Acceleration of Transient Non-Linear Electromagnetic Field Analyses Using an Automated Subspace Correction Method

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In this paper, we propose a mapping operator generation method for the subspace correction to accelerate transient and/or nonlinear eddy-current analyses. In these analyses, a sequence of linear systems whose coefficient matrices are similar or identical is solved. In our method, we automatically construct the operator using the information obtained in the preceding solution steps. The operator is generated with minimum additional memory requirement through the sampling of approximate solution vectors and the use of the Rayleigh–Ritz's method. Numerical tests using transformer and box-shield models demonstrate that the subspace correction based on the proposed method significantly reduces the number of iterations and the computational time.

Index Terms—**Finite-element method, linear iterative method, nonlinear analysis, subspace correction, transient analysis.**

I. Introduction

The subspace correction method [1] is widely recognized as an acceleration technique for the convergence of a linear iterative solver. It efficiently removes the error involved in a specified subspace, which is designated by a mapping operator (matrix). Practically, the method works well when the image of the mapping operator contains slowly convergent error vectors, which generally coincide with the eigenvectors with the smallest eigenvalues in a symmetric positive definite (s.p.d.) problem [2]. Such an operator can also be used for various convergence acceleration techniques (e.g., deflation method) [3]–[6].

An effective mapping operator for the techniques is mostly derived from knowledge of the problem to be solved. For example, a multigrid method [7] provides one of the most successful operators for various simulations including computational electromagnetics [8]. However, it is often difficult to identify slowly convergent vectors (the eigenvectors with small eigenvalues) from the view point of physics. Moreover, even if the cause of the slow convergence is identified, the construction of the operator is not always straightforward. For example, finite-element analysis using flat elements often results in worse convergence. However, when tetrahedral elements are used, it is difficult to construct the appropriate operator. In this circumstance, we propose an automatic mapping operator generation method for various worse convergence problems in transient and/or nonlinear analyses.

In transient or nonlinear analyses (e.g., transient eddy-current analysis), we solve a sequence of linear systems whose coefficient matrices are similar or identical. The key idea of our method is based on the expectation that we may obtain useful information about slowly convergent error vectors in the preceding solution step [9]. In our technique, a limited number of approximation vectors are preserved in the previous solution steps, and then a mapping operator is generated using the vectors. We expect that the subsequent iterative process is accelerated when using subspace correction with this operator. We examine the proposed technique when applied to transient nonlinear eddy-current analyses, and confirm its effectiveness.

Manuscript received October 25, 2018; revised January 24, 2019; accepted March 1, 2019. Date of publication March 28, 2019; date of current version May 16, 2019. Corresponding author: T. Iwashita (e-mail: iwashita@iic.hokudai.ac.jp).

Digital Object Identifier 10.1109/TMAG.2019.2903837

II. Subspace Correction Method

Let us consider an n-dimensional linear system of equations $A\boldsymbol{x} = \boldsymbol{b}$, where A is the coefficient matrix, $\boldsymbol{x}$ is the unknown vector, and $\boldsymbol{b}$ is the right-hand side vector.

In the subspace correction method, approximate solution vector $\tilde{\boldsymbol{x}}$ is updated as follows.

Step 1: Compute $\boldsymbol{f} = B^{\top}(\boldsymbol{b} - A\tilde{\boldsymbol{x}})$.

Step 2: Solve $(B^{\top}AB)\boldsymbol{u} = \boldsymbol{f}$.

Step 3: Update $\tilde{\boldsymbol{x}} \leftarrow \tilde{\boldsymbol{x}} + B\boldsymbol{u}$.

where B is the auxiliary (mapping) matrix, the choice of which is the key to making the method work. In some practical applications, the linear system is approximately solved in Step 2. Subspace correction is the generalized version of coarse grid correction in the multigrid method. Therefore, before Step 1 and after Step 3, the approximation vector is often updated by the smoothing process.

A. Preconditioning Based on Subspace Correction

When the subspace correction method is used with a Krylov subspace solver, it is applied to the linear system as preconditioning. Preconditioning based on subspace correction can be combined with any other (standard) preconditioning techniques in the additive/multiplicative Schwarz preconditioning manner. When the original (stand-alone) preconditioner is denoted by M and its preconditioning step is written as

$$z = M^{-1}\boldsymbol{r} \tag{1}$$

where $\boldsymbol{r}$ is the residual vector, additive Schwarz type preconditioning with subspace correction is given as follows.

Step 1: Solve $Mz = \boldsymbol{r}$.

Step 2: Compute $\boldsymbol{f} = B^{\top}\boldsymbol{r}$.

Step 3: Solve $(B^{\top}AB)\boldsymbol{u} = \boldsymbol{f}$.

Step 4: Compute $z_c = z + B\boldsymbol{u}$.

When the preconditioning step with subspace correction is denoted by $z_c = M_c^{-1}\boldsymbol{r}$, preconditioner M_c^{-1} is given by

$$M_c^{-1} = M^{-1} + B(B^\top AB)^{-1}B^\top. \quad (2)$$

III. Mapping Operator Construction Method

In this paper, we consider the application of the (preconditioned) conjugate gradient (CG) method to a symmetric positive or semi-positive definite problem, which often appears in eddy-current analyses. Subspace correction is expected to work when the mapping operator corresponds to the span of eigenvectors of small positive eigenvalues. Our main idea is to identify these eigenvectors using the information obtained in the previous nonlinear iteration or time steps.

A. Problem Description

We focus on solving a series of n-dimensional linear systems

$$A_k\boldsymbol{x}_k = \boldsymbol{b}_k, \quad (k = 1, 2, \ldots, k_t). \quad (3)$$

It is assumed that the linear systems (3) should be solved sequentially because the right-hand side vector $\boldsymbol{b}_k$ depends on the previous solution vectors. Furthermore, we consider the problem in which the coefficient matrix A_k in (3) does not significantly change. The problem often arises in various transient (nonlinear) eddy-current analyses.

B. Proposed Mapping Operator Construction Method

This section describes how to construct the mapping operator at the kth step, which is denoted by B_k. B_k is used for subspace correction in the (k+1)th solution step. During the kth solution step, m_k approximation vectors $\tilde{\boldsymbol{x}}_k^{(s)}(s = 1, 2, \ldots, m_k)$ are preserved. In our method, m_k is much smaller than n. The selection method for $\tilde{\boldsymbol{x}}_k^{(s)}$ is described in Section III-C. After the solution process is completed, the error vectors that correspond to $\tilde{\boldsymbol{x}}_k^{(s)}$ are calculated by

$$\boldsymbol{e}_k^{(s)} = \boldsymbol{x}_k - \tilde{\boldsymbol{x}}_k^{(s)}. \quad (4)$$

Based on the space spanned by $\boldsymbol{e}_k^{(s)}$, we use the Rayleigh–Ritz method to identify approximate right-singular vectors with small nonzero singular values, which are approximate eigenvectors with small eigenvalues in an s.p.d. problem.

We use the Rayleigh–Ritz method using the basis

$$E_k = \left[\boldsymbol{e}_k^{(1)}\,\boldsymbol{e}_k^{(2)}\,\cdots\,\boldsymbol{e}_k^{(m_k)}\right]. \quad (5)$$

Step 1: Solve the m_k-dimensional generalized eigenvalue problem as follows:

$$E_k^\top A_k^\top A_k E_k \boldsymbol{t} = \lambda E_k^\top E_k \boldsymbol{t}. \quad (6)$$

Step 2: When Ritz value λ is less than given threshold θ^2 ($0 < \theta \ll 1$), Ritz vector $E_k\boldsymbol{t}$ is adopted as a column vector of mapping operator B_k, where the number of Ritz values less than θ^2 is denoted by $\tilde{m}_k$ and the Ritz vector that corresponds to each small Ritz value is written as $\boldsymbol{p}_i (i = 1, 2, \ldots, \tilde{m}_k)$. Ritz vector $\boldsymbol{p}_i$ satisfies $\|A_k\boldsymbol{p}_i\|/\|\boldsymbol{p}_i\| < \theta$, and it corresponds to the approximate eigenvector whose eigenvalue is less than θ when the problem is s.p.d. Finally, the mapping operator constructed at the kth step is given by

$$B_k = (\boldsymbol{p}_1\,\boldsymbol{p}_2\,\cdots\,\boldsymbol{p}_{\tilde{m}_k}). \quad (7)$$

Although the mapping operator (7) is mainly used in the numerical test, we can efficiently use the vectors identified in the previous steps. It is an effective option for linear transient analysis in which the coefficient matrix does not vary. In this method, we use the Gram–Schmidt process, and obtain the mutually orthogonal $\bar{m}_k$ ($\leq m_k$) basis vectors that satisfy

$$\bar{\boldsymbol{e}}_k^{(d)} \in \operatorname{Span}\left\{\boldsymbol{e}_k^{(1)}\,\boldsymbol{e}_k^{(2)}\,\cdots\,\boldsymbol{e}_k^{(m_k)}\right\}, \quad d = 1, 2, \ldots, \bar{m}_k \quad (8)$$

and

$$\bar{\boldsymbol{e}}_k^{(d)} \perp \mathrm{R}(B_{k-1}), d = 1, 2, \ldots, \bar{m}_k \quad (9)$$

where $\mathrm{R}(B_{k-1})$ denotes the range of B_{k-1}. After we perform the Rayleigh–Ritz method with the above basis vectors, the mapping operator constructed at the kth step is finally obtained by

$$\boldsymbol{B}_k = (\boldsymbol{B}_{k-1}\,\boldsymbol{p}_1\,\boldsymbol{p}_2\,\cdots\,\boldsymbol{p}_{\tilde{m}_k}). \quad (10)$$

C. Selection Method for Stored Approximation Vectors

In practical analyses, to avoid an excessive additional cost (in memory space and computations), the number of stored vectors, m_k, should be substantially small. We use the selection method based on “sampling.” We intend to store approximate solution vectors with a certain interval in the solution process. Although it is difficult to predict the number of iterations necessary for convergence, Algorithm 1 can preserve m approximation solution vectors that satisfy the following property with the minimum memory space, where $\tilde{\boldsymbol{x}}_{i_{te}}$ is the approximate solution vector at the i_{te}th iteration of the linear solver. The interval in the iteration count between the vectors is 2^d or 2^{d+1}, where d is a positive integer that satisfies $d < \log_2(N/m) \leq d + 1$. In the algorithm, N is the total iteration count and $l_{\max}$ is a given parameter that satisfies $m^{l_{\max}} > N_{\max}$, where $N_{\max}$ is the maximum iteration count of the linear solver. If the iterative solver converges at the 5000th iteration (i.e., $N = 5000$) with the setting of $m = 4$, the approximate solution vectors at the 1024, 2048, 3072, and 4096th iterations are preserved.

IV. Numerical Results

A. Test Conditions

In the following two numerical tests, we conduct transient nonlinear eddy-current analyses. The A-phi formulation and finite-element method using edge and nodal elements are used in the analysis. The B–H curve is given to simulate the nonlinear magnetic property of iron parts. The source current density is properly set to obtain a consistent linear system [10]. The analysis code consists of three nested loops: time, nonlinear iteration, and linear solver iteration. The nonlinear (Newton–Raphson) iteration stops when the correction of the magnetic flux density is less than 10^{-2} in all finite elements. In the linear solver, the diagonally scaled linear system is solved using the (preconditioned) CG method.

Algorithm 1 Selection of Approximate Solution Vectors

$h = 1$
for $i_{te} = 1, 2, \cdots$ **do**
  Solver part
  Convergence check
  if $(\mathrm{mod}(i_{te}, h) == 0)$ **then**
    $i_t = \sum_{l=0}^{l_{\max}} (-1)^l \lfloor (i_{te} - 1)/m^l \rfloor$
    $j = \mathrm{mod}(i_t, m) + 1$
    $\tilde{\boldsymbol{x}}^{(j)} = \tilde{\boldsymbol{x}}_{i_{te}}$
    if $(i_{te} == h * m)$ **then**
      $h = h * 2$
    end if
  end if
end for

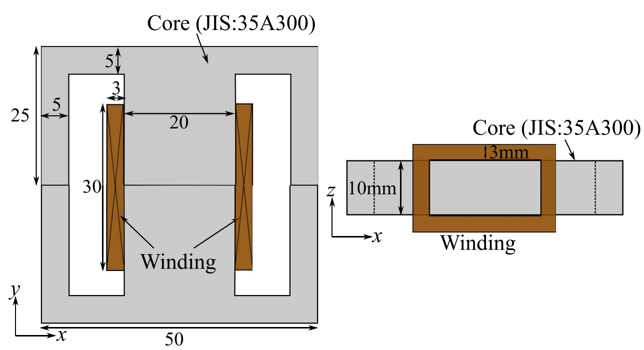

Fig. 1. Transformer test model.

TABLE I
EFFECT OF THE PROPOSED SUBSPACE CORRECTION METHOD ON TRANSFORMER MODEL ANALYSIS

Preconditioning	Total number of iterations	Total elapsed time (s)
DP	211,991	1,796
DP+SC	113,976	1,413
IC	76,065	1,490
IC+SC	41,126	1,126

(DP: diagonal preconditioning)
(SC: additive Schwarz preconditioning based on subspace correction)
(IC: incomplete Cholesky factorization preconditioning)

When the incomplete Cholesky (IC) preconditioning is used, the shifted IC factorization is used with the acceleration factor of 1.05. Threshold θ in the proposed method is set to be 10^{-3}. The analysis was conducted on a PC with Intel Core i7-6700, 3.40 GHz. The program was complied with Intel C++ compiler XE 16.0.

B. Transient Nonlinear Analysis of a Transformer Model

In this section, the EE core transformer shown in Fig. 1 is used as a test model. We analyzed the one-eighth model, considering the symmetry of the model. The tetrahedral element was used and the number of elements was 253 788. One period was divided into 30 time steps and the electromagnetic field behavior during three periods was analyzed.

Table I shows the effect of the proposed method, where DP and SC denote the diagonal preconditioning and additive Schwarz preconditioning based on the proposed subspace correction, respectively. Using the proposed method together with standard preconditioning techniques, the convergence of the iterative solver significantly accelerated. The proposed method achieved approximately 45% and 25% reduction in the number of iterations and computational time, respectively, compared with the standard incomplete Cholesky conjugate gradient (ICCG) solver.

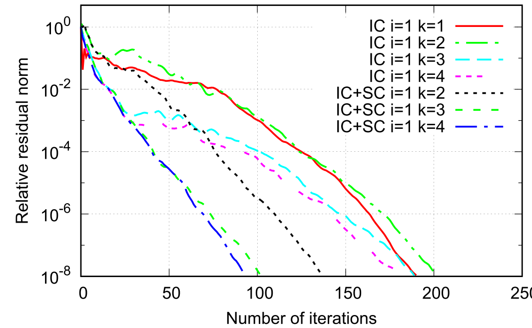

Fig. 2. Convergence behavior of the IC and IC+SC CG solvers (transformer analysis).

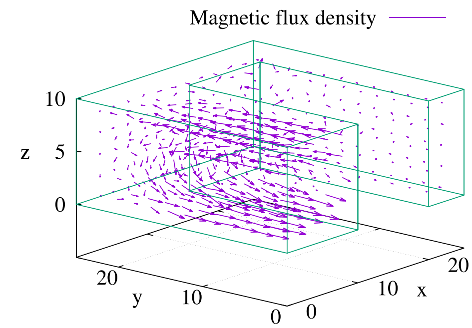

Fig. 3. Magnetic flux density distribution that corresponds to the approximate eigenvector with small eigenvalue.

Fig. 2 shows the convergence behavior of the ICCG solver with/without the proposed correction method, where k is the time step and i is the nonlinear iteration count. In the first nonlinear iteration, the normal ICCG method was used, and during the iteration process, the mapping operator for the next iteration was generated. Fig. 2 indicates that the generated mapping operator is effectively used to accelerate convergence in the subsequent iteration step.

When the proposed method was used with IC preconditioning, at most 22 (approximate) eigenvectors with small positive eigenvalues were identified. We observed that some of these vectors were identified only in specific time steps. For example, one of these vectors, which we denote by $\boldsymbol{p}_p$, was found only when the amplitude of the external voltage was relatively high. Fig. 3 demonstrates the magnetic flux density distribution that corresponds to $\boldsymbol{p}_p$. Fig. 3 indicates that vector $\boldsymbol{p}_p$ corresponds to the magnetic flux that circulates inside the central part of the EE core. However, it is difficult to identify the vector from the viewpoint of the model information because the convergence issue related to $\boldsymbol{p}_p$ does not arise in every time step. The result confirms that the advantage of the proposed method is derived from the automatic identification of the vectors that cause a worse convergence problem.

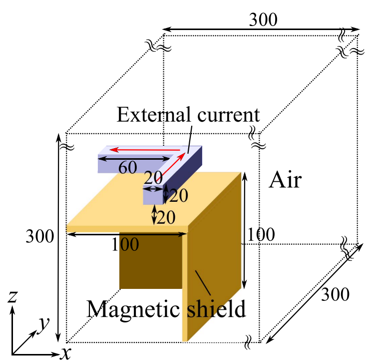

Fig. 4. Box shield test model.

TABLE II
EFFECT OF THE PROPOSED SUBSPACE CORRECTION METHOD ON BOX-SHIELD MODEL ANALYSIS

Preconditioning	Total number of iterations	Total elapsed time (s)
DP	691,241	4,015
DP+SC	255,308	2,462
IC	150,645	1,977
IC+SC	61,272	1,082

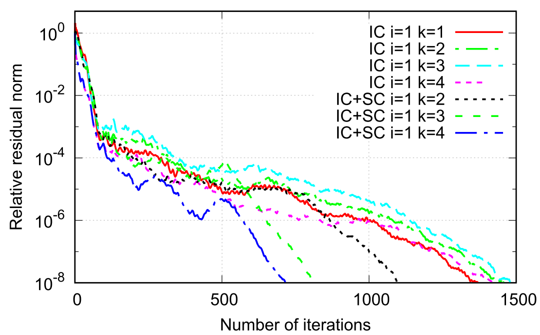

Fig. 5. Convergence behavior of the IC and IC+SC CG solvers (box shield analysis).

C. Transient Nonlinear Analysis of a Box-Shield Model

In this section, we describe the numerical result of transient nonlinear analysis for the box-shield model shown in Fig. 4. It is noted that Fig. 4 demonstrates one-eighth of the entire analysis domain. The magnetic property of the shield is set to be the same as that of the plunger in the TEAM workshop problem 20. The model was discretized by 184 042 tetrahedral elements. The electromagnetic field behavior in 60 time steps was analyzed.

Table II indicates that the proposed method significantly reduced the number of iterations and computational time in both diagonal and IC preconditioning cases. When the proposed method was used with IC preconditioning, it reduced the number of iterations and computational time by approximately 60% and 45%, respectively. Fig. 5 shows the convergence behavior of the ICCG solver with/without the proposed correction method in the first four nonlinear iterations of the first time step. The proposed method, which used information obtained in the preceding solution step, accelerated the convergence of the linear solver.

In box-shield analysis, the aspect ratio of the elements in the shield tends to be large, which results in an ill-conditioned problem. In our method, an appropriate mapping operator was generated for the problem, even when the model was discretized with tetrahedral elements. Accordingly, the proposed method can be effective for various discretized models that involve (unstructured) high aspect ratio elements.

V. CONCLUSION

In this paper, we proposed a mapping operator generation method for subspace correction to accelerate transient and/or nonlinear eddy-current analyses. In our method, the operator is automatically generated using the information obtained in the preceding time or nonlinear iteration steps. Transient nonlinear analyses with the transformer and the box shield models demonstrated that the proposed method attained a significant reduction in the number of iterations and computational time, respectively. Although the numerical result is not shown here because of a lack of space, we also confirmed the effectiveness of the proposed method in transient linear analyses.

To the best of our knowledge, the work by Gosselet *et al.* [11] is the most closely related to ours. However, in their method, all residual vectors in the previous solution steps are preserved, which requires huge additional memory space. Consequently, our method has significantly different memory requirements from Gosselet's method, which is important in the application to practical analyses.

In future work, we will examine our method for more numerical test models with various time-step sizes. Moreover, we will investigate the effect of the method when it is used with various nonlinear and linear iteration methods, for example, fixed point iterations and a multigrid method.

ACKNOWLEDGMENT

This work was supported by JSPS KAKENHI under Grant 26540052.

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