The Connections Among Hamming Metric, b-Symbol Metric, and r-th Generalized Hamming Metric

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***Abstract*— The r-th generalized Hamming metric and the b-symbol metric are two different generalizations of Hamming metric. The former is used on the wire-tap channel of Type II, and the latter is motivated by the limitations of the reading process in high-density data storage systems and applied to a read channel that outputs overlapping symbols. In this paper, we study the connections among the three metrics (that is, Hamming metric, b-symbol metric, and r-th generalized Hamming metric) mentioned above and give a conjecture about the b-symbol Griesmer Bound for cyclic codes.**

***Index Terms*— Hamming metric, b-symbol metric, r-th generalized Hamming metric, unrestricted codes, Griesmer bound.**

I. Introduction

THE concept of r-th generalized Hamming metric first appeared in the 1970s and was proposed by Helleseth et al. [17], [22]. Until 1991, in Wei's research [27] on wire-tap channel of Type II, Wei mentioned this concept again and provided a series of excellent conclusions. Subsequently, many researchers studied the weight hierarchy of several series of linear codes (e.g. RM codes [16], [27], BCH codes [5], [13], [14], trace codes [23], cyclic codes [12], [20], etc.). The bounds, asymptotic behaviour, and the duality under r-th generalized Hamming metric have been considered in [1], [18], [19], [22], [26], and [27]. In addition to its applications in wire-tap channels of type II, the r-th generalized Hamming metric is also used to address t-resilient functions and trellis or branch complexity of linear codes [26].

The b-symbol metric is another generalization of the Hamming metric that has been proposed by Cassuto and Blaum [2], [3] in recent years. It differs from r-th generalized Hamming metric in that its research motivation is not derived from data storage or cryptography but from other domains such as molecular biology and chemistry. In these domains, the information redundancy is so low that the only effective way to combat errors is to transmit the same message over and over again (overlapping symbols). Although in practical applications, consecutive symbols may affect each other, in the traditional read channel, people always assume that the adjacent symbols are individual. However, with the development of the high-density data storage technologies, this is no longer a reasonable assumption, and symbols are faced with the need to be read repeatedly (since the bit size at high-densities is small, it is hard to read the individual bits). This is why we have to pay attention to the b-symbol channel, which is a read channel suitable for the output of overlapping b-symbols. Under this new metric (or paradigm), errors are no longer single symbolic errors but b-symbolic errors. At present, the research progress of the b-symbol metric includes the bounds of codes (e.g. b-symbol Sphere Packing Bound [2], [3], b-symbol Singleton Bound [6], [10], b-symbol Linear Programming Bound [11], b-symbol asymptotic bound [4], etc), the decoding and the constructions. The research on the codes that reach the b-symbol Singleton Bound (such codes are called b-symbol MDS codes) is a hot topic, and a lot of relevant research progress has been achieved [6], [7], [8], [9], [10], [21]. It is very difficult to determine the b-symbol weight distribution or the minimum b-symbol distance of linear codes. Nevertheless, in some special cases, the b-symbol weight distributions are determined [24], [25], [30].

These two metrics receive wide attention by researchers because they are generalizations of the Hamming metric. Our motivation is to investigate the connections and differences between these two metrics. Since the r-th generalized Hamming metric has a longer history than the b-symbol metric, we can refer to the research progress of r-th generalized Hamming metric when we consider the b-symbol metric. In this paper, we first establish the connection between the Hamming metric and the b-symbol metric. Although the connection has been considered in [28] and [29], we get better results (e.g., Theorem 4 is a generalization of Lemma 1 in [28] and [29], and Theorem 6 is a generalization of Proposition 2 in [28] and [29]). Subsequently, we compare the same linear codes under the b-symbol metric and r-th generalized Hamming metric. When C is cyclic, we prove that $d_b(C) \geq \mathbf{d}_b(C)$, where $d_b(C)$ and $\mathbf{d}_b(C)$ denote the minimum b-symbol weight and the minimum b-th generalized Hamming weight of C, respectively. In fact, the two metrics have a lot

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in common when C is cyclic. We also propose a conjecture on the b-symbol Griesmer Bound for cyclic codes.

The rest of the paper is organized as follows. In Section II, we introduce some notations and definitions. In Section III, we show the connections among Hamming metric, b-symbol metric, and r-th generalized Hamming metric. Section IV concludes the paper.

II. Preliminaries

Throughout this paper, we assume and fix the following:

- $\mathbb{F}_q$: finite field with q elements.
- $\mathbb{F}_q^* = \mathbb{F}_q \backslash \{0\}$.
- $p = \mathrm{Char}(\mathbb{F}_q)$.
- $\mathbf{x}, \mathbf{y}$ are two vectors which belong to $\mathbb{F}_q^n$.
- Let α be an element of $\mathbb{F}_q$ and let $\langle \alpha^i \rangle$ be the group generated by α^i, where i is an integer.
- If C is an unrestricted code (it may be linear or non-linear), we use the notation $(n, M, d_b(C))_q$ to denote its parameters, where n is the length of C, M is the size of C, $d_b(C)$ is the b-symbol minimum distance of C, and q indicates that its alphabet is $\mathbb{F}_q$. If C is a linear code, we denote its parameters by the notation $[n, k, d_b(C)]_q$, where k denotes the dimension of the C.

A. Hamming Metric

- Hamming weight $w_H(\mathbf{x})$: the number of nonzero coordinates in $\mathbf{x}$.
- Hamming distance $d_H(\mathbf{x}, \mathbf{y})$: the number of coordinates in which $\mathbf{x}$ and $\mathbf{y}$ differ.

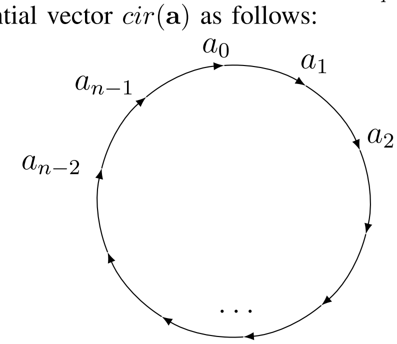
B. The b-Symbol Metric

Let b be a positive integer with $1 \le b \le n$.

- b-symbol weight $w_b(\mathbf{x})$: the Hamming weight of $\pi_b(\mathbf{x})$, where $\pi_b(\mathbf{x}) \in (\mathbb{F}_q^b)^n$ and
$$\begin{aligned}\pi_b(\mathbf{x}) \;\; = \;\; & ((x_0, \ldots, x_{b-1}), (x_1, \ldots, x_b), \cdots, \\ & (x_{n-1}, \ldots, x_{b+n-2 (\mathrm{mod}\ n)})).\end{aligned}$$
- b-symbol distance $d_b(\mathbf{x}, \mathbf{y})$:
$$d_b(\mathbf{x}, \mathbf{y}) = w_b(\mathbf{x} - \mathbf{y}).$$

When $b = 1$, $w_1(\mathbf{x}) = w_H(\mathbf{x})$ and $d_1(\mathbf{x}, \mathbf{y}) = d_H(\mathbf{x}, \mathbf{y})$. For convenience, we use $w_1(\mathbf{x})$ and $d_1(\mathbf{x}, \mathbf{y})$ to represent $w_H(\mathbf{x})$ and $d_H(\mathbf{x}, \mathbf{y})$, respectively.

For $\lambda \in \mathbb{F}_q$ and $\eta, \xi \in \mathbb{F}_q \setminus \{\lambda\}$, if $\eta, \underbrace{\lambda, \ldots, \lambda}_{m}, \xi$ appears in the sequence $\bar{\mathbf{a}}$, then we say that $\underbrace{\lambda, \ldots, \lambda}_{m}$ is a run of λ's of length m. Let $\mathbf{0}_i = (\alpha, \underbrace{0, \ldots, 0}_{i}, \beta)$, where $\alpha, \beta \in \mathbb{F}_q^*$. For any vector $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_q^n$, we define a circumferential vector $cir(\mathbf{a})$ as follows:



and $\Psi(\mathbf{a}, \mathbf{0}_i)$ denotes the number of occurrences of $\mathbf{0}_i$ on the circumferential vector $cir(\mathbf{a})$. The $\mathbf{0}$'s run distribution of $\mathbf{a}$ is defined by $\{\Psi(\mathbf{a}, \mathbf{0}_1), \Psi(\mathbf{a}, \mathbf{0}_2), \ldots, \Psi(\mathbf{a}, \mathbf{0}_n)\}$.

Example 2.1: Let $\mathbf{a} = (01001000100)$. Then the $\mathbf{0}$'s run distribution of $\mathbf{a}$ is $\{\Psi(\mathbf{a}, \mathbf{0}_2) = 1, \Psi(\mathbf{a}, \mathbf{0}_3) = 2, \Psi(\mathbf{a}, \mathbf{0}_i) = 0, i \neq 2, 3\}$.

For any vector
$$\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}_q^n,$$
by the definition of b-symbol weight, then we have
$$\begin{aligned} w_b(\mathbf{c}) \;\; = \;\; n - \Big| & \{ i | c_i = c_{i+1} = \cdots \\ & = c_{i+b-1} = 0, 0 \le i \le n-1 \} \Big|. \end{aligned}$$
If the $\mathbf{0}$'s run distribution of $\mathbf{a}$ is given, we have the following formula to calculate the b-symbol weight of $\mathbf{c}$. Sometimes we just write $\Psi(\mathbf{0}_i)$, instead of $\Psi(\mathbf{a}, \mathbf{0}_i)$, if the vector $\mathbf{a}$ is clear from the context.

Theorem 2.2: For any vector
$$\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}_q^n,$$
we have
$$w_b(\mathbf{c}) = n - \sum_{i=b}^{n-1} (i - b + 1) \cdot \Psi(\mathbf{c}, \mathbf{0}_i). \tag{1}$$

Proof: By the definition of the b-symbol metric, the zero coordinate of $\mathbf{c}$ is b cyclic consecutive zeros. For a
$$\mathbf{0}_i = (\alpha, \underbrace{0, \ldots, 0}_{i}, \beta),$$
there are $i - b + 1$ zero coordinates if $i \ge b$ and there is no zero coordinate in $\mathbf{0}_i$ if $i < b$. Counting the number of the zero coordinates, we obtain the desired result. ∎

From Theorem 2.2, one can induce the following result directly.

Corollary 2.3: Suppose $\Psi(\mathbf{x}) = \Psi(\mathbf{y})$. Then for $1 \le b \le n$, $w_b(\mathbf{x}) = w_b(\mathbf{y})$.

C. Cyclic Codes

A q-ary linear code C of length n is cyclic if C is invariant under the cyclic shift τ, i.e.,
$$\tau(c_0, c_1, \ldots, c_{n-1}) = (c_{n-1}, c_0, \ldots, c_{n-2}) \in C,$$
where $(c_0, c_1, \ldots, c_{n-1}) \in C$.

For a polynomial $f(x)$ over $\mathbb{F}_q$, the period (or order) of $f(x)$ is the least positive integer t such that $f(x) | (x^t - 1)$, denoted by $per(f) = t$.

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{F}_q[x]$ with $a_n \neq 0$. The reciprocal polynomial $f^*(x)$ is defined by
$$\begin{aligned} f^*(x) \;\; & = \;\; x^n f\left(\frac{1}{x}\right) \\ & = \;\; a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n. \end{aligned}$$

D. Linear Feedback Shift Register

A shift register is converted into a code generator by including a feedback loop, which computes a new term for the left-most stage, based on the n previous terms. The n q-ary storage elements a_i are called the stages of the shift register, and their contents $\xi_i = (a_i, a_{i+1}, \ldots, a_{i+n-1})$ are called the states of the shift register. The shift register is run by an external clock which generates a timing signal every t_0 seconds. A delay element stores one bit (from some alphabet) for one clock cycle, after which the bit is pushed out and replaced by another bit. A linear shift register is a series of delay elements; a bit enters at one end of the shift register and moves to the next delay element with each new clock cycle. A linear feedback shift register (LFSR for short) is a linear shift register in which the output is fed back into the shift register as part of the input.

Let $V(\mathbb{F}_q)$ be a set consisting of all infinite sequences whose elements are taken from $\mathbb{F}_q$; that is,

$$V(\mathbb{F}_q) = \{\overline{\mathbf{a}} = (a_0, a_1, a_2, \ldots) | a_i \in \mathbb{F}_q\}.$$

More specifically, an LFSR sequence is a sequence $\overline{\mathbf{a}} = (a_0, a_1, a_2, \ldots)$ in $V(\mathbb{F}_q)$ whose elements satisfy the linear recursive relation

$$a_{n+k} = \sum_{i=0}^{n-1} c_i a_{k+i}, \quad k = 0, 1, \ldots,$$

where $c_i \in \mathbb{F}_q$. Let $\xi_0 = (a_0, a_1, \ldots, a_{n-1})$ be the initial state of $\overline{\mathbf{a}}$, we can compute the succession of states of $\overline{\mathbf{a}}$ by the linear recursive relation.

For $\overline{\mathbf{a}} = (a_0, a_1, a_2, \ldots) \in V(\mathbb{F}_q)$, one defines (left) shift operator $\mathcal{L}^i$ as follows: $\mathcal{L}^i \overline{\mathbf{a}} = (a_i, a_{i+1}, a_{i+2}, \ldots)$. For any infinite sequence $\overline{\mathbf{a}} \in V(\mathbb{F}_q)$, if there exists a non-zero monic polynomial $f(x) \in \mathbb{F}_q[x]$ such that

$$f(\mathcal{L})\overline{\mathbf{a}} = 0,$$

then $\overline{\mathbf{a}}$ is called a linear recursive sequence. The polynomial $f(x)$ is called the characteristic polynomial of $\overline{\mathbf{a}}$ over $\mathbb{F}_q$. The reciprocal polynomial of $f(x)$ is called the feedback polynomial of $\overline{\mathbf{a}}$. For any non-zero polynomial $f(x) \in \mathbb{F}_q[x]$, we use $G(f(x))$ to denote the set consisting of all sequences in $V(\mathbb{F}_q)$ with $f(\mathcal{L})\overline{\mathbf{a}} = 0$.

If there exist integers $r > 0$ and $u > 0$ such that $a_{i+r} = a_i$ for all $i \geq u$, then the sequence $\overline{\mathbf{a}}$ is said to be ultimately periodic with parameters (r, u), and the smallest integer r is called a period of the sequence and denoted by $per(\overline{\mathbf{a}})$. If $u = 0$, then the sequence is said to be periodic. Two periodic sequences $\overline{\mathbf{a}} = \{a_i\}$ and $\overline{\mathbf{b}} = \{b_i\}$ are called cyclically shift equivalent if there exists an integer k such that $a_i = b_{i+k}$, $\forall i \geq 0$. In this case, we write $\overline{\mathbf{a}} = \mathcal{L}^k \overline{\mathbf{b}}$, or simply $\overline{\mathbf{a}} \sim \overline{\mathbf{b}}$. Otherwise, they are called cyclically shift distinct. It is not difficult to see that each linear recursive sequence is periodic.

Remark 2.4: Different initial states maybe obtain different state cycles. For instance, let $f(x) = x^4 + x^3 + x^2 + x + 1 \in \mathbb{F}_2[x]$ be a characteristic polynomial of $\overline{\mathbf{a}}$ over $\mathbb{F}_2$. When the initial state $\xi_0 = (0001)$, the next states are $\xi_1 = (0011)$, $\xi_2 = (0110)$, $\xi_3 = (1100)$. When the initial state ξ_0 does not belong to the first cycle, i.e., $\xi_0 = (0101)$, the second cycle is obtained. Four cycles are corresponding to different initial states, as shown in Fig 1.

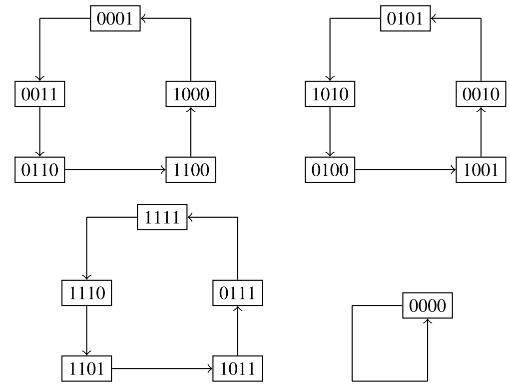


Fig. 1. The state diagram of the 4-stage LFSR.

A set in which all sequences are cyclically shifted equivalent is called a shift-equivalent class. One cyclically shift equivalent class of $G(f(x))$ corresponds to one cycle of states in the state diagram of the LFSR with characteristic polynomial $f(x)$.

Proposition 2.5 ([15, P.100]): Let $f(x)$ be an irreducible polynomial over $\mathbb{F}_q$ of degree n. Then the number of cyclically shift equivalent classes of non-zero LFSR sequences in $G(f)$ is given by $\frac{q^n-1}{per(f)}$.

In the language of the state diagram, this proposition shows that for an LFSR with an irreducible polynomial, there are $\frac{q^n-1}{per(f)}$ cycles with length $per(f)$ and one cycle of length 1 in its state diagram.

A parity check matrix $H_{q,k}$ of the Hamming code $\mathcal{H}_{q,k}$ over $\mathbb{F}_q$ is defined by choosing for its columns a nonzero vector from each one-dimensional subspace of $\mathbb{F}_q^k$. The duals of the Hamming codes $\mathcal{H}_{q,k}$ are called simplex codes $\mathcal{S}_{q,k}$, which have parameters $[\frac{q^k-1}{q-1}, k, d_1 = q^{k-1}]_q$. In fact, the nonzero codewords of $\mathcal{S}_{q,k}$ all have Hamming weight q^{k-1}.

Remark 2.6: A linear code C is called a single **0**'s run code if for any $\mathbf{c}, \mathbf{c}' \in C \backslash \{\mathbf{0}\}$, $\mathbf{c}$ and $\mathbf{c}'$ have the same **0**'s run distribution. For example, $\mathcal{S}_{q,k}$ is a single **0**'s run code. For any $\mathbf{c}$ in $\mathcal{S}_{q,k}$, we have

$$\Psi(\mathbf{c}, \mathbf{0}_i) = \begin{cases} (q-1)q^{k-2-i}, & 1 \leq i \leq k-2; \\ 1, & i = k-1; \\ 0, & i \geq k. \end{cases}$$

Its **0**'s run distribution can be deduced from [15, Page.123, R-2]. By Theorem 2.2, we have the minimum b-symbol weight of $\mathcal{S}_{q,k}$ is $\frac{(q^b-1)q^k}{q^b(q-1)}$. Obviously, $\mathcal{S}_{q,k}$ is also a constant b-symbol weight linear code.
The single **0**'s run codes are of interest in their own right. The following result gives a class of single **0**'s run codes and their parameters.

Theorem 2.7: Let Δ be a factor of $q^k - 1$. Let $n = \frac{q^k-1}{\Delta}$ and C be an irreducible cyclic code over $\mathbb{F}_q$ with parity-check polynomial $h(x)$ of degree $\deg(h(x)) = k$ and period (or order) $\mathrm{per}(h(x)) = n$.

If $\Delta | q-1$ and $\gcd(n, \Delta) = 1$, then C is a single **0**'s run code with parameters $[n, k, d_b(C) = \frac{(q^b-1)q^k}{q^b \Delta}]_q$ where $1 \leq b \leq k-1$.

Proof: We use $G(h^*(x))$ to denote the set consisting of all sequences with $h^*(\tau)\bar{\mathbf{a}} = 0$ where $\bar{\mathbf{a}} = (a_0, a_1, a_2, \dots)$ is a nonzero sequence and τ denotes the left shift operator. Let α be a primitive element of $\mathbb{F}_q$. If $\gcd(\mathrm{per}(h(x)), \Delta) = 1$, we claim that the codewords $\mathbf{x}$ and $\mathbf{y}$ are cyclically shift distinct if there exists $\delta \in \langle \alpha^{\frac{q-1}{\Delta}} \rangle \setminus \{1\} \subseteq \mathbb{F}_q^*$ such that $\mathbf{x} = \delta \cdot \mathbf{y}$. The claim can be proved by showing that for a non-zero sequence $\bar{\mathbf{a}} \in G(h^*(x))$, it is impossible for both states ξ_i and ξ_j of $\bar{\mathbf{a}}$ such that $\xi_i = \delta \cdot \xi_j$ where $i \neq j$. Assume that there are two states ξ_i and ξ_j of $\bar{\mathbf{a}}$ such that $\xi_i = \delta \cdot \xi_j$ for some $\delta \in \langle \alpha^{\frac{q-1}{\Delta}} \rangle \setminus \{1\}$. As it can be seen from the recurrence relation,

$$\begin{aligned}\xi_i &= \delta \cdot \xi_j = \delta^2 \cdot \xi_{j+j-i} \\ &= \cdots = \delta^{\Delta+1} \cdot \xi_{j+\Delta(j-i)} = \delta \cdot \xi_{j+\Delta(j-i)},\end{aligned}$$

then

$$\mathrm{per}(h(x)) | \Delta(j-i).$$

Since

$$\gcd(\mathrm{per}(h(x)), \Delta) = 1,$$

we have

$$\mathrm{per}(h(x)) | (j-i),$$

this leads to a contradiction. Therefore, the non-zero codewords

$$\mathbf{x}, \alpha^{\frac{q-1}{\Delta}}\mathbf{x}, (\alpha^{\frac{q-1}{\Delta}})^2\mathbf{x}, \dots, (\alpha^{\frac{q-1}{\Delta}})^{\Delta-1}\mathbf{x}$$

are from distinct cycles. Since there are Δ cycles (we ignore the cycle corresponding to zero sequence), C has one non-zero b-symbol weight.

We claim that for any non-zero codeword $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in C$, the $\mathbf{0}$'s run distribution of $\mathbf{c}$ is

$$\Psi(\mathbf{c}, \mathbf{0}_i) = \begin{cases} \frac{(q-1)^2}{\Delta} q^{k-2-i}, & 1 \le i \le k-2; \\ \frac{q-1}{\Delta}, & i = k-1; \\ 0, & i \ge k. \end{cases}$$

There is no $\mathbf{0}$'s run with length greater than or equal to k since $\bar{\mathbf{a}}$ is not a zero state. When $1 \le i \le k-2$, we add $k-i-2$ coordinates $(b_1, b_2, \dots, b_{k-i-2})$ behind $(a_1, \underbrace{0, \dots, 0}_{i}, a_2)$, where $a_1, a_2 \in \mathbb{F}_q^*$, $b_{j_1} \in \mathbb{F}_q, 1 \le j_1 \le k-i-2$; i.e.,

$$(a_1, \underbrace{0, \dots, 0}_{i}, a_2, b_1, b_2, \dots, b_{k-i-2}).$$

There are $q^{k-i-2}(q-1)^2$ choices of $\{a_1, a_2, b_1, \dots, b_{k-i-2}\}$. But there are only $\frac{q^{k-i-2}(q-1)^2}{\Delta}$ states which cover $\mathbf{0}_i$ that can be obtained in the same non-zero sequence $\bar{\mathbf{a}}$. When $i = k-1$, let $\xi_1 = (a, 0, \dots, 0)$ be a state of a non-zero sequence $\bar{\mathbf{a}}$, where $a \in \mathbb{F}_q^*$. According to the linear recursive relation, $\xi_2 = (0, \dots, 0, b)$, where $b \in \mathbb{F}_q^*$ and b is determined by a. Similarly, there are only $\frac{q-1}{\Delta}$ choices of a such that ξ_1 is a state of $\bar{\mathbf{a}}$.

For any non-zero codeword $\mathbf{c} \in C$, it then follows from Theorem 2.2 that

$$\begin{aligned}w_b(\mathbf{c}) &= n - \sum_{i=b}^{n-1} (i-b+1) \cdot \Psi(\mathbf{0}_i) \\ &= \frac{q^k-1}{\Delta} - \frac{(q-1)^2}{\Delta} \cdot \sum_{j_2=b+1}^{k-1} \sum_{i_2=1}^{k-j_2} q^{i_2-1} \\ &\quad - \frac{(q-1)(k-b)}{\Delta} \\ &= \frac{q-1}{\Delta}(q^{k-1} + \cdots + q^{k-b}) \\ &= \frac{(q^b-1)q^k}{q^b \Delta}.\end{aligned}$$

This completes the proof. ∎

The conditions given by Corollary 2.3 are not necessary. Here we give a sufficient and necessary condition for the b-symbol weight of two vectors to be equal. To this end, we give a generalization of the support of a vector $\mathbf{c}$. We define the b-symbol support to be

$$\mathcal{I}_b(\mathbf{c}) = supp(\pi_b(\mathbf{c})) = \bigcup_{i=0}^{b-1} supp(\tau^i(\mathbf{c})),$$

where $supp(\mathbf{c})$ denotes the support of a vector $\mathbf{c}$ and τ denotes the left shift operator. It is easy to check that $w_b(\mathbf{c}) = |\mathcal{I}_b(\mathbf{c})|$.

Theorem 2.8: Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors that belong to $\mathbb{F}_q^n$, then $w_b(\mathbf{x}) = w_b(\mathbf{y})$ if and only if $|\mathcal{I}_b(\mathbf{x})| = |\mathcal{I}_b(\mathbf{y})|$.

Proof: The desired result follows from the definition of b-symbol metric. ∎

E. The r-th Generalized Hamming Metric

Let C be an $[n,k]_q$ linear code. For any subcode $D \subset C$, then the support of D is defined to be

$$\begin{aligned}\chi(D) \;=\; &\{i : 0 \le i \le n-1 | c_i \neq 0 \text{ for some} \\ &(c_0, c_1, \dots, c_{n-1}) \in D\}.\end{aligned}$$

The r-th generalized Hamming weight of a code C is the smallest cardinality of the support of an r-dimensional subcode of C. To avoid confusion of notation, we use $\mathbf{d}_r(C)$ for r-th generalized Hamming distance of C.

The set $\{\mathbf{d}_r(C) : 1 \le r \le k\}$ is called the distance hierarchy of C. Considering that C is a linear code, this set is often referred to as the weight hierarchy. To distinguish it from the later definition, let us call it the generalized weight hierarchy in the sequel.

III. The Connections Among Hamming Metric, b-Symbol Metric, and r-th Generalized Hamming Metric

The goal of this section is to show the connections among Hamming metric, b-symbol metric and r-th generalized Hamming metric.

A. Hamming Metric and b-Symbol Metric

Let $G_b(\mathbf{c})$ be the generator matrix of the code generated by $\mathbf{c}$ and its first $b-1$ cyclic shifts.

$$G_b(\mathbf{c}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{b-1} & c_{b+1} & \cdots & c_{b-3} & c_{b-2} \end{pmatrix}. \quad (2)$$

The following result shows the connection between the b-symbol weight and Hamming weight for any vector. It is a generalization of Lemma 1 in [28]. An important observation is that $w_b(\mathbf{c})$ equals the number of nonzero columns of $G_b(\mathbf{c})$.

Theorem 3.1: Let $\mathbf{c} \in \mathbb{F}_q^n$ and denote by $V_b(\mathbf{c})$ the vectors generated by all linear combinations of $\mathbf{c}$ and its first $b-1$ cyclic shifts (i.e., generated by all linear combinations of $G_b(\mathbf{c})$). Then

$$w_b(\mathbf{c}) = \frac{1}{q^{b-1}(q-1)} \sum_{\mathbf{c}' \in V_b(\mathbf{c})} w_1(\mathbf{c}').$$

Proof: Let $V_b(\mathbf{c})$ be all q^b vectors generated by all linear combinations of $G_b(\mathbf{c})$ and consider these as row vectors in a $q^b \times n$ matrix. Then in each nonzero column all elements occur equally often i.e., q^{b-1} times each. Then, since in this matrix, the sum of the weights of all row vectors equals the sum of the weight of all column vectors, we obtain

$$\sum_{\mathbf{c}' \in V_b(\mathbf{c})} w_1(\mathbf{c}') = q^{b-1}(q-1) w_b(\mathbf{c})$$

which proves the theorem. ∎

$V_b(\mathbf{c})$ may be a multiset. For instance, if we take $\mathbf{c} = (1010) \in \mathbb{F}_2^4$, then

$$\begin{aligned} V_2(\mathbf{c}) &= \{0000, 1010, 0101, 1111\}; \\ V_3(\mathbf{c}) &= \{0000, 1010, 0101, 1111, 0000, 1010, \\ &\quad 0101, 1111\}. \end{aligned}$$

Observe that if the minimal polynomial of $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ has degree $\rho(\mathbf{c}) < b$, then $w_b(\mathbf{c}) = w_{\rho(\mathbf{c})}(\mathbf{c})$. This follows since $G_b(\mathbf{c})$ has the same row space as $G_{\rho(\mathbf{c})}(\mathbf{c})$ but has each vector with multiplicity $q^{b-\rho(\mathbf{c})}$.

In [10] and [28] the authors gave the connection between $w_b(\mathbf{c})$ and $w_1(\mathbf{c})$ for any vector $\mathbf{c} \in \mathbb{F}_q^n$ with $0 < w_1(\mathbf{c}) \leq n-(b-1)$ as:

$$w_1(\mathbf{c}) + b - 1 \leq w_b(\mathbf{c}) \leq b \cdot w_1(\mathbf{c}). \tag{3}$$

The following result generalizes Inequality (3) to the general case and gives an interesting triangle inequality about the b-symbol metric. To this end, we need the following lemma.

Lemma 3.2: Let $\mathbf{c}$ be a vector that belongs to $\mathbb{F}_q^n$. Then

$$w_b(\mathbf{c}) \geq b \sum_{i=b}^{n-1} \Psi(\mathbf{0}_i).$$

Proof: The result follows since each nonzero element before the start of a run of b $\mathbf{0}$'s or more belongs to b nonzero b-tuples. ∎

Theorem 3.3: (Triangle inequality) Let $\mathbf{c} \in \mathbb{F}_q^n$ be such that $0 < w_b(\mathbf{c}) \leq n-m$ and $0 < w_m(\mathbf{c}) \leq n-b$. Then we have

$$\begin{aligned} \max\{w_b(\mathbf{c}) + m, w_m(\mathbf{c}) + b\} &\leq w_{b+m}(\mathbf{c}) \\ &\leq \min\{w_b(\mathbf{c}) + w_m(\mathbf{c}), n\}. \end{aligned} \tag{4}$$

Proof: Let $\{\Psi(\mathbf{0}_1), \Psi(\mathbf{0}_2), \ldots, \Psi(\mathbf{0}_n)\}$ be the $\mathbf{0}$'s run distribution of $\mathbf{c}$. According to Theorem 2.2, we have

$$w_b(\mathbf{c}) = n - \sum_{i=b}^{n-1} (i-b+1) \cdot \Psi(\mathbf{0}_i); \tag{5}$$

$$w_m(\mathbf{c}) = n - \sum_{i=m}^{n-1} (i-m+1) \cdot \Psi(\mathbf{0}_i); \tag{6}$$

$$w_{b+m}(\mathbf{c}) = n - \sum_{i=b+m}^{n-1} (i-b-m+1) \cdot \Psi(\mathbf{0}_i). \tag{7}$$

- If $\sum_{i=b+m}^{n-1} \Psi(\mathbf{0}_i) = 0$, then $w_{b+m}(\mathbf{c}) = n \geq w_b(\mathbf{c}) + m$ and by symmetry $w_{b+m}(\mathbf{c}) = n \geq w_m(\mathbf{c}) + b$.
- If $\sum_{i=b+m}^{n-1} \Psi(\mathbf{0}_i) \geq 1$, then

$$\begin{aligned} w_{b+m}(\mathbf{c}) - w_b(\mathbf{c}) &= \sum_{i=b}^{b+m-1} (i-b+1) \cdot \Psi(\mathbf{0}_i) \\ &\quad + m \sum_{i=b+m}^{n-1} \Psi(\mathbf{0}_i) \geq m. \end{aligned}$$

Therefore, by symmetry, since also $w_{b+m}(\mathbf{c}) - w_m(\mathbf{c}) \geq w_b(\mathbf{c})$ we complete the proof of the inequality on the left-hand side.

Without loss of generality, assume that $b \leq m$.

Let $w_b(\mathbf{c}) + w_m(\mathbf{c}) - w_{b+m}(\mathbf{c}) = (*)$. In the light of Equations (5), (6) and (7), we obtain

$$\begin{aligned} (*) &= w_b(\mathbf{c}) + n - \sum_{i=m}^{n-1} (i-m+1) \cdot \Psi(\mathbf{0}_i) \\ &\quad - n + \sum_{i=b+m}^{n-1} (i-b-m+1) \cdot \Psi(\mathbf{0}_i) \\ &= w_b(\mathbf{c}) - b \sum_{i=m+b}^{n-1} \Psi(\mathbf{0}_i) \\ &\quad - \sum_{i=m}^{m+b-1} (i-m+1) \cdot \Psi(\mathbf{0}_i) \\ &= w_b(\mathbf{c}) - b \sum_{i=b}^{n-1} \Psi(\mathbf{0}_i) + b \sum_{i=b}^{m+b-1} \Psi(\mathbf{0}_i) \\ &\quad - \sum_{i=m}^{m+b-1} (i-m+1) \Psi(\mathbf{0}_i) \\ &= \begin{cases} w_b(\mathbf{c}) - b \sum_{i=b}^{n-1} \Psi(\mathbf{0}_i) + b \sum_{i=b}^{m-1} \Psi(\mathbf{0}_i) \\ \quad + \sum_{i=m}^{m+b-1} (b+m-1-i) \Psi(\mathbf{0}_i), & \text{if } b < m \\ w_b(\mathbf{c}) - b \sum_{i=b}^{n-1} \Psi(\mathbf{0}_i) \\ \quad + \sum_{i=m}^{m+b-1} (b+m-1-i) \Psi(\mathbf{0}_i), & \text{if } b = m \end{cases} \\ &\geq 0. \end{aligned}$$

The last inequality $(*) > 0$ is obtained according to Lemma 3.2 and the assumption $b \leq m$. Therefore, we get the desired results. ∎

Remark 3.4: According to Proposition 2.3 in [10], we have $w_{b+1}(\mathbf{c}) \geq w_b(\mathbf{c}) + 1$ if $0 < w_b(\mathbf{c}) < n$. [10, Proposition 2.3] provides a simpler way to prove the left-hand side of the inequality (4).

From Theorem 3.3, we have the following proposition.

Proposition 3.5: Let $\mathbf{c} \in \mathbb{F}_q^n$ and $0 < w_b(\mathbf{c}) < n$. Let $b = \sum_{i=1}^{t} k_i$, where $k_1, k_2, \ldots, k_t$ are positive integers.

Then we have

$$\max\{w_{k_1}(\mathbf{c})+b-k_1,\ldots,w_{k_t}(\mathbf{c})+b-k_t\} \le w_b(\mathbf{c}) \le \min\left\{\sum_{i=1}^{t} w_{k_i}(\mathbf{c}), n\right\}.$$

B. b-Symbol Metric and r-th Generalized Hamming Metric

Let $G = [s_0, s_1, \cdots, s_{n-1}]$ be a generator matrix of an $[n,k]_q$ code C with columns s_i, $i = 0,1,\ldots,n-1$. Let U be a subspace of $\mathbb{F}_q^k$, $m(U) = |\{i|s_i \in U\}|$ and

$$F_{k,l} = \{U|\dim(U) = l\}.$$

Lemma 3.6 [19]: Let C be an $[n,k]_q$ linear code. Then

$$\mathbf{d}_r(C) = n - \max\{m(U)|U \in F_{k,k-r}\}.$$

Theorem 3.7: Let C be an $[n,k]_q$ linear code. Then

$$\mathbf{d}_r(C) = \min\left\{ \frac{1}{q^{r-1}(q-1)} \sum_{\mathbf{c}\in R} w_1(\mathbf{c}) \middle| R \text{ is an } r\text{-dimensional subspace of } C \right\} \quad (8)$$

and

$$d_b(C) = \min\left\{ \frac{1}{q^{b-1}(q-1)} \sum_{\mathbf{c}'\in V_b(\mathbf{c})} w_1(\mathbf{c}') \middle| \mathbf{c} \in C \right\}. \quad (9)$$

Proof: The desired result (8) follows from the definition of the r-th generalized Hamming weight and Lemma 3.6. The desired result (9) follows from Theorem 3.1. ■

In the case of b-symbol weight, we need to compute the sum of the Hamming weights of vectors in $V_b(\mathbf{c})$ for any $\mathbf{c}$, while for the r-th generalized Hamming weight, this summation needs to be calculated for all r-dimensional subcodes.

The next goal is to give a very interesting connection between $d_b(C)$ and $\mathbf{d}_r(C)$ if C is cyclic. To this end, we need the following lemmas.

Lemma 3.8: Let C be an $[n,k]_q$ cyclic code. Let $\mathbf{c}$ be a codeword of C. Let b be a positive integer not greater than k. If $G_b(\mathbf{c})$ has rank $\rho(\mathbf{c})$, then

$$w_b(\mathbf{c}) \ge \mathbf{d}_{\rho(\mathbf{c})}(C).$$

Proof: If C is cyclic and $G_b(\mathbf{c})$ has rank $\rho(\mathbf{c})$, then the code C' generated by $G_b(\mathbf{c})$ is a subcode of C and C' is a $\rho(\mathbf{c})$-dimensional subspace of C. By Theorem 3.1, we have

$$\begin{aligned} w_b(\mathbf{c}) &= \frac{1}{q^{b-1}(q-1)} \sum_{\mathbf{c}'\in V_b(\mathbf{c})} w_1(\mathbf{c}') \\ &= \frac{q^{b-\rho(\mathbf{c})}}{q^{b-1}(q-1)} \sum_{\mathbf{c}'\in C'} w_1(\mathbf{c}') \\ &\ge \mathbf{d}_{\rho(\mathbf{c})}(C). \end{aligned}$$

Therefore, we obtain the desired result. ■

Remark: The reason why we need the restriction $b \le k$ is to ensure $\rho(\mathbf{c}) \le k$.

Assume that $\mathbf{c}$ is a codeword of C. Let $\vartheta(\mathbf{c})$ be the maximum $\mathbf{0}$'s run length of $cir(\mathbf{c})$ and $\theta = \max\{\vartheta(\mathbf{c})|\mathbf{c} \in C\backslash\{\mathbf{0}\}\}$. The parameter θ is called the maximum $\mathbf{0}$'s run length of C.

Lemma 3.9: Let C be a linear code over $\mathbb{F}_q$ with minimum b-symbol weight $d_b(C) < n$. Let $\overline{C}_b = \{\mathbf{c}|\mathbf{c} \in C \text{ and } w_b(\mathbf{c}) = d_b(C) < n\}$. Then $Rank(G_b(\mathbf{c})) = b$ for any $\mathbf{c} \in \overline{C}_b$.

Proof: We claim that $\vartheta(\mathbf{c}) \ge b$ for any $\mathbf{c} \in \overline{C}_b$. Assume that there exists a codeword $\mathbf{c}' \in \overline{C}_b$ such that $\vartheta(\mathbf{c}) \le b-1$. By Theorem 2.2, $w_b(\mathbf{c}') = n$, a contradiction. Since $\vartheta(\mathbf{c}) \ge b$, for any $\mathbf{c} \in \overline{C}_b$, $\mathbf{c}$ is of the form

$$\mathbf{c} = (\ldots, \alpha, \underbrace{0, \ldots, 0}_{\vartheta(\mathbf{c})\ge b}, \beta, \ldots),$$

where α and β belong to $\mathbb{F}_q^*$. Then the matrix $G_b(\mathbf{c})$ has a b by $\vartheta(\mathbf{c})+2$ submatrix of the form

$$G' = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 & 0 & \beta \\ 0 & 0 & 0 & \cdots & 0 & \beta & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \beta & & & \end{pmatrix}_{b\times(\vartheta(\mathbf{c})+2)}.$$

The desired result follows from $Rank(G') = b$. ■

Theorem 3.10: Let C be an $[n,k]_q$ cyclic code. Let b be a positive integer not exceeding k. Then $d_b(C) \ge \mathbf{d}_b(C)$.

Proof: The desired result follows from Lemma 3.8 and Lemma 3.9. ■

Just as the weight hierarchy of C under the r-th generalized Hamming metric, we define the weight hierarchy of C under the b-symbol metric as follows:

b-symbol

- weight hierarchy of C:

$$\{d_1(C), d_2(C), \ldots, d_\theta(C), d_{\theta+1}(C), \ldots, d_n(C)\}.$$

Theorem 3.11: Let C be an $[n,k]_q$ linear code of the maximum $\mathbf{0}$'s run length θ. The following hold:

1) $0 < d_1(C) < d_2(C) < \cdots < d_\theta(C) < d_{\theta+1}(C) = \cdots = d_n(C) = n$.
2) The b-symbol weight hierarchy of C is the same as the b-symbol weight hierarchy of CDP^i, where D is an n by n diagonal invertible matrix, and

$$P = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{n\times n}$$

and $0 \le i \le n-1$.

Proof: Let $\mathbf{c}$ be a codeword in C of minimum nonzero b-symbol weight with $2 \le b \le \theta$, then

$$d_b(C) = w_b(\mathbf{c}) \ge w_{b-1}(\mathbf{c}) + 1 \ge d_{b-1}(C) + 1.$$

Note that $d_\theta(C)$ can not equal n since there exists at least one codeword in C of b-symbol weight less than n.

Two linear codes C_1 and C_2 have the same b-symbol weight hierarchy if they have the same $\mathbf{0}$'s run distribution for any codeword. Therefore, C and CDP^i have the same b-symbol weight hierarchy. ∎

If C is cyclic, then the maximum $\mathbf{0}$'s run length of C is $k-1$. We can get the following corollary directly.

Corollary 3.12: For an $[n,k]_q$ cyclic code C, then

$$\begin{aligned} d_1(C) &< d_2(C) < \cdots < d_{k-1}(C) < d_k(C) \\ &= d_{k+1}(C) = \cdots = d_n(C) = n. \end{aligned}$$

1) Singleton Bound: Wei [27] established the generalized Singleton Bound in 1991.

Lemma 3.13 ([27] Generalized Singleton Bound): For an $[n,k,d]_q$ linear code,

$$\mathbf{d}_r(C) \le n-k+r$$

for $1 \le r \le k$.

Wei called an $[n,k]_q$ code C r-rank maximum distance separable (r-rank MDS for short) if $\mathbf{d}_r(C) = n-k+r$.

Cassuto et al. [4], [10] established the b-symbol Singleton Bound in 2013.

Lemma 3.14 ([4], [10] b-Symbol Singleton Bound): If C is an $(n,M,d_b(C))_q$ b-symbol code, then we have

$$M \le q^{n-d_b(C)+b}.$$

An $(n,M,d_b(C))_q$ b-symbol code C with $M = q^{n-d_b(C)+b}$ is called a b-symbol maximum distance separable (b-symbol MDS for short) code.

Despite of the different metrics, their Singleton-type Bounds are the same. The only difference is that when the metric is the r-th generalized Hamming metric, we need to restrict C to be linear.

Lemma 3.15 [4], [10], [27]: Let C be an MDS code with parameters (n,q^k) over $\mathbb{F}_q$. Then:

- If C is linear, then C is an r-rank MDS code and $\mathbf{d}_r(C) = d_H(C)+r-1$ for $1 \le r \le k$ (see [27]).
- C is a b-symbol MDS code and $d_b(C) = d_H(C)+b-1$ for $1 \le b \le k$ (see [4], [10]).

The following two theorems give the characterization of trivial r-rank MDS codes or b-symbol MDS codes.

From [27, Theorem 1], we can easily derive the following result.

Theorem 3.16: A cyclic code C with parameters $[n,k]_q$ is an r-rank MDS code if $r=k$.

Proof: By the definition of the r-th generalized Hamming metric, we have $\mathbf{d}_r(C) = n$ since C is cyclic. According to Lemma 3.13, C is r-rank MDS. ∎

Theorem 3.17: A cyclic code C with parameters $[n,k]_q$ is a b-symbol MDS code if $b=k$. Moreover, $d_b(C)=n$ if $b \ge k$.

Proof: From the proof of Theorem 2.7, there is no $\mathbf{0}$'s run with length greater than or equal to k since $\bar{\mathbf{a}}$ is not a zero state. According to Lemma 3.14, we get the desired result. ∎

Remark: (i) Theorem 3.17 proves the stability theorem in [24] by using a more concise way and does not need the restriction $\gcd(n,q)=1$.

(ii) The definition of the b-symbol weight does not restrict $b \le k$, but if C is cyclic, we only consider the case $b \le k$. If C is cyclic, we have $\theta = k-1$. In the sequel, we assume that b is always less than k if C is cyclic. For the same cyclic code C, it has two weight hierarchies as follows:

- generalized weight hierarchy:

$$\{\mathbf{d}_1(C), \mathbf{d}_2(C), \ldots, \mathbf{d}_k(C)\}.$$

b-symbol
- weight hierarchy:

$$\{d_1(C), d_2(C), \ldots, d_k(C)\}.$$

Further, we have $d_H(C) = d_1(C) = \mathbf{d}_1(C)$ and $d_k(C) = \mathbf{d}_k(C) = n$.

2) Griesmer Bound: In 1992, Helleseth et al. [19] established the generalized Griesmer Bound.

Lemma 3.18 ([19] Generalized Griesmer Bound): Let C be an $[n,k]_q$ linear code. Then

$$n \ge \mathbf{d}_r(C) + \sum_{i=1}^{k-r} \left\lceil \frac{q-1}{q^i(q^r-1)} \mathbf{d}_r(C) \right\rceil,$$

and

$$(q^r-1)\mathbf{d}_{r-1}(C) \le (q^r-q)\mathbf{d}_r(C).$$

We naturally expect to give the b-symbol Griesmer Bound for cyclic codes. There are two reasons why we restrict C to being cyclic. If C is cyclic, then

1) the maximum $\mathbf{0}$'s run length of C is less than the dimension of C;
2) for any codeword $\mathbf{c} \in C$, the code generated by $G_b(\mathbf{c})$ is a subcode of C.

b-consecutive positions are independent if the set of codewords takes on all q^b possible b-tuples equally often when restricted to these positions.

Lemma 3.19: Let C be a cyclic code with parameters $[n,k]$ over $\mathbb{F}_q$. Let $k = tb+s$, where t,s are two integers, $t \ge 1$, and $0 \le s < b$. Assume that any b-consecutive positions of C are independent. Then

$$\sum_{\mathbf{c} \in C} w_b(\mathbf{c}) = nq^{k-b}(q^b-1). \tag{10}$$

Moreover, we have

$$d_b(C) \le \left\lfloor \frac{q^k-q^{k-b}}{q^k-1} \cdot n \right\rfloor, \tag{11}$$

$$n \ge \left\lceil \sum_{i=0}^{t-1} \frac{d_b(C)}{q^{bi}} + \frac{d_b(C)(q^s-1)}{q^{(t-1)b+s}(q^b-1)} \right\rceil, \tag{12}$$

$$n \ge \left\lceil \sum_{i=0}^{t-1} \frac{d_b(C)}{(q^b)^i} \right\rceil \quad \text{if} \quad b|k. \tag{13}$$

Proof: The first claim follows by observing that any b-consecutive positions are information symbols, since this implies that the cyclic code restricted to these positions contains any b-tuple equally often, i.e., q^{k-b} times.

According Equality (10), we have

$$(q^k-1)d_b(C) \le nq^{k-b}(q^b-1).$$

Then Inequality (10) holds, and

$$n \ge d_b(C) \cdot \frac{(q^k-1)}{q^{k-b}(q^b-1)}$$

$$\begin{aligned}
&= d_b(C)\cdot\frac{(q^{tb+s}-1)}{q^{(t-1)b+s}(q^b-1)}\\
&= d_b(C)\cdot\frac{(q^{tb}-1)}{q^{(t-1)b}(q^b-1)}\\
&\quad +d_b(C)\cdot\frac{(q^{s}-1)}{q^{(t-1)b+s}(q^b-1)}\\
&= \left\lceil\sum_{i=0}^{t-1}\frac{d_b(C)}{q^{bi}}+\frac{d_b(C)}{q^{(t-1)b}}\cdot\frac{q^s-1}{q^s}\cdot\frac{1}{q^b-1}\right\rceil
\end{aligned}$$

Therefore, we obtain the desired result. ■

The following lemma was given in [24]. A nice relation between $d_b(C)$ and $d_{b-1}(C)$ was established, and it is the same as the relation between $\mathbf{d}_r(C)$ and $\mathbf{d}_{r-1}(C)$ which is given in Lemma 3.18.

Lemma 3.20 ([24, Inequality (42)]): Let C be a cyclic code with parameters $[n,k]$ over $\mathbb{F}_q$. Then

$$(q^b-q)d_b(C)\geq(q^b-1)d_{b-1}(C). \tag{14}$$

By Lemma 3.20, we have

$$d_b(C)\geq\left\lceil\frac{q^b-1}{q^b-q}d_{b-1}(C)\right\rceil. \tag{15}$$

Yaakobi et al. proved $d_H(C)\leq\left\lfloor\frac{2n}{3}\right\rfloor$ over $\mathbb{F}_2$ in [29, Theorem 1] to ensure that the lower bound on $d_2(C)$ does not exceed n. Notice that the lower bound on $d_b(C)$ given in (15) can not be greater than n since

$$d_{b-1}(C)\leq\left\lfloor\frac{q^k-q^{k-b+1}}{q^k-1}\cdot n\right\rfloor.$$

Further, we have

$$d_b(C)\geq\left\lceil\sum_{i=0}^{b-1}\frac{d_1(C)}{q^i}\right\rceil. \tag{16}$$

In fact, the lower bound (16) can be further reinforced into the following result.

Theorem 3.21: Let C be a cyclic code with parameters $[n,k]$ over $\mathbb{F}_q$. Then

$$d_b(C)\geq\sum_{i=0}^{b-1}\left\lceil\frac{d_1(C)}{q^i}\right\rceil. \tag{17}$$

Proof: Let $\mathbf{c}$ be a codeword with b-symbol weight $w_b(\mathbf{c})=d_b(C)$. By Lemma 3.9, the rank of $G_b(\mathbf{c})$ equals b. Let C_1 be the linear code generated by $G_b(\mathbf{c})$. Since $Rank(G_b(\mathbf{c}))=b$, the parameters of C_1 are $[n,b,d_1(C_1)]$ with $d_1(C_1)\geq d_1(C)$. Assume that

$$G_b(\mathbf{c})=\begin{pmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_n\end{pmatrix}_{b\times n},$$

where $\mathbf{r}_i$ are the columns of $G_b(\mathbf{c})$ with $1\leq i\leq n$. Let

$$G'_b(\mathbf{c})=\begin{pmatrix}\mathbf{r}_{j_1} & \mathbf{r}_{j_2} & \dots & \mathbf{r}_{j_m}\end{pmatrix}_{b\times m},$$

where $\mathbf{r}_{j_1},\ldots,\mathbf{r}_{j_m}$ are all the nonzero columns of $G_b(\mathbf{c})$. Since the number of all the nonzero columns of $G_b(\mathbf{c})$ equals $w_b(\mathbf{c})$, $m=w_b(\mathbf{c})$. Then the parameters of the linear code C_2 generated by $G'_b(\mathbf{c})$ are $[w_b(\mathbf{c}),b,d_1(C_2)=d_1(C_1)\geq d_1(C)]$. By the Griesmer Bound, we have

$$d_b(C)=w_b(\mathbf{c})\geq\sum_{i=0}^{b-1}\left\lceil\frac{d_1(C)}{q^i}\right\rceil.$$

Therefore, we obtain the desired result. ■

Assume that $b|k$ and $k=tb$. By the preceding theorem, we have

$$\sum_{i=0}^{t-1}\left\lceil\frac{d_b(C)}{q^{bi}}\right\rceil\geq\sum_{i=0}^{t-1}\left\lceil\frac{\sum_{j=0}^{b-1}\left\lceil\frac{d_1(C)}{q^j}\right\rceil}{q^{bi}}\right\rceil. \tag{18}$$

From the Griesmer Bound, we have

$$\begin{aligned}
n &\geq \sum_{i=0}^{tb-1}\left\lceil\frac{d_1(C)}{q^i}\right\rceil\\
&= \sum_{i=0}^{t-1}\sum_{j=0}^{b-1}\left\lceil\frac{d_1(C)}{q^{ib+j}}\right\rceil\\
&= \sum_{i=0}^{t-1}\sum_{j=0}^{b-1}\left\lceil\frac{\frac{d_1(C)}{q^j}}{q^{bi}}\right\rceil.
\end{aligned}\tag{19}$$

Combining the Inequalities (18) and (19), a natural motivation for us is to explore the relation between n and $\sum_{i=0}^{t-1}\left\lceil\frac{d_b(C)}{q^{bi}}\right\rceil$. Then we have the following conjecture.

Conjecture 1: (b-Symbol Griesmer Bound for cyclic codes) Assume that $b|k$ and $k=tb$. If C is a cyclic code with parameters $[n,k]$ over $\mathbb{F}_q$, then

$$n\geq\sum_{i=0}^{t-1}\left\lceil\frac{d_b(C)}{(q^b)^i}\right\rceil. \tag{20}$$

Inequality (20) holds if $b=k$ or $b=1$ from Theorem 3.17 and the Griesmer Bound. The reader is warmly invited to attack Conjecture 1 when $b>1$ and $t>1$.

IV. Conclusion and Open Problems

Many results in this paper generalize the previous work (e.g., Theorem 3.1, Theorem 3.3). In this paper, the b-symbol weight of a vector $\mathbf{c}$ is first calculated by calculating the $\mathbf{0}$'s run distribution of $\mathbf{c}$ (Theorem 2.2). This provides a new way to calculate the b-symbol weight distribution of linear codes (e.g., in Remark 2.6 and Theorem 2.7 we calculated the b-symbol weight distribution of some special codes based on their $\mathbf{0}$'s run distributions). However, for general linear codes (or cyclic codes), it is still a difficult task to determine their $\mathbf{0}$'s run distributions. A potential direction is to determine the b-symbol weight distributions of some special linear codes by exploring their $\mathbf{0}$'s run distributions.

Another highlight of this paper is that we first studied the connections and differences between b-symbol metric and r-th generalized Hamming metric. As two different generalizations of Hamming metric, they have a lot in common. When C is cyclic, a very important relation between $d_b(C)$ and $\mathbf{d}_b(C)$ is given (Theorem 3.10). It is a pity that this paper fails to give the b-symbol Griesmer Bound for cyclic codes and only gives a conjecture (Conjecture 1). It is important to note that b-symbol

Griesmer bound (if Conjecture 1 is true) does not apply to arbitrary linear codes (this is one of the differences from Griesmer Bound and generalized Griesmer Bound). However, the b-symbol Griesmer Bound is not restricted to cyclic codes; in fact, it is also applicable to constacyclic codes.

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