q,k}$ can be estimated by

$$\hat{\rho}_{q,k} = \frac{1}{LM_q} \sum_{l=1}^{L} \sum_{(i,j)\in\mathcal{P}_q} \hat{h}_{i,k}(l)\hat{h}_{j,k}^*(l), \quad i < j \tag{3}$$

where $\hat{\rho}_{q,k}$ is the estimate of $\rho_{q,k}$, L is the number of channel realizations, M_q is the number of antenna pairs in $\mathcal{P}_q$, and $\hat{h}_{i,k}$ and $\hat{h}_{j,k}(l)$ are the estimates of channel coefficients $h_{i,k}(l)$ and $h_{j,k}(l)$ respectively. $h_{i,k}(l)$ and $h_{j,k}(l)$ are the i-th and j-th channel coefficients in the sample vector of $\mathbf{h}_k$ at the l-th channel realization, relating to the i-th and j-th antenna elements at the BS.

IV. Spatially Asymptotic Behavior of TRA

When the antenna elements are correlated, for the antenna pairs in the same group, the antenna elements in different antenna pairs can be correlated, and thus the samples of the related correlation coefficients can be correlated. Considering the correlation between the correlation samples, the widely known LLN for *i.i.d.* random variables cannot be directly applied to studying the spatially asymptotic behavior of the TRA method.

In the following Lemma 1, the Khinchin's statement on LLN proposed in [16] for correlated random variables is introduced, based on which the spatially asymptotic behavior of the TRA method is studied.

Lemma 1: Let $x_1, x_2, \dots, x_n, \dots$ be a sequence of random variables. Denoting by r_{ij} the correlation coefficient between x_i and x_j and putting

$$\begin{aligned} a_k &= \mathbb{E}(x_k), \\ B_n &= \sum_{k=1}^{n} \mathbb{E}|x_k - a_k|^2, \\ C_n &= \sum_{k=0}^{n-1} \sup_{|i-j|=k} |r_{ij}|, \end{aligned} \tag{4}$$

if

$$B_n C_n = O(n^{2-\delta}) \tag{5}$$

for some $\delta > 0$, the strong LLN is

$$\frac{1}{n}\sum_{k=1}^{n}(x_k - a_k) \to 0, \quad n \to \infty, \tag{6}$$

and, for any real number $A > 0$,

$$\mathrm{P}\left\{\left|\sum_{i=1}^{n}(x_{k+i} - a_{k+i})\right| > A\right\} < \frac{2}{A^2}(B_{k+n} - B_k)C_n. \tag{7}$$

A. Spatially Asymptotic Convergence

In the following, in the light of Lemma 1, we investigate the spatially asymptotic convergence of a TRA covariance matrix estimate with the number of antenna elements for a uniform array. Note that the uniform array can be a uniform linear array (ULA) or a uniform non-linear array, *e.g.*, a uniform planar array (UPA).

Assume that, TRA is used to estimate the receiving covariance matrix at the BS with a uniform array composed of co-polarized antenna elements for UE k, and Q antenna pair groups are established with M_q antenna pairs each.

For an antenna pair set $\mathcal{P}_q$, let

$$x_n = h_{i_{n,q},k} h^*_{j_{n,q},k} - \rho_{q,k}, \quad n = 1, \ldots, M_q \tag{8}$$

where the antenna pair $(i_{n,q}, j_{n,q})$ is the n-th member in $\mathcal{P}_q$.

Since

$$\mathbb{E}(h_{i_{n,q},k} h^*_{j_{n,q},k}) = \rho_{q,k}, \tag{9}$$

we have that x_n's expectations are equal to zero, *i.e.*,

$$\mathbb{E}(\phi_1) = \mathbb{E}(\phi_2) = \ldots = \mathbb{E}(\phi_{M_q}) = 0. \tag{10}$$

Let

$$B_{M_q} = \sum_{n=1}^{M_q} \mathbb{E}|x_n|^2 \tag{11}$$

and

$$C_{M_q} = \sum_{p=0}^{M_q-1} \beta_p \tag{12}$$

with

$$\beta_p = \sup_{|l-m|=p} |\mathbb{E}(x_l {x_m}^*)|. \tag{13}$$

Let $b_{l,m,q}$ denote the shortest distance between any antenna element in the antenna pair l and any antenna element in the antenna pair m, where the antenna pairs l and m are the l-th and m-th antenna pairs in the antenna pair group $\mathcal{P}_q$. When $b_{l,m,q} \geq \delta_0$ with δ_0 an empirical value for approximating two antenna elements as uncorrelated, *e.g.*, $\delta_0 = 4\lambda$ with λ the wavelength, x_l and x_m can be approximated as uncorrelated, *i.e.*, $\mathbb{E}(x_l {x_m}^*) \approx 0$.

Without the loss of generality, the antenna pairs in $\mathcal{P}_q$ can be numbered to satisfy that $\beta_p > 0$ for $p \leq N_q - 1$ and $\beta_p \approx 0$ for $p \geq N_q$ if there is $\beta_p \approx 0$, where $N_q \in \mathbb{Z}^+$ and $1 \leq N_q \leq M_q$. Then we have that

$$N_q \min\{\beta_p > 0\} \leq C_{M_q} \leq N_q \beta_0 \tag{14}$$

Considering x_n's are zero-mean identically distributed, we have that

$$B_{M_q} = M_q \sigma^2_{q,k} \tag{15}$$

where $\sigma^2_{q,k}$ is the variance of x_n.

From (14) and (15), the upper bound of $B_{M_q} C_{M_q}$ is

$$B_{M_q} C_{M_q} \leq N_q M_q \beta_0 \sigma^2_{q,k}, \tag{16}$$

and the lower bound of $B_{M_q} C_{M_q}$ is

$$B_{M_q} C_{M_q} \geq N_q M_q \sigma^2_{q,k} \min\{\beta_p > 0\}. \tag{17}$$

For a uniform array, if its size is a strictly increasing linear or sublinear function of M, M_q increases with M and $N_q = o(M_q)$ where $o(\cdot)$ is the little-o notation. In terms of (16),

$$B_{M_q} C_{M_q} = O(M_q^{2-\delta}), \tag{18}$$

with some $\delta > 0$, which in terms of (5) in Lemma 1 is the sufficient condition for

$$\hat{\rho}_{q,k} - \rho_{q,k} \to 0, \quad M_q \to \infty, \tag{19}$$

with

$$\hat{\rho}_{q,k} = \frac{1}{M_q} \sum_{n=0}^{M_q-1} h_{i_{n,q},k} h^*_{j_{n,q},k}. \tag{20}$$

On the other side, if the size of a uniform array is fixed while M grows, M_q increases with M and $N_q \sim M_q$. In terms of (17),

$$B_{M_q} C_{M_q} = O(M_q^{2+\delta'}), \tag{21}$$

with some $\delta' \geq 0$, which is not a sufficient condition in terms of (5) in Lemma 1.

Additionally, it is seen that, for a uniform array, if its size is a strictly increasing linear or sub-linear function of the number of antenna elements M, the number of antenna pair groups Q approaches a constant as M increases.

Therefore, for a uniform array, if its size is a strictly increasing linear or sublinear function of M, under the assumption of ideal channel coefficient estimation, the spatially asymptotic convergence of the TRA estimate to the true covariance matrix occurs within one single channel realization, *i.e.*,

$$\hat{\mathbf{R}}_k \to \mathbf{R}_k, \quad M \to \infty. \tag{22}$$

B. Impact of Spatial Correlation Level

In the following, in the light of the inequality (7) in Lemma 1, we investigate the impact of the spatial correlation level on the performance of the TRA method.

From the derivation in Section IV-A and (7) in Lemma 1, we have that, for any real number $A > 0$,

$$P(\gamma_{q,k}) = P\left\{\left|M_q \gamma_{q,k} - h_{i_{1,q},k} h^*_{j_{1,q},k}\right| > A\right\} < \frac{2\sigma^2_{q,k}}{A^2} C_{M_q}, \tag{23}$$

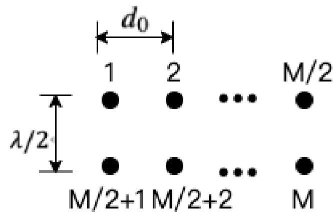

Fig. 1. An M-antenna uniform planar array (UPA).

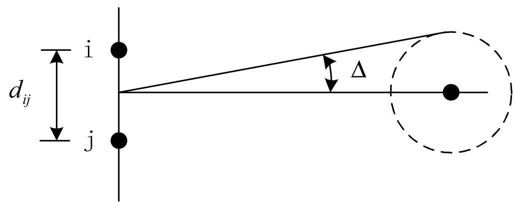

Fig. 2. The one-ring model.

with

$$\gamma_{q,k} = \hat{\rho}_{q,k} - \rho_{q,k}. \tag{24}$$

We can see that when the correlation level $|\mathbb{E}(x_l x_m{}^*)|$ decreases, C_{M_q} decreases and thus, the upper bound of $P(\gamma_{q,k})$ in (23) decreases.

Since the above conclusion can be applied to any M_q, $h_{i_{n,q},k}$ and $\rho_{q,k}$, we deduce that the squared error $|\gamma_{q,k}|^2$ decreases with the spatial correlation level.

Therefore, for a uniform array, if its size is a strictly increasing linear or sublinear function, the mean squared error of a TRA covariance matrix estimate $\|\hat{\mathbf{R}} - \mathbf{R}\|_{\mathrm{F}}^2/M^2$ decreases as spatial correlation decreases; that is, the TRA method performs better with lower spatial correlation. Note that $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm of a matrix.

V. Numerical Evaluation

In our simulation, we assume that an M-antenna uniform planar array (UPA), as shown in Fig. 1, is installed at the BS with M an even number, and a single-antenna MS on the boresight of the UPA is in the center of a scatter ring composed of unlimited scatters. The UPA is composed of two rows of antenna elements with $M/2$ antenna elements in each row. The spacing between the two rows is $\lambda/2$. The antenna spacing between the antenna elements in the same row is denoted by d_0.

The one-ring model in [17], as shown in Fig. 2, is used to generate the true spatial covariance matrix $\mathbf{R}$, whose correlation coefficients are generated as

$$r_{ij} = J_0\left(\frac{2\pi}{\lambda}\Delta d_{ij}\right), \tag{25}$$

where d_{ij} is the distance between antennas i and j, Δ is the angle spread determined by the radius of the scatter ring and the distance between the BS and the MS, and $J_0(\cdot)$ is the zeroth order Bessel function.

With the true covariance matrix $\mathbf{R}$, the l-th channel realization can be represented by

$$\mathbf{h}(l) = \mathbf{R}^{\frac{1}{2}}\mathbf{h}_{\mathrm{w}}(l), \quad l = 1, \ldots, L \tag{26}$$

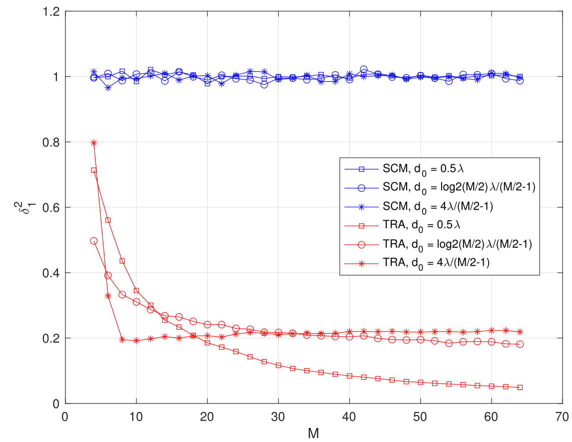

Fig. 3. Average MSE vs the number of antenna elements. $L = 1$, $\Delta = 15°$.

where L is the number of channel realizations, $\mathbf{h}_{\mathrm{w}}(l)$ is a M-length column vector whose entries are zero-mean unit-variance *i.i.d.* complex Gaussian distributed random variables $\mathcal{CN}(0,1)$ and $\mathbf{R}^{\frac{1}{2}}$ is the matrix squared root of $\mathbf{R}$,

$$\mathbf{R} = \mathbf{R}^{\frac{1}{2}}\mathbf{R}^{\frac{1}{2}\mathrm{H}}. \tag{27}$$

In our simulation, the performance metric is the average mean squared error (MSE) between the true spatial covariance matrix $\mathbf{R}$ and the covariance matrix estimate $\hat{\mathbf{R}}$ in terms of the Frobenius norm,

$$\delta_L^2 = \mathbb{E}\left(\frac{\|\mathbf{R} - \hat{\mathbf{R}}\|_{\mathrm{F}}^2}{M^2}\right). \tag{28}$$

The Monte Carlo simulation method is used to obtain the approximation of δ_L^2.

For a UPA shown in Fig. 1, its true covariance matrix $\mathbf{R}$ is neither a Toeplitz matrix nor a block-Toeplitz matrix, but there is correlation redundancy in it. We evaluate the spatially asymptotic behaviors of the TRA and SCM methods as both of them can be applied to covariance matrix estimation for UPAs. We assume that true channel coefficients are used in covariance matrix estimation; that is, $\hat{\mathbf{h}}(l) = \mathbf{h}(l)$.

In the SCM method,

$$\hat{\mathbf{R}} = \frac{1}{L}\sum_{l=1}^{L}\hat{\mathbf{h}}(l)\hat{\mathbf{h}}^H(l), \tag{29}$$

where $\hat{\mathbf{h}}(l)$ is the channel vector estimate at the l-th channel realization. For acquiring a good covariance matrix estimate by the SCM method, the number of channel realizations L has to be quite large, creating large pilot overhead and causes concerns on time delay and resource efficiency.

In Fig. 3, the asymptotic behaviors of the TRA and SCM methods are evaluated in three settings within one single channel realization. The three settings are: 1) d_0 is fixed to 0.5λ while antenna elements grows, that is, the physical size of UPA is a strictly increasing linear function of M; 2) $d_0 = \lambda \log 2(M/2)/(M/2-1)$, that is, the physical length of UPA array is $\lambda \log_2(M/2)$ which is a strictly increasing sublinear function of M; 3) $d_0 = 4\lambda/(M/2-1)$, that is, the physical length of UPA is fixed to 4λ. The Mont Carlo number is 10,000, and the angle spread is $\Delta = 15°$.

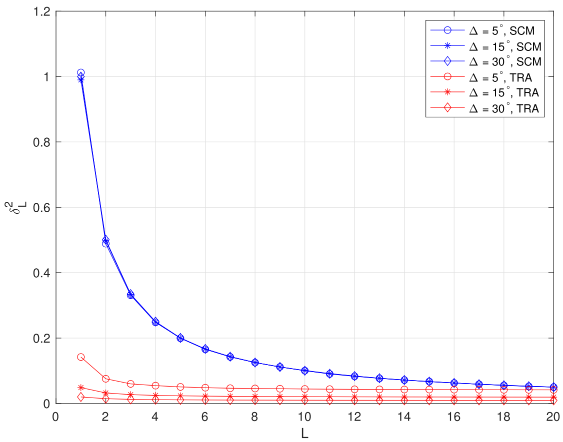

Fig. 4. Average MSE vs the number of channel realizations. $d_0 = 0.5\lambda$, $M = 64$.

From Fig. 3, it is seen that, within one single channel realization, the performance of the SCM method is poor in all three settings and does not improve as M grows. When the physical length of UPA is a strictly increasing linear or sublinear function of M, the performance of the TRA method improves, and the tendency of spatially asymptotic convergence to the true covariance matrix estimation is shown; whereas, when the physical length of UPA is fixed, although the performance of the TRA method is still much better than the SCM method in the given range of M, the tendency of spatially asymptotic convergence is not shown.

In Fig. 4, under the assumption that antenna spacing is fixed to 0.5λ, the performances of the TRA and SCM methods are evaluated with different angle spreads. As given in Eq. (25), the Mont Carlo number is 1,000. Angle spread can represent the spatial correlation level: the wider angle spread means the lower spatial correlation. Fig. 4 shows that the TRA method outperforms the SCM method for all spatial correlation levels and performs better with lower spatial correlation, while the spatial correlation level does not influence the performance of SCM much.

VI. Conclusion and Discussion

In this letter, we investigated the spatially asymptotic behavior of the TRA method for structured covariance matrix estimation. As the samples of correlation coefficients in the TRA method can be correlated, the widely known LLN for *i.i.d.* random variables cannot be directly used to analyze the asymptotic behavior of the TRA method. Resorting to Khinchin's statements on LLN for correlated random variables. we derived that for a uniform array, if its physical size is strictly linear or sublinear function of the number of antenna elements, the spatially asymptotic convergence of the TRA estimate to the true covariance matrix occurs within one single channel realization. Moreover, in the light of the inequality of Khinchin's statement, we derived that lower spatial correlation leads to increased estimation performance in terms of average mean squared error. Finally, we demonstrated our theoretical analysis results for the TRA method by the simulation on UPAs with the SCM method as a reference.

Similar to the known claim that the asymptotic orthogonality helps the simple MRC receiver and conjugate precoding to improve their performance in massive MIMO systems [18], our investigation on the asymptotic behavior of the TRA method shows that the simple TRA method, which can be applied to various array configurations, can be sufficient for spatial covariance matrix estimation in massive MIMO systems within one single channel realization.

References

[1] H. Yin, D. Gesbert, M. Filippou, and Y. Liu, "A coordinated approach to channel estimation in large-scale multiple-antenna systems," *IEEE J. Sel. Areas Commun.*, vol. 31, no. 2, pp. 264–273, Feb. 2013.

[2] M. Barzegar Khalilsarai, S. Haghighatshoar, X. Yi, and G. Caire, "FDD massive MIMO via UL/DL channel covariance extrapolation and active channel sparsification," *IEEE Trans. Wireless Commun.*, vol. 18, no. 1, pp. 121–135, Jan. 2019.

[3] H. Krim and M. Viberg, "Two decades of array signal processing research: The parametric approach," *IEEE Signal Process. Mag.*, vol. 13, no. 4, pp. 67–94, Jul. 1996.

[4] O. Ozdogan, E. Bjornson, and E. G. Larsson, "Massive MIMO with spatially correlated rician fading channels," *IEEE Trans. Commun.*, vol. 67, no. 5, pp. 3234–3250, May 2019.

[5] S. Qiu, D. Gesbert, D. Chen, and T. Jiang, "A covariance-based hybrid channel feedback in FDD massive MIMO systems," *IEEE Trans. Commun.*, vol. 67, no. 12, pp. 8365–8377, Dec. 2019.

[6] M. Matthaiou, P. J. Smith, H. Q. Ngo, and H. Tataria, "Does massive MIMO fail in Rician channels?" *IEEE Wireless Commun. Lett.*, vol. 8, no. 1, pp. 61–64, Feb. 2019.

[7] C. T. Kurisummoottil, W. Tabikh, D. Slock, and Y. Yuan-Wu, "Non-coherent multi-user MIMO communications using covariance CSIT," in *Proc. 51st Asilomar Conf. Signals, Syst., Comput.*, Oct. 2017, pp. 1248–1252.

[8] W. Cui, X. Zhang, and Y. Liu, "Covariance matrix estimation from linearly-correlated Gaussian samples," *IEEE Trans. Signal Process.*, vol. 67, no. 8, pp. 2187–2195, Apr. 2019.

[9] J. Lin and H. V. Poor, "A systematic approach to deriving the covariance matrix of correlated Nakagami-m fading channels," *IEEE Trans. Veh. Technol.*, vol. 69, no. 2, pp. 1612–1625, Feb. 2020.

[10] F. Bouchard, A. Breloy, G. Ginolhac, and F. Pascal, "Riemannian framework for robust covariance matrix estimation in spiked models," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process. (ICASSP)*, May 2020, pp. 5979–5983.

[11] E. Ollila, D. P. Palomar, and F. Pascal, "Shrinking the eigenvalues of M-estimators of covariance matrix," *IEEE Trans. Signal Process.*, vol. 69, pp. 256–269, 2021.

[12] D. A. Linebarger and D. H. Johnson, "The effect of spatial averaging on spatial correlation matrices in the presence of coherent signals," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 38, no. 5, pp. 880–884, May 1990.

[13] T. T. Cai, Z. Ren, and H. H. Zhou, "Estimating structured high-dimensional covariance and precision matrices: Optimal rates and adaptive estimation," *Electron. J. Statist.*, vol. 10, no. 1, pp. 1–59, Jan. 2016.

[14] A. P. Shikhaliev, L. C. Potter, and Y. Chi, "Low-rank structured covariance matrix estimation," *IEEE Signal Process. Lett.*, vol. 26, no. 5, pp. 700–704, May 2019.

[15] X. Du, A. Aubry, A. De Maio, and G. Cui, "Toeplitz structured covariance matrix estimation for radar applications," *IEEE Signal Process. Lett.*, vol. 27, pp. 595–599, 2020.

[16] A. Khintchine, "Sur la loi forte des grands nombres," *CR Acad. Sci. Paris Ser. I, Math.*, vol. 186, pp. 285–287, 1928.

[17] D.-S. Shiu, G. J. Foschini, M. J. Gans, and J. M. Kahn, "Fading correlation and its effect on the capacity of multielement antenna systems," *IEEE Trans. Commun.*, vol. 48, no. 3, pp. 502–513, Mar. 2000.

[18] T. L. Marzetta, "Noncooperative cellular wireless with unlimited numbers of base station antennas," *IEEE Trans. Wireless Commun.*, vol. 9, no. 11, pp. 3590–3600, Nov. 2010.