Advances in Flexible Robotic Manipulator Systems—Part I: Overview and Dynamics Modeling Methods

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***Abstract*—Flexible robotic manipulators (FRMs) exhibit advantages over traditional rigid-body manipulators, including lower energy consumption, faster response, enhanced safety, reduced spatial footprint, and greater operational flexibility. This article reviews the advancements in the modeling, planning, and control technologies of FRMs, along with their applications and perspectives. Compared with previous survey papers about FRMs, this work has the following highlights. First, modeling methods w.r.t. joint flexibility, link flexibility, and link-joint coupling effects are systematically reviewed; second, FRM motion planning methods are reviewed solely instead of being treated as part of FRM control methods; third, this survey presents the recent advancements in FRMs that incorporate artificial intelligence (AI), acknowledging AI as a prevalent research trend; and finally, this is the only survey to systematically explore the rapidly evolving applications of FRM systems over the past five years. Our work is divided into two independent articles. This article, which is the first one, includes the basic background of FRMs, an overview of previous FRM-related surveys, and dynamic modeling methods for FRMs. The second article will review the prevalent motion planning and control methods for an FRM, together with FRM applications and perspectives.**

***Index Terms*—Artificial intelligence (AI), dynamic modeling, flexibility, flexible robotic manipulator (FRM), vibration suppression.**

Manuscript received 23 November 2023; revised 15 December 2023; accepted 23 January 2024. Date of publication 15 February 2024; date of current version 18 April 2024. Recommended by Technical Editor S. Zhuang and Senior Editor H. Gao. This work was supported in part by the National Natural Science Foundation of China under Grant 62303010 and Grant 62103139, in part by the Lotus Youth Talent Program of Hunan Province, China under Grant 2023RC3115, in part by the Anhui Provincial Key Research Program of Universities under Grant 2022AH050087, in part by the Natural Science Foundation of Anhui Province under Grant 2208085QF207, and in part by the Fundamental Research Funds for the Central Universities under Grant 531118010509. *(Corresponding authors: Hejia Gao; Fei-Yue Wang.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TMECH.2024.3359067.

Digital Object Identifier 10.1109/TMECH.2024.3359067

I. Introduction

ROBOTIC manipulators refer to mechanical arms designed to perform tasks that mimic human arm functions in skill-intensive, dangerous, repetitive, or tedious jobs. They find extensive utilization across diverse domains, including manufacturing, aerospace, agriculture, healthcare, logistics, and operations in hazardous environments [1], [2]. Traditional robotic manipulators are characterized by rigid links and joints, possessing high stiffness conducive to precise positioning and reduced vibrations [3]. Nevertheless, these rigid and sizable manipulators incur inherent drawbacks such as high power consumption, low response speed, and high cost.

In response to these challenges, lightweight manipulators, known as flexible robotic manipulators (FRMs), have emerged since the 1970s [4]. An FRM denotes a robotic manipulator designed with flexible links and/or flexible joints, with flexibility primarily manifesting in link bending, torsion, and joint elasticity [5]. In contrast to the conventional bulky and rigid-body manipulators, FRMs offer a spectrum of advantages, including reduced weight, lower energy consumption, enhanced operational speed, a superior payload-to-robot-weight ratio, compact workspace footprints, and heightened portability [6], [7], [8]. These attributes bolster their economic viability, as they often have lower operational costs. Safety constitutes another pivotal advantage of FRMs, particularly as robots increasingly share spaces with humans. The reduced inertia of FRMs renders them safer for close proximity operation with people [9]. The introduced flexibility also enables easier maintenance, further enhancing the practicality of an FRM system.

The advancement of FRMs has opened up new application opportunities that were not feasible with traditional bulky manipulators [10]. In the aerospace and outer space industries, the lightweight design and low energy consumption of FRMs are crucial for missions like satellite deployment and space station operations [11]. In manufacturing, FRMs are used for handling chemical materials and in semiconductor production, where precision and safety are paramount [12]. FRMs also play a pivotal role in hazardous environments such as nuclear power plants, contributing to waste remediation and protecting human workers from high radiation zones [13]. FRMs also play a pivotal role in hazardous environments such as nuclear plants, contributing to waste remediation and safeguarding human workers from high radiation zones [13]. In the military domain, FRMs are under development for search and rescue operations [14]. In healthcare, the diminutive size and cost-effectiveness of FRMs position them as prime candidates for future advancements in surgical procedures [15]. Furthermore, FRMs are gaining traction in agriculture and homecare, particularly for aiding the elderly [16].

Notwithstanding their advantages over bulky manipulators, FRMs lead to undesirable outcomes such as decreased precision and increased oscillations attributable to their inherent flexibility. These issues can compromise the stability of an FRM system and even lead to structural fatigue if the adopted tracking controller is inadequately designed. Existing controllers suitable for bulky manipulators are unsuitable for FRMs due to their distributed parameter nature and infinite degrees of freedom (DOFs) [17]. The inherent complexity in the dynamics of FRMs is further exacerbated by factors such as underactuation, time variability, nonminimum phase (unstable zero dynamics), uncertainty, high nonlinearity, and multiple input and multiple output characteristics [18]. Therefore, prior to devising efficient controllers for FRMs, it is crucial to formulate accurate dynamic models capable of simulating authentic deformations induced by bending, torsional, and twisting forces. In scenarios where an FRM operates within a complex workspace replete with complex constraints like vibration suppression and collision avoidance, open-loop motion planning becomes an indispensable precursor to implementing closed-loop tracking control.

This work aims to provide a systematic review of methodologies for modeling, planning, and control, along with the prevalent applications of FRMs and their future development chances. Unlike previous surveys in this domain, this survey distinguishes itself through several key attributes. First, it covers three types of FRM modeling methods, i.e., joint flexibility, link flexibility, and link-joint coupling effects. Second, this is the first time that FRM motion planning methods are systematically reviewed instead of being treated as part of FRM control methods. Third, it includes the latest research trend, extending up to the year 2024, which involves the integration of artificial intelligence (AI) techniques with FRMs. Lastly, this is the only survey that systematically explores the rapidly evolving applications of FRM systems over the past five years. As detailed in Section II, these distinctive features set this work apart from prior surveys in the field of FRMs.

The rest of the article is structured as follows. Section II provides an overview of previous surveys on FRM techniques, whereas Section III concentrates on the current FRM modeling methods. Finally, Section IV concludes the article. In the second part of this survey, we will continue to investigate FRM motion planning and control methodologies accompanied by an examination of FRM applications and perspectives.

II. Related Works

This section offers an overview of prior surveys related to FRMs, each bringing its own unique classification principles and insights to the community of FRMs.

The earliest survey paper about FRM was published in 1986 [19], laying a foundational groundwork for FRM categorization in terms of modeling and control. That work classified FRMs into two distinct types: those featuring flexible links and those with flexible joints, and proceeded to review the modeling and control methodologies designed for either FRM type. Building upon this groundwork, Spong [20] in 1990 focused on a specific subset of FRMs—those with rigid links but flexible joints, often referred to as flexible joint manipulators (FJMs). His work was the first to exclusively review the controllers designed for an FJM. Also in 1990, Book [21] reviewed the existing modeling methods for an FRM with flexible links or flexible joints. In that paper, he also reviewed the trajectory planners and tracking controllers for an FJM. That is the first survey that systematically reviewed the trajectory planners for an FJM. In 1993, Hu [22] reviewed the experimental methods for FRM modeling and control. That is the first survey paper to report experimental approaches and performances. In the same year, Piedboeuf et al. [23] reviewed the existing modeling and control methods for an FRM with flexible links and/or joints.

A decade later, Benosman and Le Vey in 2004 [24] offered a classification of FRM controllers based on their control goals, categorizing them into four groups: end-effector regulation, end-effector to rest motion in a fixed time, joint-trajectory tracking, and end-effector trajectory tracking. That paper particularly focused on FRMs with multiple links. Dwivedy and Eberhard in 2006 [3] provided a review of FRM modeling methods by categorizing all FRMs into single-link, two-link, and multilink manipulators and flexible-joint manipulators. That paper also touched upon the applications of FRMs. In the same year, Ozgoli and Taghirad [9] reviewed the modeling and control methods for FJMs. That work emphasized the necessity of considering joint flexibility and classified related studies based on whether Spong's assumptions were adopted or not.

Fast forward to 2014, Rahimi and Nazemizadeh [25] reviewed the FRM controllers that particularly use intelligent control methods, including fuzzy logic, neural networks, and genetic algorithms. Kiang et al. [10] in 2015 divided the FRM controllers into model-based and model-free ones, further categorizing the latter into active, passive, composite, and intelligent types. In 2016, Sayahkarajy et al. [26] focused on modeling and control methods for two-link FRMs because such FRMs offer greater maneuvering applicability and specificity than other FRM types. In the same year, Lochan et al. [27] reviewed the modeling and control methods of a two-link FRM, emphasizing its suitability in various applications such as the manufacturing industry, aerospace, and military. They also highlighted some experimentally validated studies.

In 2020, Subedi et al. [28] reviewed the modeling methods for FRMs with flexible links, dividing the involved modeling techniques into dynamic models and motion equations. That

TABLE I
OVERVIEW OF HISTORICAL FRM SURVEY PUBLICATIONS

Survey paper	Review Coverage							
	Modeling	Planning	Control	Application	Link flexibility	Joint flexibility	Literature amount	Range of literature years
Kanoh et al. (1986)	√	×	√	×	√	√	33	1975–1986
Spong (1990)	×	×	√	×	×	√	99	1979–1990
Book (1990)	√	√	√	×	√	√	45	1975–1990
Hu (1993)	√	×	√	×	√	×	18	1984–1990
Piedboeuf et al. (1993)	√	×	√	×	√	×	31	1967–1993
Green and Sasiadek (2002)	×	×	√	×	√	×	14	1984–2002
Benosman and Vey (2004)	×	×	√	×	√	√	119	1983–2003
Dwivedy and Eberhard (2006)	√	×	√	√	√	√	433	1974–2005
Ozgoli and Taghirad (2006)	√	×	√	×	×	√	167	1983–2005
Rahimi and Nazemizadeh (2014)	√	×	√	×	√	×	115	1970–2013
Kiang et al. (2015)	√	×	√	√	√	√	167	1981–2013
Sayahkarajy et al. (2016)	√	×	√	×	√	×	146	1944–2016
Lochan et al. (2016)	√	×	√	√	√	√	204	1974–2016
Subedi et al. (2020)	√	×	√	×	√	×	159	1985–2020
Alandoli and Lee (2020)	×	×	√	×	√	×	237	2000–2019
Lee and Alandoli (2020)	√	×	×	×	√	×	170	1980–2002
Mishra and Singh (2021)	√	×	√	×	√	×	153	1975-2021
Gao et al. (2023)	√	×	√	×	√	√	42	2017–2023
This work	√	√	√	√	√	√	223	1974–2024

work also divided the existing controllers into model-based and model-free ones after summarizing eight FRM control schemes. In 2020 and 2021, Alandoli and Lee reviewed the FRM modeling [29] and FRM control [30] methods, respectively. In 2023, Gao et al. [31] classified the existing FRM modeling methods by the usage of ordinary differential equations (ODEs) or partial differential equations (PDEs) and divided the existing controllers into classical and intelligent types. By the summer of 2023, that paper was the most up-to-date survey in the domain of FRM.

The aforementioned survey papers, together with similar ones [32], [33], outlined the historical developments of FRMs, which are summarized in Table I. Notably, there also exist survey papers about modeling and/or control of a flexible multibody system [34], [35], [36], [37], which are related to the theme of this work but not discussed in details.

Previous surveys on FRMs exhibit certain limitations, which are summarized as follows. First, there is a notable lack of coverage on recent advancements in FRMs, especially in the context of burgeoning AI-related technologies such as virtual reality and large language models. These innovations have opened new avenues for FRM systems, yet this AI-embedded trend has not been adequately explored in surveys from the past five years. Notably, the most recent FRM survey, i.e., [31], just reviewed 42 references, thus falls short in this aspect. There is a clear need for a new survey guiding FRM scholars on the AI-driven opportunities in FRM development.

Second, the importance of FRM motion planners has been underestimated in most previous surveys, where they were often subsumed under FRM controllers. However, motion planners are vital in executing collision-free maneuvers in complex workspaces, which is beyond the capability of a tracking controller. The advancement of motion planners is crucial, as it alleviates the design challenges faced by FRM controllers and is likely to be a key factor in shaping the future of FRM research.

Lastly, a systematic investigation concerning the practical applications of FRMs in the past five years is absent. A thorough review of these applications is essential to gain a complete understanding of the field and to lay the groundwork for targeted future research. This gap highlights the importance of a comprehensive survey that can bridge these areas and contribute to the evolving landscape of FRM technology.

III. FRM MODELING METHODS

This section delves into the existing FRM modeling methods. The flexibility of an FRM manifests in its links, joints, or both. Link flexibility refers to the torsion, bending, and compression deformations of a link. To describe such flexible characteristics, distributed parameter models are typically employed to capture the infinite DOFs in a link's spatially continuous deformation. Joint flexibility refers to the torsional deformation of a transmission unit and/or rotation shaft. In contrast to link flexibility presented by infinite-dimensional distributed parameter models, joint flexibility is described by concentrated models, e.g., regarding a flexible joint as a pure spring that stores only potential energy. Regarding link flexibility, Section III-A examines the nominal infinite-dimensional models governed by nonlinear PDEs. Obtaining closed-form solutions to such PDEs is a challenge, which complicates the analysis of FRM systems and the design of controllers. To conquer this challenge, it becomes pertinent to truncate the infinite-dimensional flexible dynamics through finite-dimensional modeling techniques, as elaborated in Section III-B. Flexible joint models are reviewed in Section III-C. If an FRM involves both flexible links and joints, the coupling effects should be modeled, which are reviewed in Section III-D. An overall diagram of the reviewed FRM model types is depicted in Fig. 1.

A. Infinite-Dimensional Modeling of Link Flexibility

A flexible link in an FRM is characterized by continuous nonlinear dynamical systems with infinite DOFs described by

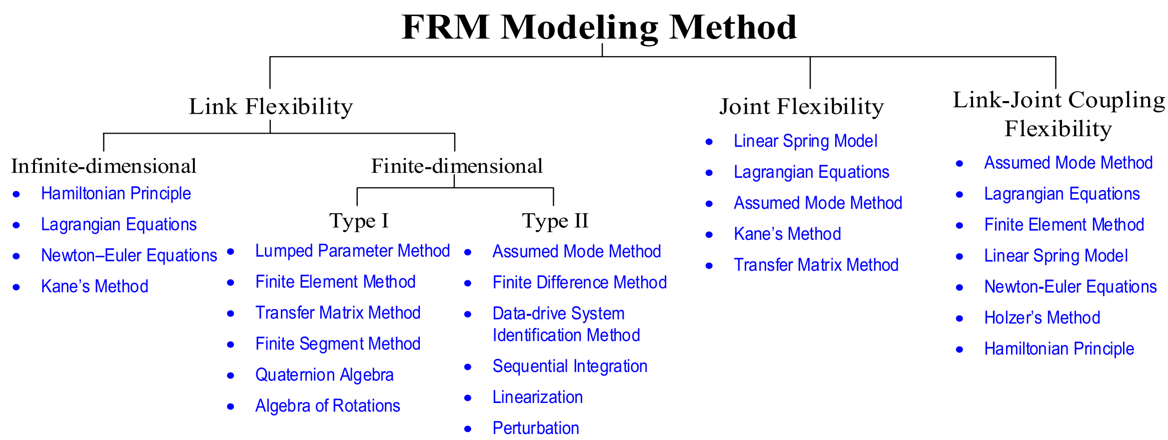

Fig. 1. Overall diagram of FRM modeling method types.

complex but accurate PDEs. This section reviews the PDE-based models for link flexibility in an FRM.

1) Hamiltonian Principle: The Hamiltonian principle serves as a foundational tool for modeling the dynamics of a distributed parameter system. Initially, this method requires the formulation of kinetic, potential energy, and virtual work equations about link flexibility [38]. After this step, the derived equations are further processed by the Hamiltonian principle to yield a series of temporal ODEs and spatio–temporal PDEs. Given that this method analyzes FRM dynamics from an energy perspective, internal forces within the system are effectively eliminated, thus making the derived infinite-dimensional model simple. The derived model offers two benefits: 1) it is conducive to model-based controller design, and 2) it ensures modeling accuracy by eliminating truncation errors [39], [40].

Bodley [41] used the Hamiltonian principle to model a nonlinearly constrained two-link flexible manipulator with a tip mass. Ji et al. [42] used the Hamiltonian principle to model the elastic vibration feature in a three-dimensional (3-D) Euler–Bernoulli beam. Lee [43] used the Hamiltonian principle to model a one-link flexible robot without elongation effects. Zhao et al. [44] used the Hamiltonian principle to model a flexible Timoshenko arm with a strong coupling of elastic deformation, shear deformation, angle position, and load dynamics. Cao and Liu [45] employed the Hamiltonian principle to nonlinearly model a 3-D two-link flexible manipulator integrated into an excavator. Xing et al. [46] adopted the Hamiltonian principle to formulate a model for a flexible two-blade aero propeller. Yao et al. [47] applied the Hamiltonian principle to develop a model for a flexible beam, constituting part of a flexible spacecraft. Wang et al. [48] used the Hamiltonian principle to construct a model for a 3-D FRM. In addition, Mattioni et al. [49] leveraged the Hamiltonian principle to model a two-link flexible robotic arm known as Canadarm2, which found deployment on the International Space Station.

2) Lagrangian Equations: Much like the aforementioned Hamiltonian principle, the Lagrangian equations describe the dynamics of a flexible-link system from the perspective of mechanical energy. This facilitates the dynamic modeling of a flexible-link system because internal forces can be safely ignored. The Lagrangian equations are particularly suitable for flexible systems with simple structures [50].

Cannon Jr. and Schmitz [51] adopted the Lagrangian equations to model the dynamics of a flexible Euler–Bernoulli beam. Bodley [41] formulated a dynamic model for a generic multi-flexible-body system using Lagrangian equations and D'Alembert's virtual work principle. A similar modeling method using Lagrangian equations and D'Alembert's principle was proposed by Surdilović and Vukobratović [52], which improved the computational efficiency of dynamic analysis. Densborn and Sawodny [53] employed the Lagrangian equations to establish the vertical dynamics model for an aerial rescue ladder, where each ladder segment was regarded as a flexible link.

3) Newton–Euler Equations: The Newton–Euler method begins by segregating individual bodies within a multibody flexible system. For translational bodies, equilibrium equations are established via Newton's second law, whereas rotational bodies are analyzed via Euler's theorem [54]. This leads to the formulation of dynamic equations for each flexible link in an FRM. By integrating the constraints among these flexible links, a complete set of dynamic equations for the entire multibody flexible system is derived. This method, rooted in both Newton's laws and Euler's theorem, is thus referred to as the Newton–Euler modeling method. Its simplicity in derivation and clarity in physical meaning make it effective for modeling small systems. However, its application to high-DOF systems is difficult.

Rakhsha and Goldenberg [55] adopted the Newton–Euler equations to model a single-link flexible manipulator. Boyer and Khalil [56] modeled the inverse dynamics of flexible manipulators using generalized Newton–Euler equations. Hwang [57] proposed nonlinear generalized Newton–Euler equations to model flexible links with large translational and rotational displacements. Xu et al. [58] used Newton–Euler equations to formulate an accurate dynamics model for each flexible cable in a cable-driven multibody manipulator.

4) Kane's Method: Kane's method, initially proposed by Prof. T. R. Kane from Stanford University in the 1950s, distinguishes itself from both the Newton–Euler equations and Lagrangian equations by being straightforward and computationally efficient [59], [60]. Kane's method introduces new variables

TABLE II
STRENGTHS AND LIMITATIONS OF INFINITE-DIMENSIONAL FRM MODELING METHODS

Modeling method	Advantages	Disadvantages
Hamiltonian Principle	● Analyzes FRM dynamics from an energy perspective with internal forces eliminated, thus facilitating the modeling process ● Facilitates model-based controller design ● Ensures modeling accuracy without truncation errors	◆Fails to handle FRM systems that are non-conservative or those with significant damping
Lagrangian Equations	● Analyzes FRM dynamics from an energy perspective with internal forces eliminated, thus facilitating the modeling process ● Acquires closed-form symbolic equations directly	◆Renders complex and lengthy formulations in the derivation of Lagrangians ◆Computationally expensive in dealing with complex FRM systems with high dimensions ◆The additionally introduced generalized coordinates make the DOFs larger than the actuator number, thus complicating the FRM controller design
Newton–Euler Equations	● Provides simplified but unified formulation independent of rigid body geometry, inertia, or motion constraints ● Offers real-time inverse dynamic analysis capability ● Well-suited for the implementation of model-based control schemes	◆Difficult to determine workless constraint forces especially when the FRM body number is large ◆Low computational efficiency in handling high-dimensional multi-body FRMs
Kane's Method	● Eliminates workless constraint forces ● Derives FRM model equations in an automated way with minimal labor efforts ● Suitable for dynamic analysis of closed-chain FRM systems without the need to break the chain structures	◆Requires calculation of body center of mass accelerations, which is time-consuming

namely generalized speeds, partial angular velocities, partial velocities, generalized inertia forces, and generalized active forces, and then models the dynamics of a flexible multibody system via D'Alembert's principle [61]. A merit of this method is that the computation is lightweight, thus enabling symbolic programs to automatically produce explicit dynamic equations for a multibody system [62], [63], [64], especially when the system is complex.

Singh et al. [65] developed software based on Kane's method that can automatically generate the motion equations for multibody flexible structures. Meghdari and Fahimi [66] integrated Kane's method and a first-order decoupling method to model the dynamics equations of a multibody system consisting of an arbitrary number of flexible bodies. Lee et al. [67] used Kane's method to establish a dynamic model for a flexible beam attached to a rigid shaft undergoing free rotational motions. Feng et al. [68] used Kane's method to formulate dynamic equations for a rigid-liquid-flexible spacecraft under large-amplitude sloshing motions. Cibicik and Egeland [69] used Kane's method and dual screw transformation matrices to linearize the kinematic and dynamic model of an FRM. Hu and Zhang [70] used Kane's method to model the dynamics equations of a flexible space manipulator equipped with control moment gyros.

Table II summarizes the strengths and limitations of the aforementioned infinite-dimensional modeling methods.

B. Finite-Dimensional Modeling of Link Flexibility

The preceding section has reviewed the PDE-based infinite-dimensional methods for modeling a flexible-link FRM. However, such models are complex, thus making dynamic behavior analysis and controller design challenging [71]. To fix this issue, a natural idea is to convert the infinite-dimensional models into finite-dimensional forms.

There are two types of methods to realize such a conversion. The first type is to discretize a spatially continuous flexible link into a finite number of pieces that are featured by being simple in their dynamics. The dynamic model of each discretized piece is computed recursively based on boundary conditions and spatial connection relations. The derived dynamic models of the finite pieces present the dynamics of the entire flexible link, which would typically presented as a series of ODEs. The second type is to nominally formulate an infinite-dimensional model before discretizing or truncating the nominal model in a finite form, which is also commonly presented as a series of ODEs just like the first type. This section reviews the prevalent finite-dimensional methods for modeling a flexible-link FRM.

1) Lumped Parameter Method (LPM): The LPM is a conventional type-I modeling approach for a flexible-link manipulator. LPM distributes the whole mass of a flexible link to finite equidistant nodes, which are interconnected by massless elastic elements (such as springs) to reflect the manipulator's flexibility. Starting from one end of the flexible link and progressing sequentially, the dynamics of each node is modeled. LPM renders high modeling accuracy if the number of distributed nodes is large, but that would also cause a high computational burden, which hinders real-time modeling and control. Thus, a balance should be made between modeling accuracy and speed in using LPM. Another issue in LPM is how to rationally model the dynamics of each connective elastic element between adjacent nodes.

Sun et al. [72] used LPM to model the dynamics of a flexible-link manipulator. Sarkhel et al. [73] used LPM to model a fishing-rod-like flexible manipulator, the free end of which is linked to the payload via a string, thus yielding sudden and sinusoidal loadings. Yap et al. [74] used LPM to model a flexible multibody system. Khajiyeva et al. [75] used LPM to model the nonlinear vibrations of flexible drill strings in a supersonic gas flow. İlman et al. [76] used LPM to model a multilink flexible manipulator. Gadringer et al. [77] used LPM to describe a 3-DOF flexible-link robot. Mangalasseri et al. [78] used LPM to model the vibrational dynamics of a magneto-electro-elastic beam.

2) Assumed Mode Method (AMM): The AMM is a type-II modeling approach for a flexible-link manipulator. AMM describes the nominal infinite-dimensional dynamic motion model as a series consisting of spatial eigen-mode functions multiplied by time-varying mode amplitudes. The series is truncated to its first few low-frequency terms because the remainder terms have a slim contribution to improving the dynamic model accuracy. One of the salient features of AMM is its explicit representation of natural frequencies in a flexible link, which facilitates dynamic analysis and controller design. Dynamic models derived by AMM are typically in low order, making them computationally easy. AMM is suitable for modeling flexible links with uniform cross sections, but it cannot deal with links with spatially varying cross sections.

De Luca and Siciliano [79] used Lagrangian equations to derive a dynamic model for a multilink flexible robot before converting the modeling into finite dimensions via AMM. A similar work is given in [80]. Mishra and Singh [81] used AMM to model the dynamics of a two-link flexible manipulator. Meghdari and Fahimi [66] used AMM and Kane's method to formulate the dynamic equations for a two-link flexible manipulator. Sun et al. [82] used AMM to model the elastic deformations in a single-link flexible manipulator. Lochan et al. [83] used AMM to model a two-link flexible manipulator.

3) Finite-Element Method (FEM): The FEM is a type-I modeling approach for a flexible-link manipulator. In this approach, a flexible link is segmented into elementary units, which are then interconnected at designated points. Each of these units is defined by an equation, formulated through an approximation method such as polynomial interpolation [84]. Compiling these equations results in a dynamic model of the whole flexible-link system. One advantage of FEM over the infinite-dimensional modeling methods is its ability to address nonlinear and mixed boundary conditions. FEM is particularly effective in modeling flexible links that exhibit complex and irregular cross-sectional geometries. A disadvantage of FEM is that it yields an overestimated stiffness matrix, which causes unstable closed-loop responses if a model-based controller is used [85]. Also, FEM is computationally heavier than AMM.

Usoro et al. [86] used Lagrangian equations and FEM to model a two-link flexible manipulator. Bayo [87] used the Hamiltonian principle to derive a dynamic model for a single-link flexible robot before discretizing it into ODEs via FEM. Yang and Sadler [88] used FEM to model the dynamics of flexible planar linkages under large deformations. Naganathan and Soni [89] used FEM and Newton–Euler equations to model the nonlinear dynamics of a flexible-link manipulator when link flexibility and stiffness are coupled. Alandoli et al. [90] used FEM and Lagrangian equations to model a single-link flexible manipulator. A similar work is done by Ge et al. [91]. Qiu et al. [92] used FEM and Lagrangian equations to model a coupling system with four flexible beams connected by springs.

4) Transfer Matrix Method (TMM): The TMM is a type-I modeling approach for flexible link manipulators. TMM is particularly suitable for large systems made up of several subsystems, each of which consists of simple elastic and dynamic elements. In TMM, each subsystem's motion equation is represented by a transfer matrix, which conveys a state vector from one end of the subsystem to the other [93]. The overall system dynamic model is derived by multiplying individual matrices [94]. Advantages of TMM-based models include being independent of the overall system dimension, lightweight in memory usage, fast in computational speed, and easy for computer implementation [95].

Rong et al. [93] used FEM and a discrete-time TMM to model the dynamics of spacecraft with flexible appendages. Zhang et al. [96] decomposed a flexible-link manipulator into elements and developed a state vector for each element via Newton–Euler equations and TMM; the dynamic model of the concerned manipulator is built by concatenating the state vectors from the base to the end-effector. Shi et al. [97] used TMM to model the vibration suppression dynamics of a rigid-flexible link manipulator. Lu et al. [98] used TMM to model the linear vibration in a flexible multibody system.

5) Finite Segment Method (FSM): The FSM is a type-I modeling method similar to FEM. A flexible-link system is segmented into finite pieces with known length and inertia, which are sequentially linked with spring-damper elements [99], [100]. As opposed to FEM, FSM can describe the time-varying process of a flexible system with nonlinear deformations. FSM is particularly suitable to model flexible systems with slender links/beams.

He et al. [101] used FSM and TMM to model the dynamics of a flexible beam undergoing drastic motions. Hamper et al. [102] regarded railroad tracks as deformable and flexible links before modeling their dynamics via FSM.

6) Finite Difference Method (FDM): The FDM is a type-II modeling method, which converts an infinite-dimensional model that is presented by PDEs into a finite-dimensional model presented as ODEs.

Zhou et al. [103] adopted FDM to model a flexible-link manipulator. Doan and Nishi [104] used FDM to model the dynamic motion of a flexible riser pipe under vortex-induced vibrations. Esfandiar and Daneshmand [105] employed the Hamiltonian principle to formulate the dynamics of flexible links as PDEs before converting them into ODEs via FDM.

7) Data-Drive System Identification Method: The aforementioned finite-dimensional modeling methods aim to explore the physical principle of FRM dynamics. In contrast, a data-driven system identification method regards FRM dynamics as a black box in a standardized parametric or nonparametric structure before identifying the coefficients of the structure based on samples collected from experiments. Obviously, the data-driven system identification method is a type-II modeling approach.

Around the year 2000, the field of FRM modeling predominantly employed parametric identification methods such as the least mean squares and evolutionary optimizers [106]. Concurrently, nonparametric approaches were gaining traction, notably through the use of neural networks [107] and autoregressive moving average with exogenous inputs embedded with neural networks [108]. As the field evolved, deep learning methods began to surface [109], [110]. An emerging research trend is the physics-inspired deep network [111], which combines data and FRM physical rules, thus showing better interpretability

TABLE III
STRENGTHS AND LIMITATIONS OF FINITE-DIMENSIONAL FRM MODELING METHODS

Modeling method	Advantages	Disadvantages
LPM	● Simplifies complex systems into basic spring-mass elements ● Facilitates matrix-based modeling operations (such as matrix inversion) by ensuring the mass matrices diagonal or semi-diagonal ● Being the simplest approach for modeling FRM dynamics	◆Causes modeling inaccuracy if spring stiffness is not well estimated ◆Fails to accurately model FRMs with non-uniform mass density distributions ◆Requires excessive lumped masses in modeling FRMs with high fidelity
AMM	● Reduces modeling complexity by approximating continuous systems as discrete systems with finite vibrational modes ● Provides an insightful understanding of natural frequencies in FRM dynamics ● Renders low-order models suitable for control design	◆Unable to handle FRM links with varying cross-sections ◆Increases modeling complexity with more assumed modes ◆Requires careful selection of boundary conditions
FEM	● Able to handle complex and nonlinear boundary conditions ● Models irregularly shaped FRMs in an automated and unified way ● Enhances the modeling interpretability using physically meaningful generalized coordinates	◆Overestimates the stiffness matrix, thus leading to unstable closed-loop tracking responses ◆Computationally expensive due to the usage of excessive state-space equations ◆Unable to reflect natural frequencies explicitly in FRM dynamic analysis
TMM	● Avoids model reduction, thus being free from the modal spillover risk ● Well-suited for control designs in the frequency domain ● Maintains low matrix orders regardless of the FRM system dimension ● Easy computer-aided implementation	◆Limited to linear systems with infinite-dimensional transfer functions, thus complicating time-domain dynamic analysis ◆Difficult to model interactions between rigid-body and flexible-body dynamics in an FRM system
FSM	● Effective in modeling large FRM deformations ● Determines FRM modeling precision flexibly by adjusting the segment number	◆Introduces fictitious excitations when the number of deployed finite segments is overly large, which would reduce the modeling accuracy ◆Introduces spikes due to discontinuities in contact force estimations
FDM	● Approximates differential equations via algebraic equations ● Computationally cheap for modeling FRMs with uniform structures ● Supports real-time dynamic analysis	◆Unable to model irregularly shaped FRMs ◆Causes numerical instability in FRM dynamic analysis if the time step or spatial step is set overly large
Data-drive System Identification Method	● Models the FRM dynamics in a standardized parametric or non-parametric structure ● Suitable for modeling highly nonlinear FRM dynamics ● Supports real-time dynamic analysis	◆Relies strongly on the quantity and quality of sampling data but collecting such data might be costly

and fewer learning efforts than purely data-driven modeling methods [112].

Table III summarizes the strengths and limitations of the aforementioned finite-dimensional modeling methods. Other finite-dimensional modeling approaches include the sequential integration (type II) [113], [114], quaternion algebra (type I) [115], linearization (type II) [116], algebra of rotations (type I) [117], and perturbation (type II) [118].

C. Modeling of Joint Flexibility

Besides link flexibility, joint flexibility also exists in an FRM. In most previous studies, a joint of an FRM is typically simplified as a 1-DOF revolute element around its rotation axis while joint motions in other DOFs are ignored. However, joint flexibility accounts for major compliance of an FRM, which improves safety when a manipulator works under physical interactions with humans [119]. The compliance effect is nonnegligible when an FRM has heavy-duty rigid links [120]. Besides that, even minor joint flexibility would largely influence the frequency of a flexible system [121]. Therefore, flexible joints deserve to be accurately modeled. This section reviews the methods that model the dynamics of flexible joints in an FRM.

Spong [122] modeled the flexible joints in an FRM as linear torsional springs with constant stiffness. Moberg [123] improved that linear-spring model by replacing the torsional springs with spring-damper pairs. Nonlinear cubic functions were used to improve the joint flexibility modeling accuracy [124]. Moberg and Hanssen [125] used Kane's method to model a rigid-link flexible-joint robotic manipulator. Dal Verme et al. [126] proposed a four-element approach to replace the conventional linear springs in modeling a cable-driven manipulator. Cui et al. [127] used TMM to build the dynamic model of an industrial robot with 6-DOF flexible joints. Zhang et al. [128] used Spong's linear-spring assumption and Lagrangian equations to model the dynamics of a flexible-joint space manipulator. Shang et al. [129] used AMM to derive the dynamic model for a coupled elastic joint-flexible load drive system, which is composed of a servo motor, an elastic joint, and a flexible load with a tip payload. Do et al. [130] proposed a unified model for a flexible-joint robot that encompasses both prismatic and revolute joints; the motion equations are initially formed by the Lagrangian equations before being linearized via the Taylor series expansion; this linearization facilitates the computation of the flexible robot's modal parameters.

D. Coupled Modeling of Link Flexibility and Joint Flexibility

Link flexibility and joint flexibility often couple together and transform mutually in an FRM, making its dynamics complicated [131]. Thus, it deserves to investigate how to simultaneously model link flexibility and joint flexibility. This section reviews the modeling methods that consider the coupling effects.

Subudhi and Morris [132] proposed a dynamic modeling method for a manipulator with flexible links and flexible joints using Euler–Lagrange formulation and AMM. Yu and Elbestawi [133] employed AMM and Lagrangian equations to develop a comprehensive dynamic model for a two-link planar manipulator, accounting for link and joint flexibility, internal structural damping, joint friction, and gear backlash. A dynamic model for a flexible-link flexible-joint manipulator with an attached payload possessing rotary inertia is derived using Lagrangian equations and FEM [134]. In analyzing the dynamic motions of a flexible-link flexible-joint robot, Zhang and Zhou [135] modeled each flexible joint via Spong's linear-spring assumption and modeled each flexible link via AMM; the complete motion equations are derived via Kane's method. Similarly, Huang et al. [136] modeled each flexible joint via Spong's linear-spring assumption and modeled each flexible link via FEM before deriving the complete motion equations via the Lagrangian principle. Xi and Fenton [137] used Newton–Euler equations to describe link flexibility and used Holzer's method to describe joint flexibility before synthesizing them via AMM. Vakil et al. [138] decoupled a flexible-link flexible-joint manipulator into two subsystems; the first subsystem is a flexible-link and rigid-joint manipulator, whereas the second subsystem reflects joint flexibility and rotors' moment of inertia; the two subsystems are integrated to form a closed-form dynamic model. In deriving the governing motion equations for a manipulator with flexible links and joints, Kumar and Pratiher [139] adopted an extended Hamiltonian principle to integrate link flexibility models and joint flexibility models. Wei et al. [140] introduced a global mode method to derive a concise analytical dynamic model for a flexible-link flexible-joint manipulator; motion equations for both the flexible link and joint are first derived via the Hamiltonian principle and then truncated to low-dimensional ODEs.

IV. Conclusion

This article, as the first part of our survey on FRM systems, has presented the basic background of FRMs, a survey of previous surveys, as well as the existing FRM dynamic modeling methods. In the background introduction, we focus on analyzing the advantages of a flexible manipulator over a rigid-body one. In reviewing previous surveys, we emphasize the new materials that this survey work would present. Regarding the FRM modeling methods, we focus on three FRM types, namely the flexible-link manipulators, flexible-joint manipulators, and manipulators with both link flexibility and joint flexibility. Majority of our review efforts are devoted to the flexible-link manipulators because link flexibility is capturing the attention of scholars more prominently than the other two types. Models for a flexible-link manipulator are divided into infinite-dimensional and finite-dimensional ones, wherein the finite-dimensional models are further classified into two subgroups based on the adopted discretization strategy. Our review indicates that AMM and FEM are applied more widely than other methods in modeling link flexibility. A tradeoff has to be made between modeling accuracy and complexity, which is a common limitation among the existing models. Detailed strengths and limitations are presented in Tables II and III. Besides that, physics-inspired deep network is an emerging research trend, which is promising to conquer the existing challenges.

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