One-Shot Triple-Resource Trade-Off
in Quantum Channel Coding

Eyuri Wakakuwa and Yoshifumi Nakata

Abstract—We analyze a task in which classical and quantum messages are simultaneously communicated via a noisy quantum channel, assisted with a limited amount of shared entanglement. We derive direct and converse bounds for the one-shot capacity region, represented by the smooth conditional entropies and the error tolerance. The proof is based on the randomized partial decoupling theorem, which is a generalization of the decoupling theorem. The two bounds match in the asymptotic limit of infinitely many uses of a memoryless channel and coincide with the previous result obtained by Hsieh and Wilde. Direct and converse bounds for various communication tasks are obtained as corollaries, both for the one-shot and asymptotic scenarios.

Index Terms—Quantum channel capacity, one-shot, decoupling.

I. INTRODUCTION

ONE of the major goals of quantum communication theory is to investigate the ultimate capacities of a noisy quantum channel for transmitting classical and quantum information (see, e.g., [1], [2]). In an asymptotic limit of infinitely many uses of a memoryless channel, the classical capacity of a noisy quantum channel was obtained by Holevo [3], Schumacher and Westmoreland [4]; the quantum capacity by Lloyd [5], Shor [6] and Devetak [7]; the entanglement-assisted classical capacity by Bennet et al. [8], [9] and Shor [10]; the entanglement-assisted quantum capacity by Devetak et al. [11], [12]; and the capacity of a quantum channel for simultaneously transmitting classical and quantum messages was obtained by

Devetak and Shor [13]. One of the most general results in this direction was shown by Hsieh and Wilde [14], which provides a complete characterization of the triple trade-off among the resources of classical communication, quantum communication, and entanglement in quantum channel coding. As proved in [14], the capacity theorems in the special cases mentioned above are obtained from their result as corollaries.

A modern approach addresses the same communication tasks under the assumption that the channel is used only once, which is referred to as the *one-shot* regime. Along this line, the classical capacity of a noisy quantum channel was first investigated by Mosonyi et al. [15] and was developed by Renes et al. [16] and Wang et al. [17]; the quantum capacity was obtained by Buscemi et al. [18]; the entanglement-assisted classical and quantum capacities by Datta et al. [19]; and the capacity region for simultaneously transmitting classical and quantum information was obtained by Salek et al. [20]. In the asymptotic limit of infinitely many uses of a memoryless channel, all these results (except [15]) are known to coincide with the preexisting results presented above. Numerous studies have been done to develop these approaches further (see e.g. [21], [22], [23], [24], [25], [26], [27], [28], [29]). However, the one-shot capacity theorem that reduces in the asymptotic limit to the result by Hsieh and Wilde [14] has not been obtained so far.

In this paper, we derive a one-shot generalization of the three-dimensional rate region obtained in [14]. Namely, we consider a task in which classical and quantum messages are transmitted via a noisy quantum channel with the assistance of a limited amount of shared entanglement, all in the one-shot scenario. The main result is that we derive inner and outer bonds for the one-shot rate region, which are represented in terms of the smooth conditional entropy of the channel. To our knowledge, this is the first time that trade-off relations that apply simultaneously to all the three resources of classical communication, quantum communication, and shared entanglement has been obtained in the one-shot channel coding scenario. In the asymptotic limit of infinitely many uses of the channel, the direct and converse bounds match and coincide with the achievable rate region obtained in [14]. As proved in [14], the aforementioned capacity theorems in the special cases, such as the Holevo-Schumacher-Westmoreland theorem [3], [4] and the Lloyd-Shor-Devetak theorem [5], [6], [7], are obtained from their result as corollaries. Hence, our one-shot result also recovers the capacity theorems in the asymptotic limit in those special cases. Besides, the direct and converse bounds for various communication tasks in the

one-shot scenario readily follow from our result. It should be noted, however, that in special cases, our result is not necessarily as strong as those in the previous literature. To compare our results quantitatively with those in the previous literature is left as a future work.

The proof is based on the notion of *randomized partial decoupling* [30], which is a generalization of decoupling [31], [32], [33], [34], [35], [36]. Here, we consider a scenario in which a bipartite quantum state on system AR is subject to a unitary operation on A, followed by the action of a linear completely positive (CP) map. Unlike the usual setting of decoupling, we assume that the subsystem is decomposed into a direct-sum form, and the unitary is chosen at random from the set of unitaries that are block-diagonal under the decomposition. In [30], we proved the *randomized partial decoupling theorem*, which shows that the distance between the final state and the averaged one is bounded in terms of smooth conditional entropies of the initial state and the channel. The previous results of the one-shot decoupling theorem [35] and the dequantization theorem [21] are obtained from this result as corollaries, up to changes in the smoothing and error parameters (see Section III D in [30] for the details). The current paper applies the randomized partial decoupling theorem to the channel coding scenario for proving the one-shot capacity theorems.

This paper is organized as follows. In Section II, we introduce notations and definitions that will be used throughout this paper. In Section III, we present formulations of the problem and state the main results. In Section IV, we summarize the statement of the randomized partial decoupling theorem. The proofs of the main theorems are provided in Section V and VI. In Section VII, we apply the one-shot results to the asymptotic scenario of infinitely many uses of the channel and obtain a three-dimensional achievable rate region that is equivalent to the one obtained by Hsieh and Wilde. Proof of the asymptotic result is provided in Section VIII. In Section IX, we apply our results to special cases where one or two of the three resources are assumed to be zero and compare the results to those in the previous literature. Conclusions are presented in Section X.

II. PRELIMINARIES

We summarize notations and definitions that will be used throughout this paper.

A. Notations

We denote the set of linear operators and that of Hermitian operators on a Hilbert space $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$ and $\text{Her}(\mathcal{H})$, respectively. For positive semidefinite operators, density operators and sub-normalized density operators, we use the following notations, respectively:

$$\mathcal{P}(\mathcal{H}) = \{\rho \in \text{Her}(\mathcal{H}) : \rho \geq 0\}, \tag{1}$$
$$\mathcal{S}_=(\mathcal{H}) = \{\rho \in \mathcal{P}(\mathcal{H}) : \text{Tr}[\rho] = 1\}, \tag{2}$$
$$\mathcal{S}_\leq(\mathcal{H}) = \{\rho \in \mathcal{P}(\mathcal{H}) : \text{Tr}[\rho] \leq 1\}. \tag{3}$$

A Hilbert space associated with a quantum system A is denoted by $\mathcal{H}^A$, and its dimension is denoted by d_A. A system composed of two subsystems A and B is denoted by AB. When M and N are linear operators on $\mathcal{H}^A$ and $\mathcal{H}^B$, respectively, we denote $M \otimes N$ as $M^A \otimes N^B$ for clarity. In the case of pure states, we often abbreviate $|\psi\rangle^A \otimes |\phi\rangle^B$ as $|\psi\rangle^A |\phi\rangle^B$. For $X^{AB} \in \mathcal{L}(\mathcal{H}^{AB})$, X^A represents $\text{Tr}_B[X^{AB}]$. We denote $|\psi\rangle\langle\psi|$ simply by ψ. The maximally entangled state between A and A', where $\mathcal{H}^A \cong \mathcal{H}^{A'}$, is defined by

$$|\Phi\rangle^{AA'} := \frac{1}{\sqrt{d_A}} \sum_{\alpha=1}^{d_A} |\alpha\rangle^A |\alpha\rangle^{A'} \tag{4}$$

with respect to a fixed orthonormal basis $\{|\alpha\rangle\}_{\alpha=1}^{d_A}$.

The identity operator is denoted by I. We denote $(M^A \otimes I^B)|\psi\rangle^{AB}$ as $M^A|\psi\rangle^{AB}$ and $(M^A \otimes I^B)\rho^{AB}(M^A \otimes I^B)^\dagger$ as $M^A\rho^{AB}M^{A\dagger}$. When $\mathcal{T}$ is a supermap from $\mathcal{L}(\mathcal{H}^A)$ to $\mathcal{L}(\mathcal{H}^B)$, we denote it by $\mathcal{T}^{A\to B}$. When $A = B$, we use $\mathcal{T}^A$ for short. We also denote $(\mathcal{T}^{A\to B} \otimes \text{id}^C)(\rho^{AC})$ by $\mathcal{T}^{A\to B}(\rho^{AC})$. When a supermap is given by a conjugation of a unitary U^A or a linear operator $W^{A\to B}$, we denote it by its calligraphic font as $\mathcal{U}^A(X^A) := (U^A)X^A(U^A)^\dagger$ and $\mathcal{W}^{A\to B}(X^A) := (W^{A\to B})X^A(W^{A\to B})^\dagger$. In that case, the adjoint map of $\mathcal{W}^{A\to B}$ is defined by $\mathcal{W}^{\dagger B\to A}(\cdot) := (W^{A\to B})^\dagger(\cdot)(W^{A\to B})$.

For any linear CP map $\mathcal{T}^{A\to B}$, there exist a finite dimensional quantum system E and a linear operator $\Gamma_\mathcal{T}^{A\to BE}$ such that $\mathcal{T}^{A\to B}(\cdot) = \text{Tr}_E[\Gamma_\mathcal{T}(\cdot)\Gamma_\mathcal{T}^\dagger]$. The operator $\Gamma_\mathcal{T}$ is called the Stinespring dilation of $\mathcal{T}^{A\to B}$ [37], and the linear CP map defined by $\text{Tr}_B[\Gamma_\mathcal{T}(\cdot)\Gamma_\mathcal{T}^\dagger]$ is called the *complementary map* of $\mathcal{T}^{A\to B}$. With a slight abuse of notation, we denote the complementary map by $\mathcal{T}^{A\to E}$.

B. Norms and Distances

For a linear operator X, the trace norm is defined as $\|X\|_1 = \text{Tr}[\sqrt{X^\dagger X}]$. The trace distance between two unnormalized states $\rho, \rho' \in \mathcal{P}(\mathcal{H})$ is defined by $\frac{1}{2}\|\rho - \rho'\|_1$. For subnormalized states $\rho, \rho' \in \mathcal{S}_\leq(\mathcal{H})$, the generalized fidelity and the purified distance are defined by

$$\bar{F}(\rho, \rho') := \|\sqrt{\rho}\sqrt{\rho'}\|_1 + \sqrt{(1 - \text{Tr}[\rho])(1 - \text{Tr}[\rho'])}, \tag{5}$$
$$P(\rho, \rho') := \sqrt{1 - \bar{F}(\rho, \rho')^2}, \tag{6}$$

respectively [38]. The trace distance and the purified distance are related as

$$\frac{1}{2}\|\rho - \rho'\|_1 \leq P(\rho, \rho') \leq \sqrt{2\|\rho - \rho'\|_1} \tag{7}$$

for any $\rho, \rho' \in \mathcal{S}_\leq(\mathcal{H})$. The epsilon ball of a subnormalized state $\rho \in \mathcal{S}_\leq(\mathcal{H})$ is defined by

$$\mathcal{B}^\epsilon(\rho) := \{\rho' \in \mathcal{S}_\leq(\mathcal{H})| P(\rho, \rho') \leq \epsilon\}. \tag{8}$$

C. One-Shot Entropies

For any subnormalized state $\rho \in \mathcal{S}_\leq(\mathcal{H}^{AB})$ and normalized state $\varsigma \in \mathcal{S}_=(\mathcal{H}^B)$, define

$$H_{\min}(A|B)_{\rho|\varsigma} := \sup\{\lambda \in \mathbb{R} | 2^{-\lambda}I^A \otimes \varsigma^B \geq \rho^{AB}\}, \tag{9}$$
$$H_{\max}(A|B)_{\rho|\varsigma} := \log \|\sqrt{\rho^{AB}}\sqrt{I^A \otimes \varsigma^B}\|_1^2. \tag{10}$$
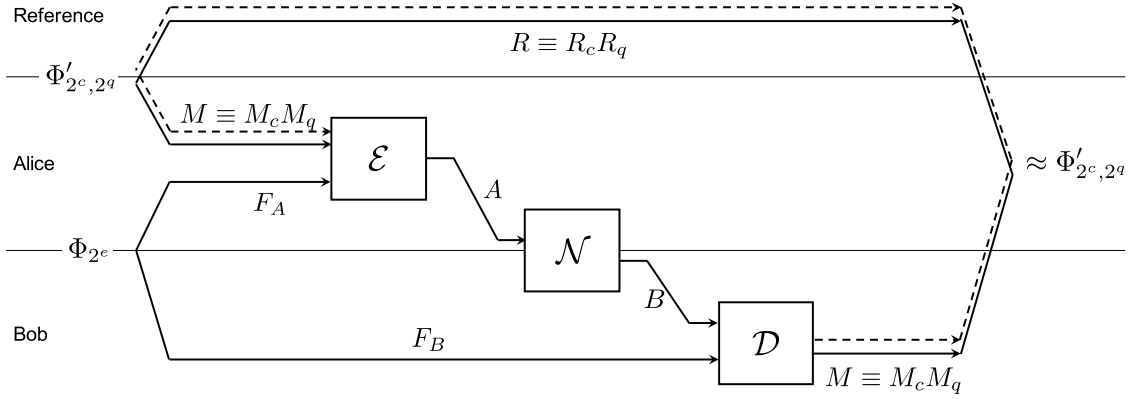
Fig. 1.   One-shot channel coding is depicted. The normal arrows represent quantum systems, and the dotted arrows represent the classical part of the source state.

The conditional min- and max- entropies (see e.g. [39]) are defined by

$$H_{\min}(A|B)_\rho := \sup_{\varsigma^B \in \mathcal{S}_=(\mathcal{H}^B)} H_{\min}(A|B)_{\rho|\varsigma}, \quad (11)$$

$$H_{\max}(A|B)_\rho := \sup_{\varsigma^B \in \mathcal{S}_=(\mathcal{H}^B)} H_{\max}(A|B)_{\rho|\varsigma}, \quad (12)$$

respectively. The smooth conditional min- and max-entropies are defined by

$$H_{\min}^\epsilon(A|B)_\rho := \sup_{\hat\rho^{AB} \in \mathcal{B}^\epsilon(\rho)} H_{\min}(A|B)_{\hat\rho}, \quad (13)$$

$$H_{\max}^\epsilon(A|B)_\rho := \inf_{\hat\rho^{AB} \in \mathcal{B}^\epsilon(\rho)} H_{\max}(A|B)_{\hat\rho} \quad (14)$$

for a smoothing parameter $\epsilon \geq 0$. The properties of the smooth conditional entropies used in this paper are summarized in Appendix B.

D. Choi-Jamiolkowski Representation

Let $\mathcal{T}^{A \to B}$ be a linear supermap from $\mathcal{L}(\mathcal{H}^A)$ to $\mathcal{L}(\mathcal{H}^B)$ and let $\Phi^{AA'}$ be the maximally entangled state between A and A'. A linear operator $\mathfrak{J}(\mathcal{T}^{A \to B}) \in \mathcal{L}(\mathcal{H}^{AB})$ defined by $\mathfrak{J}(\mathcal{T}^{A \to B}) := \mathcal{T}^{A' \to B}(\Phi^{AA'})$ is called the *Choi-Jamiołkowski representation of* $\mathcal{T}$ [40], [41]. The representation is an isomorphism. For an operator $X^{AB} \in \mathcal{L}(\mathcal{H}^{AB})$, the inverse map is given by

$$\mathfrak{J}_A^{-1}(X^{AB})(\varsigma^A) = d_A \mathrm{Tr}_A\big[(\varsigma^{A^T} \otimes I^B) X^{AB}\big], \quad (15)$$

where A^T denotes the transposition of A with respect to the Schmidt basis of $\Phi^{AA'}$. When $\mathcal{T}$ is completely positive, $\mathfrak{J}(\mathcal{T}^{A \to B})$ is an unnormalized state on AB and is called the *Choi-Jamiołkowski state of* $\mathcal{T}$.

E. Haar Measure

For a unitary group of finite degree, there exists the unique left- and right- unitarily invariant probability measure, known as the Haar measure. We denote it by H. The Haar measure satisfies the property that, for any unitary U and a set of unitaries $\mathcal{V}$,

$$\mathsf{H}(U\mathcal{V}) = \mathsf{H}(\mathcal{V}U) = \mathsf{H}(\mathcal{V}). \quad (16)$$

When a unitary U is chosen uniformly at random with respect to the Haar measure, we denote it by $U \sim \mathsf{H}$.

III. ONE-SHOT CAPACITY THEOREMS

Consider a noisy quantum channel $\mathcal{N}^{A \to B}$. Suppose that the sender, Alice, transmits c-bits of classical and q-qubits of quantum messages simultaneously to the receiver, Bob, through a noisy quantum channel assisted by e-ebits of shared entanglement (see Figure 1). We assume that Bob initially has no side information about the messages. Our goal is to obtain the conditions for this task to be achievable within error tolerance δ, in terms of c, q and e.

To be more precise, let M_q be a 2^q-dimensional quantum system that represents the quantum message, and R_q be a reference system with the same dimension that are inaccessible to Alice and Bob. The quantum message is described by the maximally entangled state $\Phi_{2^q}^{M_q R_q}$ with Schmidt rank 2^q. With F_A and F_B being 2^e-dimensional quantum registers, the entanglement resource is given by the maximally entangled state $\Phi_{2^e}^{F_A F_B}$ with Schmidt rank 2^e. A communication protocol is represented by a set of encoding CPTP maps $\{\mathcal{E}_j^{M_q F_A \to A}\}_{j=1}^{2^c}$ and a decoding instrument $\{\mathcal{D}_j^{B F_B \to M_q}\}_{j=1}^{2^c}$. Let m and $\hat{m}$ be the original classical message and the decoded one, respectively. The probability that the decoded message is $\hat{m} = k$ when the original message is $m = j$ is

$$p(\hat{m} = k | m = j) = \mathrm{Tr}[\mathcal{D}_k \circ \mathcal{N} \circ \mathcal{E}_j(\Phi_{2^q}^{M_q R_q} \otimes \Phi_{2^e}^{F_A F_B})]. \quad (17)$$

When the decoded message is $\hat{m} = k$ and the original message is $m = j$, the state of the quantum part after the decoding is given by

$$\hat\Phi_{j,k}^{M_q R_q} := \frac{1}{p(\hat{m} = k | m = j)} \mathcal{D}_k \circ \mathcal{N} \circ \mathcal{E}_j(\Phi_{2^q}^{M_q R_q} \otimes \Phi_{2^e}^{F_A F_B}). \quad (18)$$

The average probability of error in decoding the classical message is calculated to be

$$\delta_C := \frac{1}{2^c} \sum_{j=1}^{2^c} p(\hat{m} \neq j | m = j), \quad (19)$$

and that for the quantum message is

$$\delta_Q := \frac{1}{2^c} \sum_{j,k=1}^{2^c} p(\hat{m} = k | m = j) \left\| \hat\Phi_{j,k}^{M_q R_q} - \Phi_{2^q}^{M_q R_q} \right\|_1. \quad (20)$$
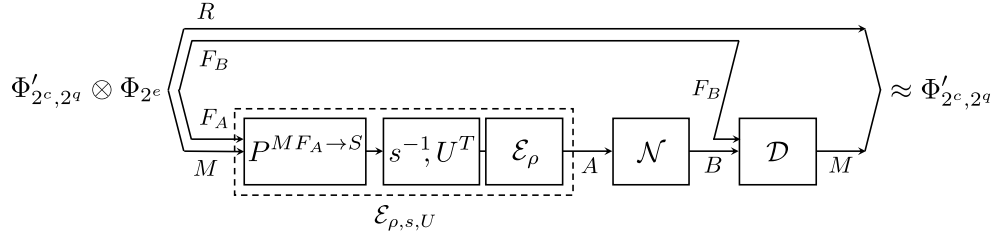
Fig. 2. The protocol for communication over the channel $\mathcal{N}$ constructed in terms of randomized partial decoupling is depicted. The encoding operation $\mathcal{E}_{\rho,s,U}$ is composed of (i) a linear isometry $P^{MF_A \to S}$ that embeds the message system M and Alice's share of the entanglement resource F_A to a larger system S, (ii) the permutation s and the unitary U that appear in randomized partial decoupling, and (iii) a linear CPTP map $\mathcal{E}_\rho$ that is obtained from ρ^{SA} by the Choi-Jamiolkowski correspondence. The explicit form of the decoder $\mathcal{D}$ is left open, because we only prove the *existence* of a proper decoder $\mathcal{D}$ in the proof of the direct part.

Both the classical and quantum errors shall be small, i.e., we want δ_C and δ_Q to be below a certain threshold $\delta > 0$.

A rigorous definition of a code is given as follows. Here, we treat the classical and quantum parts of the message as one large quantum system, because it is more convenient in our analysis.

Definition 1: Consider the setting described above. Let M_c and R_c be 2^c-dimensional quantum system with a fixed orthonormal basis $\{|j\rangle\}_{j=1}^{2^c}$. We denote $M_c M_q$ by M and $R_c R_q$ by R for brevity. Let $\Phi'^{MR}_{2^c,2^q}$ be a source state defined by

$$\Phi'^{MR}_{2^c,2^q} = \frac{1}{2^c} \sum_{j=1}^{2^c} |j\rangle\langle j|^{M_c} \otimes |\Phi_{2^q}\rangle\langle\Phi_{2^q}|^{M_q R_q} \otimes |j\rangle\langle j|^{R_c}. \quad (21)$$

A pair of an encoding CPTP map $\mathcal{E}^{MF_A \to A}$ and a decoding CPTP map $\mathcal{D}^{BF_B \to M}$ is called a (c,q,e,δ) code for the channel $\mathcal{N}$ if it holds that

$$\left\| \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}(\Phi'^{MR}_{2^c,2^q} \otimes \Phi^{F_A F_B}_{2^e}) - \Phi'^{MR}_{2^c,2^q} \right\|_1 \leq \delta \quad (22)$$

and

$$\mathcal{E}^{MF_A \to A} = \mathcal{E}^{MF_A \to A} \circ \mathcal{C}^{M_c}, \quad (23)$$
$$\mathcal{D}^{BF_B \to M} = \mathcal{C}^{M_c} \circ \mathcal{D}^{BF_B \to M}, \quad (24)$$

where $\mathcal{C}$ is the completely dephasing operation on M_c with respect to the basis $\{|j\rangle\}_{j=1}^{2^c}$.

Note that the condition (22) is equivalent to the condition that $\delta_C, \delta_Q \leq \delta$, up to a constant multiplication. The correspondence between the encoding and decoding operations is given by

$$\mathcal{E}_j^{M_q F_A \to A}(\cdot) = \mathcal{E}^{MF_A \to A}(|j\rangle\langle j|^{M_c} \otimes (\cdot)^{M_q F_A}), \quad (25)$$
$$\mathcal{D}_j^{BF_B \to M_q}(\cdot) = \langle j|^{M_c} \mathcal{D}^{BF_B \to M}(\cdot)|j\rangle^{M_c} \quad (26)$$

and

$$\mathcal{E}^{MF_A \to A}(\cdot) = \sum_{j=1}^{2^c} \mathcal{E}_j(\langle j|^{M_c}(\cdot)|j\rangle^{M_c}), \quad (27)$$

$$\mathcal{D}^{BF_B \to M}(\cdot) = \sum_{j=1}^{2^c} |j\rangle\langle j|^{M_c} \otimes \mathcal{D}_j^{BF_B \to M_q}(\cdot). \quad (28)$$

It should be noted that the capacity theorems obtained in terms of the average probability of error, as in (19), (20) and (22), are translated into those based on the worst-case

error, up to halving of the message length. See, for example, Corollary 1 in [18] and Theorem 11 in [19] for the quantum part and Lemma 1 in [16] for the classical part. The latter is known as the expurgation trick (see e.g. [1]).

A. Channel Capacity With Limited Entanglement

First, we consider the situation in which the amount of the resource of shared entanglement is limited.

Definition 2: A triplet (c,q,e) is said to be achievable within the error δ for the channel $\mathcal{N}^{A \to B}$ if there exists a (c,q,e,δ) code for $\mathcal{N}^{A \to B}$.

The direct part is represented by the following theorem. The proof is based on the direct part of randomized partial decoupling (Theorem 8), and will be provided in Section V. A protocol that achieves the direct bound is depicted in Figure 2.

Theorem 3: Let S_r be a finite dimensional quantum system and let S_c be a quantum system with a fixed orthonormal basis $\{|j\rangle\}_{j=1}^{d_{S_c}}$ such that $d_{S_c} \geq 2$. We denote $S_c S_r$ by S. Consider a state in the form of

$$\rho^{SA} = \frac{1}{d_{S_c}} \sum_{j=1}^{d_{S_c}} |j\rangle\langle j|^{S_c} \otimes \rho_j^{S_r A}, \quad (29)$$

where $\{\rho_j\}_{j=1}^{d_{S_c}}$ is a set of normalized states on $S_r A$ such that ρ^S is the full-rank maximally mixed state on S. For any such S_c, S_r, ρ^{SA}, any $\delta_1, \delta_2 > 0$ and $\epsilon \geq 0$, a triplet (c,q,e) is achievable within the error

$$\delta = 2\sqrt{\sqrt{\delta_1} + \sqrt{\delta_2} + 4\epsilon} \quad (30)$$

for the channel $\mathcal{N}^{A \to B}$ if $d_{S_c} \geq 2^c$ and the following three inequalities hold:

$$q + e \leq \log d_{S_r}, \quad (31)$$
$$c + q - e \leq -H_{\max}^\epsilon(S|B)_{\mathcal{N}(\rho)}$$
$$+ \log(d_{S_c} - 1) + \log \delta_1, \quad (32)$$
$$q - e \leq -H_{\max}^\epsilon(S_r|BS_c)_{\mathcal{N}(\rho)} + \log \delta_2. \quad (33)$$

The same statement holds in the cases of $(c = 0, d_{S_c} = 1)$ and $(q = e = 0, d_{S_r} = 1)$. In the former case, the condition (32) is removed and δ_1 in (30) is assumed to be zero. In the latter, the condition (33) is removed and δ_2 in (30) is considered to be zero.

The converse part is stated by the following theorem, which will be proved in Section VI based on the converse part for randomized partial decoupling (Theorem 9).

Theorem 4: Suppose that a triplet (c, q, e) is achievable within the error δ for the channel $\mathcal{N}^{A \to B}$. Then, there exist a quantum system S satisfying $d_S \leq 2^{c+q+e}$ and a state ρ^{SA} such that the following conditions hold. First, S is composed of finite dimensional quantum systems S_c and S_r, where S_c is equipped with an orthonormal basis $\{|j\rangle\}_{j=1}^{d_{S_c}}$. Second, the state ρ^{SA} is in the form of

$$\rho^{SA} = \frac{1}{d_{S_c}} \sum_{j=1}^{d_{S_c}} |j\rangle\langle j|^{S_c} \otimes \rho_j^{S_r A}, \qquad (34)$$

where $\{\rho_j\}_{j=1}^{d_{S_c}}$ is a set of normalized states on $S_r A$ and ρ^S is the full-rank maximally mixed state on S. Third, for any $\iota \in (0, 1]$, it holds that

$$q + e \leq \log d_{S_r}, \qquad (35)$$

$$c + q - e \leq -H_{\max}^{\lambda}(S|B)_{\mathcal{N}(\rho)} + \log d_{S_c} - \log \iota, \qquad (36)$$

$$q - e \leq -H_{\max}^{\lambda'}(S_r|BS_c)_{\mathcal{N}(\rho)} - \log \iota. \qquad (37)$$

The smoothing parameters λ and λ' are given by

$$\lambda := 2\sqrt{\iota + 2x^2} + x + 2x^2, \qquad (38)$$

$$\lambda' := \sqrt{4\sqrt{\iota + 2x} + 2\sqrt{x} + (4\sqrt{\iota + 8} + 24)x} \qquad (39)$$

and $x := 2\sqrt[8]{\delta}$.

B. Channel Capacity With Free Entanglement

Second, we consider the situation in which the resource of shared entanglement is freely available. Following [19], we assume that the entanglement resource is given in the form of the maximally entangled state.

Definition 5: A pair (c, q) is said to be achievable within the error δ for the channel $\mathcal{N}^{A \to B}$ with the assistance of entanglement if there exists $e \geq 0$ such that a triplet (c, q, e) is achievable within the error δ for $\mathcal{N}^{A \to B}$.
The direct and converse bounds for the scenario of free entanglement immediately follow from the direct bound and the converse bound for the case of limited entanglement, i.e., from Theorem 3 and Theorem 4.

Corollary 6: For any $\epsilon \in [0, 1/2)$ and $\delta' \in (0, 1 - 2\epsilon]$, a pair (c, q) is achievable within the error

$$\delta = 2\sqrt{\sqrt{2\delta'} + \sqrt{\delta'} + 4\epsilon} \qquad (40)$$

for the channel $\mathcal{N}^{A \to B}$ with the assistance of entanglement, if there exist a quantum system S and a state ρ^{SA} such that ρ^S is the full-rank maximally mixed state on S and the following inequality holds:

$$c + 2q \leq \log d_S - H_{\max}^{\epsilon}(S|B)_{\mathcal{N}(\rho)} + \log \delta'. \qquad (41)$$

Corollary 7: Suppose that a pair (c, q) is achievable within the error δ for the channel $\mathcal{N}^{A \to B}$ with the assistance of entanglement. Then, there exist a quantum system S and a

state ρ^{SA} such that ρ^S is the full-rank maximally mixed state on S and for any $\iota \in (0, 1]$, it holds that

$$c + 2q \leq \log d_S - H_{\max}^{\lambda}(S|B)_{\mathcal{N}(\rho)} - \log \iota. \qquad (42)$$

The smoothing parameter λ is given by (38).

Proof of Corollaries: Corollary 7 immediately follows from Inequalities (35) and (36) in Theorem 4. To prove Corollary 6 from Theorem 3, suppose that there exists a state ρ^{SA} that satisfies the conditions in Corollary 6. Let S_c' be a system such that $d_{S_c'} \geq 2^c$. Define $S_r' := S$, $S' := S_c' S_r'$ and consider a state

$$\rho'^{S'A} := \frac{1}{d_{S_c'}} \sum_{j=1}^{d_{S_c'}} |j\rangle\langle j|^{S_c'} \otimes \rho^{S_r'A}. \qquad (43)$$

Due to the property of the smooth max entropy for product states (Lemma 24), we have

$$H_{\max}^{\epsilon}(S_r'|BS_c')_{\mathcal{N}(\rho')} = H_{\max}^{\epsilon}(S|B)_{\mathcal{N}(\rho)} \qquad (44)$$

$$\geq H_{\max}^{\epsilon}(S'|B)_{\mathcal{N}(\rho')} - \log d_{S_c'}. \qquad (45)$$

It follows from (44) and (41) that

$$q - \log d_{S_r'} \leq -H_{\max}^{\epsilon}(S_r'|BS_c')_{\mathcal{N}(\rho')} + \log \delta' - c - q. \qquad (46)$$

Thus, there exists $e \in \mathbb{R}$ such that

$$\begin{cases} q + e & \leq \log d_{S_r'}, \\ c + q - e & \leq -H_{\max}^{\epsilon}(S_r'|BS_c')_{\mathcal{N}(\rho')} + \log \delta'. \end{cases} \qquad (47)$$

We may assume that $e \geq 0$, since $q \leq \log d_{S_r'}$. This is because the dimension bound for the smooth max entropy (see Lemma 25 in Appendix B) imply

$$2q \leq \log d_S - H_{\max}^{\epsilon}(S|B)_{\mathcal{N}(\rho)} \qquad (48)$$

$$\leq 2 \log d_S + \log\left(\frac{\delta'}{1 - 2\epsilon}\right). \qquad (49)$$

The second inequality in (47) leads to

$$c + q - e \leq -H_{\max}^{\epsilon}(S'|B)_{\mathcal{N}(\rho')} + \log(d_{S_c'} - 1) + \log 2\delta' \qquad (50)$$

due to Inequality (45) and the relation $d_{S_c'}/(d_{S_c'} - 1) \leq 2$, and to

$$q - e \leq -H_{\max}^{\epsilon}(S_r'|BS_c')_{\mathcal{N}(\rho')} + \log \delta' \qquad (51)$$

since $c \geq 0$. Combining (50), (51) and the first inequality in (47) with Theorem 3, we complete the proof. ∎

IV. RANDOMIZED PARTIAL DECOUPLING

In this section, we briefly review a task that we call randomized partial decoupling [30] and present the direct and converse bounds for it. This is a generalization of the decoupling theorem in the version of [35]. For the details and proofs, see the paper by the same authors [30].

Randomized partial decoupling is a task in which a bipartite quantum state Ψ^{AR} is transformed by a unitary operation on A and then is subject to the action of a linear CP map $\mathcal{T}^{A \to E}$. We assume that the Hilbert space $\mathcal{H}^A$ is decomposed into a direct-sum form as $\mathcal{H}^A = \bigoplus_{j=1}^{J} \mathcal{H}_j^A$, where each $\mathcal{H}_j^A$ $(j = 1, \cdots, J)$ has the same dimension r. Let $\mathcal{H}^{A_c}$ be a J-dimensional Hilbert space with a fixed orthonormal
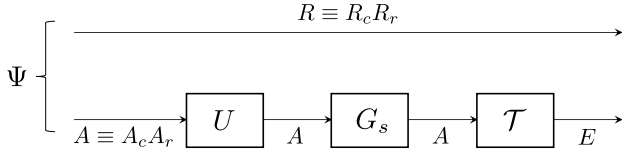
Fig. 3. The procedure used in the randomized partial decoupling is depicted. For a given initial state Ψ^{AR}, a random unitary U in the form of (54) and a permutation G_s are first applied to A. Then, the system A is mapped to another one E by $\mathcal{T}$.

basis $\{|j\rangle\}_{j=1}^J$ and $\mathcal{H}^{A_r}$ be an r-dimensional Hilbert space. The Hilbert space $\mathcal{H}^A$ is isomorphic to a tensor product Hilbert space $\mathcal{H}^{A_c} \otimes \mathcal{H}^{A_r}$, i.e., $A \cong A_c A_r$. In terms of this decomposition, any state Ψ^{AR} is written as

$$\Psi^{AR} = \sum_{j,k=1}^J |j\rangle\langle k|^{A_c} \otimes \Psi_{jk}^{A_r R}, \qquad (52)$$

where $\Psi_{jk}^{A_r R} := \Pi_j^A \Psi^{AR} \Pi_k^A$ with Π_j being the projection onto $\mathcal{H}_j^A$. By properly choosing orthonormal bases, a maximally entangled state $|\Phi\rangle^{AA'}$ is represented by

$$|\Phi\rangle^{AA'} = \left(\frac{1}{\sqrt{J}} \sum_{j=1}^J |j\rangle^{A_c} |j\rangle^{A'_c} \right) \otimes \left(\frac{1}{\sqrt{r}} \sum_{\alpha=1}^r |\alpha\rangle^{A_r} |\alpha\rangle^{A'_r} \right), \quad (53)$$

where $A_c \cong A'_c$ and $A_r \cong A'_r$.

Consider a random unitary U on A in the form of

$$U := \sum_{j=1}^J |j\rangle\langle j|^{A_c} \otimes U_j^{A_r}. \qquad (54)$$

where U_j is independently chosen for each j from the Haar measure H on the unitary group on $\mathcal{H}^{A_r}$. For any state Ψ^{AR}, the averaged state after the action of this random unitary is given by

$$\Psi_{\mathrm{av}}^{AR} := \mathbb{E}_{U_1,\dots,U_J \sim \mathsf{H}}[U^A(\Psi^{AR}) U^{\dagger A}] \qquad (55)$$

$$= \sum_{j=1}^J p_j |j\rangle\langle j|^{A_c} \otimes \pi^{A_r} \otimes \Psi_j^R. \qquad (56)$$

Here, π^{A_r} is the maximally mixed state on $\mathcal{H}^{A_r}$, $\{p_j\}_{j=1}^J$ is a probability distribution defined by $p_j := \mathrm{Tr}[\langle j|^{A_c} \Psi^{AR} |j\rangle^{A_c}]$, and Ψ_j^R is a normalized state on $\mathcal{H}^R$ defined by $\Psi_j^R := p_j^{-1} \Psi_{jj}^R$. In the following, we denote $\mathbb{E}_{U_1,\dots,U_J \sim \mathsf{H}}$ simply by $\mathbb{E}_U$ when there is no ambiguity. Let $\mathbb{P}$ be the permutation group on $\{1, \cdots, J\}$ and define a unitary G_s for each $s \in \mathbb{P}$ by

$$G_s := \sum_{j=1}^J |s(j)\rangle\langle j|^{A_c} \otimes I^{A_r}. \qquad (57)$$

The permutation s is chosen at random according to the uniform distribution on $\mathbb{P}$. Our concern is how close the final state $\mathcal{T}^{A \to E} \circ \mathcal{G}_s^A \circ \mathcal{U}^A(\Psi^{AR})$ is, on average over all U, to the averaged final state $\mathcal{T}^{A \to E} \circ \mathcal{G}_s^A(\Psi_{\mathrm{av}}^{AR})$, for typical choices of the permutation s (see Figure 3 as well).

For simplicity of analysis, we assume that $R \cong R_c R_r$, where R_c is a quantum system with dimension J. We also

assume that Ψ^{AR} is decomposed in the form of

$$\Psi^{AR} = \sum_{k,l=1}^J |k\rangle\langle l|^{A_c} \otimes \psi_{kl}^{A_r R_r} \otimes |k\rangle\langle l|^{R_c}, \qquad (58)$$

where $\psi_{kl} \in \mathcal{L}(\mathcal{H}^{A_r} \otimes \mathcal{H}^{R_r})$ for each k and l. Such states are called *classically coherent states* [21].

The following theorem is the direct part of the randomized partial decoupling theorem.

Theorem 8 (Theorem 3 in [30]): Consider a linear CP map $\mathcal{T}^{A \to E}$ and a state $\Psi^{AR} \in \mathcal{S}_=(\mathcal{H}^{AR})$ that is decomposed as (58). Let U and G_s be random unitaries defined by (54) and (57), respectively. Define the partial decoupling error $\Delta_{s,U}(\mathcal{T}, \Psi)$ by

$$\Delta_{s,U}(\mathcal{T}, \Psi) := \left\| \mathcal{T}^{A \to E} \circ \mathcal{G}_s^A (\mathcal{U}^A(\Psi^{AR}) - \Psi_{\mathrm{av}}^{AR}) \right\|_1, \quad (59)$$

where $\Psi_{\mathrm{av}}^{AR} := \mathbb{E}_{U_1,\dots,U_J \sim \mathsf{H}}[\mathcal{U}^A(\Psi^{AR})]$. Then, for any $\epsilon, \mu \geq 0$, it holds that

$$\mathbb{E}_{s,U}[\Delta_{s,U}(\mathcal{T}, \Psi)] \leq \theta_I + \theta_{II} + 4(\epsilon + \mu + \epsilon\mu). \qquad (60)$$

The terms θ_I and θ_{II} are represented by

$$\theta_I = \begin{cases} 2^{-\frac{1}{2}H_I} & (J \geq 2) \\ 0 & (J = 1) \end{cases}, \quad \theta_{II} = \begin{cases} 2^{-\frac{1}{2}H_{II}} & (d_{A_r} \geq 2) \\ 0 & (d_{A_r} = 1) \end{cases}, \quad (61)$$

where the exponents H_I and H_{II} are given by

$$H_I = \log(J-1) + H_{\min}^\epsilon(A|R)_\Psi - H_{\max}^\mu(A|C)_{\mathcal{C}(\tau)}, \quad (62)$$

$$H_{II} = H_{\min}^\epsilon(A|R)_{\mathcal{C}(\Psi)} - H_{\max}^\mu(A_r|CA_c)_{\mathcal{C}(\tau)}, \qquad (63)$$

respectively. Here, $\mathcal{C}$ is the completely dephasing operation on A_c with respect to the basis $\{|j\rangle\}_{j=1}^J$ and τ is the Choi-Jamiolkowski state of the complementary map $\mathcal{T}^{A \to C}$ of $\mathcal{T}^{A \to E}$, i.e. $\tau = \mathfrak{J}(\mathcal{T}^{A \to C})$.

In [30], we also obtained a converse bound for randomized partial decoupling, which is stated by the following theorem.

Theorem 9 (Theorem 4 in [30]): Consider a linear trace-preserving CP map $\mathcal{T}^{A \to E}$ and a state $\Psi^{AR} \in \mathcal{S}_=(\mathcal{H}^{AR})$ that is decomposed as (58). Suppose that, for $\delta > 0$, there exists a normalized state in the form of

$$\Omega^{ER} := \sum_{j=1}^J p_j \varsigma_j^E \otimes \Psi_j^{R_r} \otimes |j\rangle\langle j|^{R_c}, \qquad (64)$$

such that

$$\left\| \mathcal{T}^{A \to E}(\Psi^{AR}) - \Omega^{ER} \right\|_1 \leq \delta. \qquad (65)$$

Then, for any $\upsilon \in [0, 1/2)$ and $\iota \in (0, 1]$, it holds that

$$H_{\min}^\lambda(A|R)_\Psi - H_{\min}^\upsilon(BR|C)_{\mathcal{T} \circ \mathcal{C}(\Psi)} + \log J \geq \log \iota, \quad (66)$$

$$H_{\min}^{\lambda'}(A|R)_{\mathcal{C}(\Psi)} - H_{\min}^\upsilon(BR_r|CR_c)_{\mathcal{T} \circ \mathcal{C}(\Psi)} \\ \geq \log \iota + \log(1 - 2\upsilon). \qquad (67)$$

The second terms in the L.H.S.s of (66) and (67) are for a purification $|\Psi\rangle^{ABR}$ of Ψ^{AR} and the complementary channel $\mathcal{T}^{A \to C}$ of $\mathcal{T}^{A \to E}$, with $\mathcal{C}$ being the completely dephasing channel on A_c. The smoothing parameters λ and λ' are given by

$$\lambda := 2\sqrt{\iota + 4\sqrt{20\upsilon + 2\delta}} + \sqrt{2\sqrt{20\upsilon + 2\delta}}$$
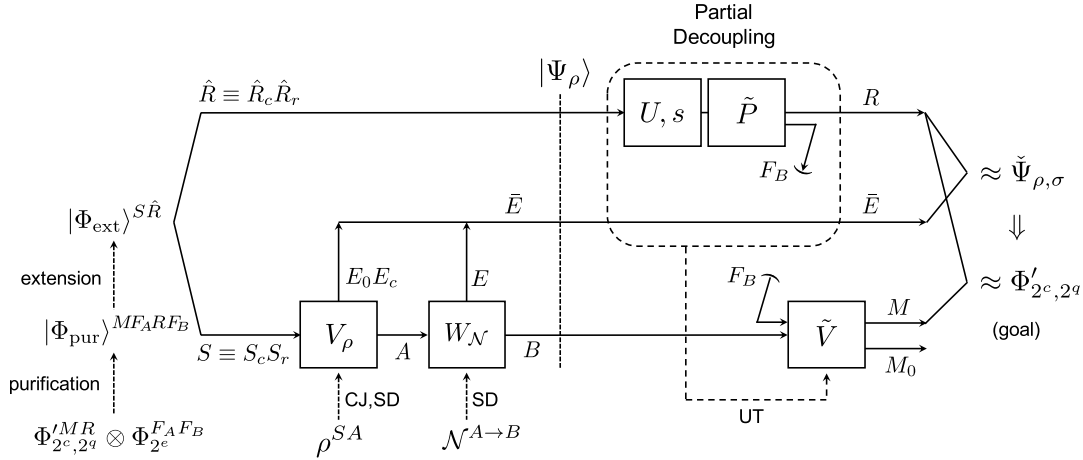
Fig. 4. The definition of the state $|\Psi_\rho\rangle$ given by (87) and its transformation by partial decoupling are depicted. CJ, SD and UT stand for the Choi-Jamiolkowski correspondense, the Stinespring dilation and Uhlmann's theorem, respectively. Note that $R \equiv R_c R_q$ and $M \equiv M_c M_q$. In composing the encoding operation, we use the fact that the actions of U, s and $\tilde{\mathcal{P}}$ on system $\hat{R}$ are replaced by those of $\tilde{\mathcal{P}}$, G_s^T and U^T in this order, on system $\hat{S}$ before applying V_ρ. This trick was introduced in [33] to analyze the quantum channel capacity in an asymptotic scenario.

$$+ 2\sqrt{2\delta} + 2\sqrt{20\upsilon + 2\delta} + 3\upsilon, \tag{68}$$

$$\lambda' := \upsilon + \sqrt{4\sqrt{\iota + 2x} + 2\sqrt{x} + (4\sqrt{\iota + 8} + 24)x} \tag{69}$$

and $x := \sqrt{2}\sqrt[4]{24\upsilon + 2\delta}$.

V. PROOF OF THE DIRECT PART (THEOREM 3)

We prove the direct part of the capacity theorem (Theorem 3) based on the direct part of the randomized partial decoupling theorem (Theorem 8). We follow the idea of Ref. [33] that the problem of finding a good code for a quantum channel is equivalent to the problem of finding a good way to decouple a bipartite state constructed from the channel.

Fix an arbitrary triplet (c, q, e), $\delta > 0$, system $S \equiv S_c S_r$ and a state ρ^{SA} that satisfy the conditions in Theorem 3. We prove achievability of the triplet (c, q, e) within the error δ along the following lines: First, we construct a state Ψ_ρ from the state ρ^{SA}, the channel $\mathcal{N}^{A \to B}$ and a state obtained by "purifying" and "extending" the source state and the resource state. Second, we prove that, if a CP map achieves randomized partial decoupling of the state Ψ_ρ for a particular choice of s and U in high precision, there exist an encoder $\mathcal{E}_{\rho,s,U}$ and a decoder $\mathcal{D}$ that accomplish the communication with a small error. Finally, we evaluate the precision of the randomized partial decoupling based on Theorem 8, by which we complete the proof of Theorem 3.

A. Definitions of States and Operations

We embed M_c and $M_q F_A$ into S_c and S_r, respectively. This is possible because of the conditions $\dim S_c \geq 2^c$ and $\dim S_r \geq 2^{q+e}$ in Theorem 3. Similarly, we embed R_c and $R_q F_B$ to larger systems $\hat{R}_c$ and $\hat{R}_r$, respectively, such that $\hat{R}_c \cong S_c$ and $\hat{R}_r \cong S_r$. The embeddings are represented by linear isometries $P^{M_c \to S_c}$, $P^{M_q F_A \to S_r}$, $P^{R_c \to \hat{R}_c}$ and $P^{R_q F_B \to \hat{R}_r}$. For the simplicity of notations, we denote $\hat{R}_c \hat{R}_r$

by $\hat{R}$. In total, the systems $M F_A$ and $R F_B$ are embedded into systems S and $\hat{R}$ by linear isometries

$$P^{M F_A \to S} := P^{M_c \to S_c} \otimes P^{M_q F_A \to S_r}, \tag{70}$$

$$P^{R F_B \to \hat{R}} := P^{R_c \to \hat{R}_c} \otimes P^{R_q F_B \to \hat{R}_r}, \tag{71}$$

respectively. The adjoint map corresponding to these isometries are given by

$$\mathcal{P}^{\dagger S \to M F_A}(\cdot) := (P^{M F_A \to S})^\dagger (\cdot)(P^{M F_A \to S}), \tag{72}$$

$$\mathcal{P}^{\dagger \hat{R} \to R F_B}(\cdot) := (P^{R F_B \to \hat{R}})^\dagger (\cdot)(P^{R F_B \to \hat{R}}). \tag{73}$$

Define a "purified" source-resource state by

$$|\Phi_{\text{pur}}\rangle^{M F_A R F_B} := |\Phi_{2^{c+q}}\rangle^{MR} |\Phi_{2^e}\rangle^{F_A F_B} \tag{74}$$

$$= \frac{1}{\sqrt{2^c}} \sum_{j=1}^{2^c} |j\rangle^{M_c} |j\rangle^{R_c} |\Phi_{2^q}\rangle^{M_q R_q} |\Phi_{2^e}\rangle^{F_A F_B}. \tag{75}$$

Note that

$$\mathcal{C}^{M_c}(\Phi_{\text{pur}}) = \Phi'^{MR}_{2^c, 2^q} \otimes \Phi^{F_A F_B}_{2^e}, \tag{76}$$

where $\mathcal{C}^{M_c}$ the completely dephasing operation on M_c with respect to the basis $\{|j\rangle\}_{j=1}^{2^c}$. We also introduce an "extended" one by

$$|\Phi_{\text{ext}}\rangle^{S\hat{R}} := \frac{1}{\sqrt{d_{S_c}}} \sum_{j=1}^{d_{S_c}} |j\rangle^{S_c} |j\rangle^{\hat{R}_c} |\Phi_{d_{S_r}}\rangle^{S_r \hat{R}_r}. \tag{77}$$

We properly choose the embedding isometries so that

$$|\Phi_{\text{pur}}\rangle^{M F_A R F_B}$$
$$= \sqrt{\frac{d_S}{2^{c+q+e}}} (P^{M F_A \to S} \otimes P^{R F_B \to \hat{R}})^\dagger |\Phi_{\text{ext}}\rangle^{S\hat{R}} \tag{78}$$

and

$$P^{M_c \to S_c}|j\rangle^{M_c} = |j\rangle^{S_c}, \quad P^{R_c \to \hat{R}_c}|j\rangle^{R_c} = |j\rangle^{\hat{R}_c} \tag{79}$$

for any $j = 1, \cdots, 2^c$. Using

$$\tilde{P}^{\hat{R} \to R F_B} := \sqrt{\frac{d_S}{2^{c+q+e}}} (P^{R F_B \to \hat{R}})^\dagger, \tag{80}$$

the purified source-resource state and the extended one are simply related as

$$P^{MFA \to S}|\Phi_{\text{pur}}\rangle^{MFARFB} = \tilde{P}^{\hat{R} \to RFB}|\Phi_{\text{ext}}\rangle^{S\hat{R}}. \tag{81}$$

Let $\mathcal{E}_\rho^{S \to A}$ and $\mathcal{E}_{\rho_j}^{S_r \to A}$ be linear maps defined by the Choi-Jamiolkowski correspondence from ρ^{SA} and $\rho_j^{S_r A}$, respectively. That is, $\mathcal{E}_\rho^{S \to A} := \mathfrak{J}_S^{-1}(\rho^{SA})$ and $\mathcal{E}_{\rho_j}^{S_r \to A} := \mathfrak{J}_{S_r}^{-1}(\rho_j^{S_r A})$. Due to the condition that ρ^S is the full-rank maximally mixed state, the two maps are completely positive and trace-preserving. From the decomposition (29), it follows that

$$\mathcal{E}_\rho^{S \to A}(\tau) = \sum_{j=1}^{d_{S_c}} \mathcal{E}_{\rho_j}^{S_r \to A}(\langle j|^{S_c} \tau |j\rangle^{S_c}). \tag{82}$$

We denote by $V_{\rho_j}^{S_r \to AE_0}$ the Stinespring dilation of $\mathcal{E}_{\rho_j}^{S_r \to A}$ for each j. Introducing a quantum system E_c with a fixed orthonormal basis $\{|j\rangle\}_{j=1}^{d_{S_c}}$, the Stinespring dilation V_ρ of $\mathcal{E}_\rho^{S \to A}$ is given by

$$V_\rho^{S \to AE_0 E_c} = \sum_{j=1}^{d_{S_c}} |j\rangle^{E_c} \langle j|^{S_c} \otimes V_{\rho_j}^{S_r \to AE_0}. \tag{83}$$

It is straightforward from (82) that

$$\mathcal{E}_\rho^{S \to A} \circ \mathcal{C}^{S_c} = \mathcal{E}_\rho^{S \to A}, \tag{84}$$

with $\mathcal{C}^{S_c}$ being the completely dephasing operation on S_c with respect to the basis $\{|j\rangle\}_{j=1}^{d_{S_c}}$.

Let $W_\mathcal{N}^{A \to BE}$ be the Stinespring dilation of $\mathcal{N}^{A \to B}$. Using the extended source-resource state (77), we define the following pure state, where $\bar{E} \equiv E_c EE_0$ (see Figure 4):

$$|\Psi_\rho\rangle^{\hat{R}B\bar{E}} := W_\mathcal{N}^{A \to BE} \circ V_\rho^{S \to AE_0 E_c}|\Phi_{\text{ext}}\rangle^{S\hat{R}}. \tag{85}$$

Defining the state

$$|\rho_j\rangle^{\hat{R}_r AE_0} := V_{\rho_j}^{S_r \to AE_0}|\Phi_{d_{S_r}}\rangle^{S_r \hat{R}_r}, \tag{86}$$

it follows from (83) that

$$|\Psi_\rho\rangle^{\hat{R}B\bar{E}} = \frac{1}{\sqrt{d_{S_c}}} \sum_{j=1}^{d_{S_c}} |j\rangle^{\hat{R}_c}|j\rangle^{E_c} W_\mathcal{N}^{A \to BE}|\rho_j\rangle^{\hat{R}_r AE_0}. \tag{87}$$

We trace out system B to obtain the state $\Psi_\rho^{\hat{R}\bar{E}} := \text{Tr}_B[|\Psi_\rho\rangle\langle\Psi_\rho|]$.

B. Construction of Encoding and Decoding Operations

We consider randomized partial decoupling of the "bipartite" state $\Psi_\rho^{\hat{R}\bar{E}}$ by a linear CP map $\text{Tr}_{F_B} \circ \tilde{\mathcal{P}}^{\hat{R} \to RF_B} : \hat{R} \to RF_B$ (see Figure 4), where $\tilde{P}^{\hat{R} \to RF_B}$ is defined by (80). For a unitary $U^{\hat{R}} = \sum_{j=1}^{d_{S_c}} |j\rangle\langle j|^{\hat{R}_c} \otimes U_j^{\hat{R}_r}$ and a permutation s on $\{1, \cdots, d_{S_c}\}$, define the *partial decoupling error* $\Delta_{s,U}$ by

$$\Delta_{s,U} := \left\| \text{Tr}_{F_B} \circ \tilde{\mathcal{P}}^{\hat{R} \to RF_B} \circ \mathcal{G}_s^{\hat{R}} \circ \mathcal{U}^{\hat{R}}(\Psi_\rho^{\hat{R}\bar{E}}) - \tilde{\Psi}_{\rho,s}^{R\bar{E}} \right\|_1. \tag{88}$$

Here, we have defined

$$\tilde{\Psi}_{\rho,s}^{R\bar{E}} := \text{Tr}_{F_B} \circ \tilde{\mathcal{P}}^{\hat{R} \to RF_B} \circ \mathcal{G}_s^{\hat{R}}(\Psi_{\rho,\text{av}}^{\hat{R}\bar{E}}), \tag{89}$$

where

$$\Psi_{\rho,\text{av}}^{\hat{R}\bar{E}} := \mathbb{E}_U[\mathcal{U}^{\hat{R}}(\Psi_\rho^{\hat{R}\bar{E}})]. \tag{90}$$

An evaluation of the partial decoupling error $\Delta_{s,U}$ will be given in the next subsection, based on the direct part of the randomized partial decoupling theorem (Theorem 8). We introduce the operator

$$\tilde{P}_{s,U}^{\hat{R} \to RF_B} := \tilde{P}^{\hat{R} \to RF_B} G_s^{\hat{R}} U^{\hat{R}}, \tag{91}$$

by which (88) is simply represented as

$$\Delta_{s,U} := \left\| \text{Tr}_{F_B} \circ \tilde{\mathcal{P}}_{s,U}^{\hat{R} \to RF_B}(\Psi_\rho^{\hat{R}\bar{E}}) - \tilde{\Psi}_{\rho,s}^{R\bar{E}} \right\|_1. \tag{92}$$

Let $|\tilde{\Psi}_{\rho,s}\rangle^{MR\bar{E}M_0}$ be a purification of $\tilde{\Psi}_{\rho,s}^{R\bar{E}}$ with MM_0 being a purifying system. Due to Uhlmann's theorem ([42]; see also e.g. Chapter 9 in [1]) and (92), there exists a linear isometry $\tilde{V}^{BF_B \to MM_0}$ such that

$$\left\| \tilde{\mathcal{V}}^{BF_B \to MM_0} \otimes \tilde{\mathcal{P}}_{s,U}^{\hat{R} \to RF_B}(\Psi_\rho^{\hat{R}B\bar{E}}) - \tilde{\Psi}_{\rho,s}^{MR\bar{E}M_0} \right\|_1$$
$$\leq 2\sqrt{\Delta_{s,U}}. \tag{93}$$

Note that $\tilde{V}^{BF_B \to MM_0}$ depends on ρ, s and U in general. Defining

$$\mathcal{D}^{BF_B \to M} := (\text{Tr}_{M_0} \otimes \mathcal{C}^{M_c}) \circ \tilde{\mathcal{V}}^{BF_B \to MM_0}, \tag{94}$$

and tracing out systems $\bar{E}$ and M_0 in (93), we obtain

$$\left\| \mathcal{D}^{BF_B \to M} \otimes \tilde{\mathcal{P}}_{s,U}^{\hat{R} \to RF_B}(\Psi_\rho^{\hat{R}B}) - \mathcal{C}^{M_c}(\tilde{\Psi}_{\rho,s}^{MR}) \right\|_1$$
$$\leq 2\sqrt{\Delta_{s,U}}. \tag{95}$$

To obtain an explicit form of the state $\tilde{\Psi}_{\rho,s}^{MR}$, we use (87) and (90) to have

$$\Psi_{\rho,\text{av}}^{\hat{R}\bar{E}} = \frac{1}{d_{S_c}} \sum_{j=1}^{d_{S_c}} |j\rangle\langle j|^{\hat{R}_c} \otimes |j\rangle\langle j|^{E_c} \otimes \pi^{\hat{R}_r} \otimes \mathcal{N}^{A \to E}(\rho_j^{AE_0}).$$

Combining this with (89), and by using (80), we obtain

$$\tilde{\Psi}_{\rho,s}^{R\bar{E}} = \frac{1}{2^c} \sum_{j=1}^{2^c} |j\rangle\langle j|^{R_c} \otimes |s^{-1}(j)\rangle\langle s^{-1}(j)|^{E_c}$$
$$\otimes \pi_{2^q}^{R_q} \otimes \mathcal{N}^{A \to E}(\rho_{s^{-1}(j)}^{AE_0}). \tag{96}$$

A purification of this state is given by

$$|\tilde{\Psi}_{\rho,s}\rangle^{MR\bar{E}M_0} :=$$
$$\frac{1}{\sqrt{2^c}} \sum_{j=1}^{2^c} |j\rangle^{M_c}|j\rangle^{R_c}|s^{-1}(j)\rangle^{E_c}|\Phi_{2^q}\rangle^{M_q R_q}|\varrho_{s^{-1}(j)}\rangle^{E_0 EM_0},$$

with $|\varrho_j\rangle^{E_0 EM_0}$ being a purification of $\mathcal{N}^{A \to E}(\rho_j^{AE_0})$. Thus, we trace out $\bar{E}M_0$ to obtain $\tilde{\Psi}_{\rho,s}^{MR} = \Phi_{2^c,2^q}'^{MR}$. Substituting this to (95), and noting that $\mathcal{C}^{M_c}(\Phi_{2^c,2^q}'^{MR}) = \Phi_{2^c,2^q}'^{MR}$, we arrive at

$$\left\| \mathcal{D}^{BF_B \to M} \otimes \tilde{\mathcal{P}}_{s,U}^{\hat{R} \to RF_B}(\Psi_\rho^{\hat{R}B}) - \Phi_{2^c,2^q}'^{MR} \right\|_1$$
$$\leq 2\sqrt{\Delta_{s,U}}. \tag{97}$$

The first term in the L.H.S. of (97) is calculated as follows. Note that the state $|\Phi_{\text{ext}}\rangle^{S\hat{R}}$ defined by (77) is the maximally
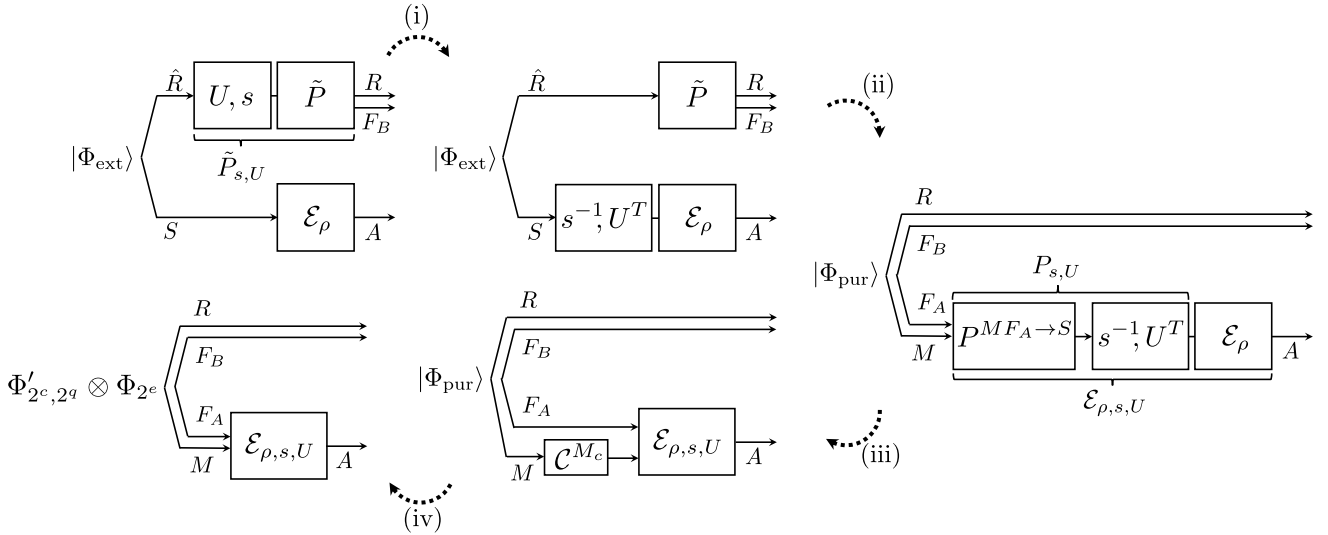
Fig. 5. Transformation of the protocol for randomized partial decoupling of the state Ψ_ρ into that for communication over the channel $\mathcal{N}$ is depicted. We first trace out $\bar{E}$ in Figure 4 and apply these transformations to obtain Figure 2. (i) is due to the fact that Φ_{ext} is the maximally entangled state between S and $\hat{R}$. (ii) is from Equality (81), and (iii) is because $\mathcal{E}_{\rho,s,U}^{MF_A \to A} = \mathcal{E}_{\rho,s,U}^{MF_A \to A} \circ \mathcal{C}^{M_c}$. Note that $|\Phi_{\text{pur}}\rangle^{\bar{M}\bar{R}} = |\Phi_{2^{c+q}}\rangle^{MR}|\Phi_{2^e}\rangle^{F_A F_B}$ as (74). (iv) follows from (76).

entangled state on $S\hat{R}$. Thus, from (91) and (81) (see Figure 5), it holds that

$$\tilde{P}_{s,U}^{\hat{R} \to RF_B}|\Phi_{\text{ext}}\rangle^{S\hat{R}}$$

$$= (G_s U)^{S^T} \otimes \tilde{P}^{\hat{R} \to RF_B}|\Phi_{\text{ext}}\rangle^{S\hat{R}} \qquad (98)$$

$$= (G_s U)^{S^T} P^{MF_A \to S}|\Phi_{\text{pur}}\rangle^{MF_A RF_B} \qquad (99)$$

$$= P_{s,U}^{MF_A \to S}|\Phi_{\text{pur}}\rangle^{MF_A RF_B}, \qquad (100)$$

where we have defined a linear isometry

$$P_{s,U}^{MF_A \to S} := U^{S^T} G_{s^{-1}}^S P^{MF_A \to S}. \qquad (101)$$

Note that $G_s^{S^T} = G_{s^{-1}}^S$. Combining (100) with (85), we obtain

$$\tilde{P}_{s,U}^{\hat{R} \to RF_B}|\Psi_\rho\rangle^{\hat{R}B\bar{E}}$$

$$= W_{\mathcal{N}}^{A \to BE} \circ V_\rho^{S \to AE_0 E_c} \circ P_{s,U}^{MF_A \to S}|\Phi_{\text{pur}}\rangle^{MF_A RF_B}. \qquad (102)$$

We now construct an encoding operation $\mathcal{E}_{\rho,s,U}$ by

$$\mathcal{E}_{\rho,s,U}^{MF_A \to A} := \mathcal{E}_\rho^{S \to A} \circ \mathcal{P}_{s,U}^{MF_A \to S} \qquad (103)$$

$$= \text{Tr}_{E_0 E_c} \circ \mathcal{V}_\rho^{S \to AE_0 E_c} \circ \mathcal{P}_{s,U}^{MF_A \to S}. \qquad (104)$$

Tracing out $\bar{E} = EE_0 E_c$ in (102) yields

$$\tilde{\mathcal{P}}_{s,U}^{\hat{R} \to RF_B}(\Psi_\rho^{\hat{R}B}) = \mathcal{N}^{A \to B} \circ \mathcal{E}_{\rho,s,U}^{MF_A \to A}(\Phi_{\text{pur}}^{MF_A RF_B}). \qquad (105)$$

Substituting this to (97), we arrive at

$$\left\| \mathcal{D}^{BF_B \to M} \circ \mathcal{N}^{A \to B} \circ \mathcal{E}_{\rho,s,U}^{MF_A \to A}(\Phi_{\text{pur}}^{MF_A RF_B}) - \Phi_{2^c,2^q}'^{MR} \right\|_1$$

$$\leq 2\sqrt{\Delta_{s,U}}. \qquad (106)$$

It remains to prove that the encoding operation defined by (103) satisfies the condition (23) (see Figure 5), i.e.,

$$\mathcal{E}_{\rho,s,U}^{MF_A \to A} = \mathcal{E}_{\rho,s,U}^{MF_A \to A} \circ \mathcal{C}^{M_c}. \qquad (107)$$

From (70), (79) and (101), we have

$$P_{s,U}^{MF_A \to S} = \sum_{j=1}^{2^c} |s^{-1}(j)\rangle^{S_c} \langle j|^{M_c} \otimes U_{s^{-1}(j)}^{S_r^T} P^{M_q F_A \to S_r}. \qquad (108)$$

Thus, it holds that

$$\mathcal{P}_{s,U}^{MF_A \to S} \circ \mathcal{C}^{M_c} = \mathcal{C}^{S_c} \circ \mathcal{P}_{s,U}^{MF_A \to S}, \qquad (109)$$

with $\mathcal{C}^{S_c}$ being the completely dephasing operation on S_c with respect to the basis $\{|j\rangle\}_{j=1}^J$. Combining this with (84), we obtain

$$\mathcal{E}_\rho^{S \to A} \circ \mathcal{P}_{s,U}^{MF_A \to S} \circ \mathcal{C}^{M_c} = \mathcal{E}_\rho^{S \to A} \circ \mathcal{P}_{s,U}^{MF_A \to S}, \qquad (110)$$

which implies (107). Substituting this to (106), and by using the relation (76), we finally arrive at

$$\left\| \mathcal{D}^{BF_B \to M} \circ \mathcal{N}^{A \to B} \circ \mathcal{E}_{\rho,s,U}^{MF_A \to A}(\Phi_{2^c,2^q}'^{MR} \otimes \Phi_{2^e}^{F_A F_B}) - \Phi_{2^c,2^q}'^{MR} \right\|_1$$

$$\leq 2\sqrt{\Delta_{s,U}}. \qquad (111)$$

The deformations of expressions that we have done in this subsection are depicted in Figure 5. The figure explains the way how the communication protocol depicted in Figure 2 is obtained from the randomized partial decoupling protocol in Figure 4.

C. Evaluation of the Errors

To evaluate the partial decoupling error $\Delta_{s,U}$ defined by (88), we apply the direct part of the randomized partial decoupling theorem (Theorem 8) under the following correspondence:

$$A_c, A_r, R_c, R_r \to \hat{R}_c, \hat{R}_r, E_c, EE_0 \qquad (112)$$

$$A, R, E, C \to \hat{R}, \bar{E}, R, F_B \qquad (113)$$

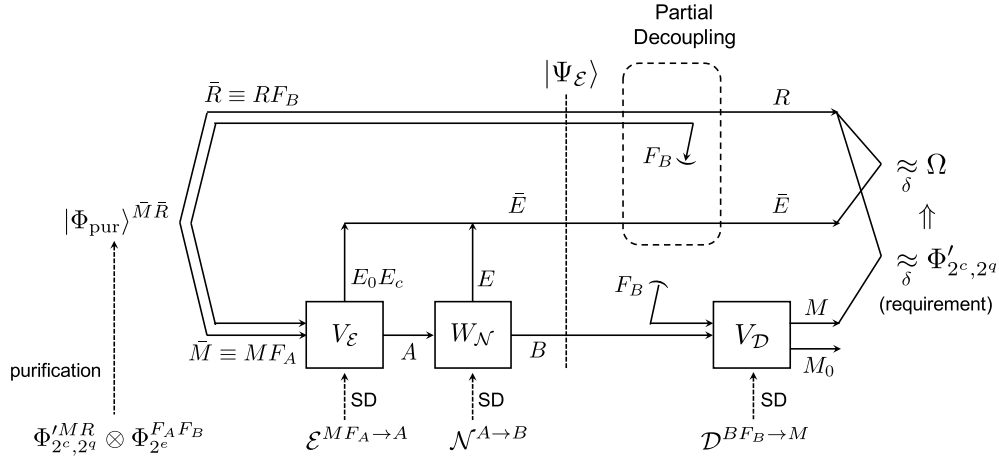$$\Psi^{AR} \to \Psi_\rho^{\hat{R}\bar{E}} \qquad (114)$$

Fig. 6. The definition of the state $\Psi_{\mathcal{E}}$ given by (137) and its transformation by partial decoupling are depicted. SD stands for the Stinespring dilation. Note that $R \equiv R_c R_q$, $M \equiv M_c M_q$ and that $|\Phi_{\mathrm{pur}}\rangle^{\bar{M}\bar{R}} = |\Phi_{2^{c+q}}\rangle^{MR}|\Phi_{2^e}\rangle^{F_A F_B}$.

$$\mathcal{T}^{A \to E} \to \mathrm{Tr}_{F_B} \circ \tilde{\mathcal{P}}^{\hat{R} \to RF_B} \tag{115}$$

$$\mu = 0. \tag{116}$$

It follows that there exist a unitary $U^{\hat{R}} = \sum_{j=1}^{d_{S_c}} |j\rangle\langle j|^{R_c} \otimes U_j^{R_q F_B}$ and a permutation s such that the partial decoupling error is bounded as

$$\Delta_{s,U} \leq \begin{cases} 2^{-\frac{1}{2}H_I} + 2^{-\frac{1}{2}H_{II}} + 4\epsilon & (d_{S_c}, d_{S_r} \geq 2) \\ 2^{-\frac{1}{2}H_I} + 4\epsilon & (q = e = 0, d_{S_c} \geq 2, d_{S_r} = 1) \\ 2^{-\frac{1}{2}H_{II}} + 4\epsilon & (c = 0, d_{S_c} = 1, d_{S_r} \geq 2). \end{cases} \tag{117}$$

Here, the exponents H_I and H_{II} are given by

$$H_I = \log(d_{S_c} - 1) + H_{\min}^\epsilon(\hat{R}|\bar{E})_{\Psi_\rho} \\ - H_{\max}(\hat{R}|F_B)_{\mathcal{C}(\tau)}, \tag{118}$$

$$H_{II} = H_{\min}^\epsilon(\hat{R}|\bar{E})_{\mathcal{C}(\Psi_\rho)} - H_{\max}(\hat{R}_r|F_B\hat{R}_c)_{\mathcal{C}(\tau)}, \tag{119}$$

and $\tau^{\hat{R}F_B}$ is the Choi-Jamiolkowski state of the complementary channel of $\mathrm{Tr}_{F_B} \circ \tilde{\mathcal{P}}^{\hat{R} \to RF_B}$. Using (75)-(78), it holds that

$$\tau^{\hat{R}F_B} := \mathrm{Tr}_{F_B} \circ \tilde{\mathcal{P}}^{\hat{R}' \to RF_B}(\Phi_{\mathrm{ext}}^{\hat{R}\hat{R}'}) \tag{120}$$

$$= \pi_{2^c}^{\hat{R}_c} \otimes \pi_{2^q}^{\hat{R}_q} \otimes \Phi_{2^e}^{\hat{F}_B F_B}. \tag{121}$$

A simple calculation yields

$$H_{\max}(\hat{R}|F_B)_{\mathcal{C}(\tau)} = c + q - e, \tag{122}$$

$$H_{\max}(\hat{R}_r|F_B\hat{R}_c)_{\mathcal{C}(\tau)} = q - e. \tag{123}$$

Using the duality of the conditional smooth entropies, we have

$$H_{\min}^\epsilon(\hat{R}|\bar{E})_{\Psi_\rho} = -H_{\max}^\epsilon(\hat{R}|B)_{\Psi_\rho}, \tag{124}$$

$$H_{\min}^\epsilon(\hat{R}|\bar{E})_{\mathcal{C}(\Psi_\rho)} = -H_{\max}^\epsilon(\hat{R}_r|BR_c)_{\Psi_\rho}. \tag{125}$$

Noting that $\hat{R}_c$ and $\hat{R}_r$ are isomorphic to S_c and S_r, respectively, it follows from the definition (87) of Ψ_ρ that

$$\Psi_\rho^{\hat{R}B} \cong \mathcal{N}^{A \to B}(\rho^{SA}), \tag{126}$$

which leads to

$$H_{\max}^\epsilon(\hat{R}|B)_{\Psi_\rho} = H_{\max}^\epsilon(S|B)_{\mathcal{N}(\rho)}, \tag{127}$$

$$H_{\max}^\epsilon(\hat{R}_r|B\hat{R}_c)_{\Psi_\rho} = H_{\max}^\epsilon(S_r|BS_c)_{\mathcal{N}(\rho)}. \tag{128}$$

Substituting all these equalities to (118) and (119), we obtain

$$H_I = -c - q + e - H_{\max}^\epsilon(S|B)_{\mathcal{N}(\rho)} + \log(d_{S_c} - 1), \tag{129}$$

$$H_{II} = -q + e - H_{\max}^\epsilon(S_r|BS_c)_{\mathcal{N}(\rho)}. \tag{130}$$

We now use the conditions (32) and (33) to have

$$H_I \geq -\log \delta_1, \quad H_{II} \geq -\log \delta_2. \tag{131}$$

Substituting these inequalities to (117), we finally arrive at

$$\Delta_{s,U} \leq \begin{cases} \sqrt{\delta_1} + \sqrt{\delta_2} + 4\epsilon & (d_{S_c}, d_{S_r} \geq 2) \\ \sqrt{\delta_1} + 4\epsilon & (q = e = 0, d_{S_c} \geq 2, d_{S_r} = 1) \\ \sqrt{\delta_2} + 4\epsilon & (c = 0, d_{S_c} = 1, d_{S_r} \geq 2) \end{cases} \tag{132}$$

and complete the proof of Theorem 3. ∎

VI. PROOF OF THE CONVERSE PART (THEOREM 4)

We prove the converse part of the capacity theorem (Theorem 4) based on the converse part of randomized partial decoupling (Theorem 9). The proof proceeds as follows: First, we construct a state $\Psi_{\mathcal{E}}$ from the source state $\Phi_{2^c,2^q}'^{MR}$, the resource state Φ_{2^e}, the channel $\mathcal{N}$ and an encoding operation $\mathcal{E}$. Second, we prove that the small error condition (22) implies that a partial-trace operation achieves partial decoupling of the state $\Psi_{\mathcal{E}}$. Applying the converse part for randomized partial decoupling, we obtain a set of inequalities represented in terms of the conditional entropies of the state $\Psi_{\mathcal{E}}$. Finally, we evaluate the entropies of the state to complete the proof of Theorem 4.

A. Application of the Converse Bound for Randomized Partial Decoupling

Suppose that a triplet (c, q, e) is achievable within the error δ for the channel $\mathcal{N}$. By definition, there exist an encoding operation $\mathcal{E}^{MF_A \to A}$ and a decoding operation $\mathcal{D}^{BF_B \to M}$ that satisfy the conditions

$$\left\| \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}(\Phi'^{MR}_{2^c,2^q} \otimes \Phi^{F_A F_B}_{2^e}) - \Phi'^{MR}_{2^c,2^q} \right\|_1 \leq \delta \quad (133)$$

and

$$\mathcal{E}^{MF_A \to A} = \mathcal{E}^{MF_A \to A} \circ \mathcal{C}^{M_c}. \quad (134)$$

Let $\mathcal{V}_{\mathcal{D}}^{BF_B \to MM_0}$ and $\mathcal{W}_{\mathcal{N}}^{A \to BE}$ be the Stinespring dilations of $\mathcal{D}$ and $\mathcal{N}$, respectively. Let E_c be a 2^c-dimensional quantum system with a fixed orthonormal basis $\{|j\rangle\}_{j=1}^{2^c}$. Due to (134), a Stinespring dilation $\mathcal{V}_{\mathcal{E}}^{MF_A \to AE_0 E_c}$ of $\mathcal{E}$ is given by

$$V_{\mathcal{E}}^{MF_A \to AE_0 E_c} = \sum_{j=1}^{2^c} |j\rangle^{E_c} \langle j|^{M_c} \otimes V_{\mathcal{E},j}^{M_q F_A \to AE_0}, \quad (135)$$

where $V_{\mathcal{E},j}$ is a linear isometry for each j. We introduce notations $\bar{R} \equiv RF_B$, $\bar{M} \equiv MF_A$, and define a "purified" source-resource state $\Phi_{\text{pur}}^{\bar{M}\bar{R}}$ (see Figure 6) by

$$|\Phi_{\text{pur}}\rangle^{\bar{M}\bar{R}} := \frac{1}{\sqrt{2^c}} \sum_{j=1}^{2^c} |j\rangle^{M_c} |j\rangle^{R_c} |\Phi_{2^q}\rangle^{M_q R_q} |\Phi_{2^e}\rangle^{F_A F_B}. \quad (136)$$

Denoting $E_0 E E_c$ by $\bar{E}$, we define a pure state $|\Psi_{\mathcal{E}}\rangle$ by

$$|\Psi_{\mathcal{E}}\rangle^{\bar{R}B\bar{E}} := W_{\mathcal{N}}^{A \to BE} \circ V_{\mathcal{E}}^{\bar{M} \to AE_0 E_c} |\Phi_{\text{pur}}\rangle^{\bar{M}\bar{R}}. \quad (137)$$

Note that $|\Psi_{\mathcal{E}}\rangle$ is classically coherent in $R_c E_c$, in the sense of (58). A purification of the state $\mathcal{D} \circ \mathcal{N} \circ \mathcal{E}(\Phi'^{MR}_{2^c,2^q} \otimes \Phi^{F_A F_B}_{2^e})$ is then given by $\mathcal{V}_{\mathcal{D}}^{BF_B \to MM_0}(\Psi_{\mathcal{E}}^{\bar{R}B\bar{E}})$, with $M_0 \bar{E}$ being a purifying system. Due to the small error condition (133) and Uhlmann's theorem (see also Lemma 17 in Appendix A), there exists a purification $|\Omega\rangle^{R\bar{E}MM_0}$ of $\Phi'^{MR}_{2^c,2^q}$ in the form of

$$|\Omega\rangle = \frac{1}{\sqrt{2^c}} \sum_{j=1}^{2^c} |j\rangle^{M_c} |j\rangle^{R_c} |j\rangle^{E_c} |\omega_j\rangle^{E_0 M_0 E} |\Phi_{2^q}\rangle^{M_q R_q}, \quad (138)$$

where $|\omega_j\rangle$ are normalized pure states, and satisfies

$$\left\| \mathcal{V}_{\mathcal{D}}^{BF_B \to MM_0}(\Psi_{\mathcal{E}}^{\bar{R}B\bar{E}}) - \Omega^{R\bar{E}MM_0} \right\|_1 \leq 2\sqrt{\delta}. \quad (139)$$

Tracing out MM_0, it follows that

$$\left\| \text{Tr}_{F_B}[\Psi_{\mathcal{E}}^{\bar{R}\bar{E}}] - \Omega^{R\bar{E}} \right\|_1 \leq 2\sqrt{\delta}. \quad (140)$$

From (138), we have

$$\Omega^{R\bar{E}} = \frac{1}{2^c} \sum_{j=1}^{2^c} |j\rangle\langle j|^{R_c} \otimes |j\rangle\langle j|^{E_c} \otimes \omega_j^{E_0 E} \otimes \pi_q^{R_q}. \quad (141)$$

Thus, the condition (140) implies that the map $\text{id}^R \otimes \text{Tr}_{F_B} : \bar{R} \to R$ achieves partial decoupling of the state $\Psi_{\mathcal{E}}^{\bar{R}\bar{E}}$ (see Figure 6).

We apply the converse bound for randomized partial decoupling (Theorem 9) under the following correspondence:

$$A_c, A_r, R_c, R_r \to R_c, R_q F_B, E_c, E_0 E \quad (142)$$

$$A, B, R, E, C \to \bar{R}, B, \bar{E}, R, F_B \quad (143)$$

$$|\Psi_{\mathcal{E}}\rangle^{ABR} \to |\Psi_{\mathcal{E}}\rangle^{\bar{R}B\bar{E}} \quad (144)$$

$$\mathcal{T}^{A \to E} \to \text{id}^R \otimes \text{Tr}_{F_B} \quad (145)$$

$$\delta \to 2\sqrt{\delta} \quad (146)$$

$$\upsilon = 0. \quad (147)$$

Noting that the complementary channel of $\text{id}^R \otimes \text{Tr}_{F_B}$ is given by $\text{Tr}_R \otimes \text{id}^{F_B}$, we obtain

$$H_{\min}^{\lambda}(\bar{R}|\bar{E})_{\Psi_{\mathcal{E}}} - H_{\min}(B\bar{E}|F_B)_{\mathcal{C}(\Psi_{\mathcal{E}})} + \log d_{R_c}$$
$$\geq \log \iota, \quad (148)$$

$$H_{\min}^{\lambda'}(\bar{R}|\bar{E})_{\mathcal{C}(\Psi_{\mathcal{E}})} - H_{\min}(BE_0 E|F_B E_c)_{\mathcal{C}(\Psi_{\mathcal{E}})}$$
$$\geq \log \iota, \quad (149)$$

where $\mathcal{C}$ is the completely dephasing operation on E_c. Substituting $\upsilon = 0$ to (68) and (69), the smoothing parameters λ and λ' are given by (38) and (39), respectively.

B. Evaluation of Entropies

By using the duality of the smooth conditional entropies [38] (see also Lemma 27 and 29 in [30]), the first terms in (148) and (149) are calculated to be

$$H_{\min}^{\lambda}(\bar{R}|\bar{E})_{\Psi_{\mathcal{E}}} = -H_{\max}^{\lambda}(\bar{R}|B)_{\Psi_{\mathcal{E}}}, \quad (150)$$

$$H_{\min}^{\lambda'}(\bar{R}|\bar{E})_{\mathcal{C}(\Psi_{\mathcal{E}})} = -H_{\max}^{\lambda'}(R_q F_B|BR_c)_{\Psi_{\mathcal{E}}}. \quad (151)$$

To calculate the second term in (148), note that (135), (136) and (137) imply

$$\mathcal{C}^{E_c}(\Psi_{\mathcal{E}}^{B\bar{E}F_B}) = \Psi_{\mathcal{E}}^{B\bar{E}F_B} \quad (152)$$

$$= \mathcal{W}_{\mathcal{N}}^{A \to BE} \circ \mathcal{V}_{\mathcal{E}}^{MF_A \to AE_0 E_c}(\Phi'^{M}_{2^c,2^q} \otimes \Phi^{F_A F_B}_{2^e}). \quad (153)$$

Thus, due to the isometric invariance of the conditional max entropy (Lemma 15 in [38]), we have

$$H_{\min}(B\bar{E}|F_B)_{\mathcal{C}(\Psi_{\mathcal{E}})}$$
$$= H_{\min}(MF_A|F_B)_{\Phi'_{2^c,2^q} \otimes \Phi_{2^e}} \quad (154)$$
$$= c + q - e. \quad (155)$$

Similarly, the second term in (149) is calculated to be

$$H_{\min}(BE_0 E|F_B E_c)_{\mathcal{C}(\Psi_{\mathcal{E}})}$$
$$= H_{\min}(M_q F_A|F_B M_c)_{\Phi'_{2^c,2^q} \otimes \Phi_{2^e}} \quad (156)$$
$$= q - e. \quad (157)$$

Substituting all these equalities into (148) and (149), we arrive at

$$-H_{\max}^{\lambda}(\bar{R}|B)_{\Psi_{\mathcal{E}}} - c - q + e + \log d_{R_c} \geq \log \iota, \quad (158)$$

$$-H_{\max}^{\lambda'}(R_q F_B|BR_c)_{\mathcal{C}(\Psi_{\mathcal{E}})} - q + e \geq \log \iota. \quad (159)$$

Finally, we calculate the reduced state of $\Psi_{\mathcal{E}}$ by using (135) and (137) to obtain

$$\Psi_{\mathcal{E}}^{RBF_B} = \mathcal{C}^{R_c}(\Psi_{\mathcal{E}}^{RBF_B}) \quad (160)$$

$$= \frac{1}{2^c} \sum_{j=1}^{2^c} |j\rangle\langle j|^{R_c} \otimes \mathcal{N}^{A \to B}(\rho_j^{R_q F_B C}), \quad (161)$$

where

$$\rho_j^{R_q F_B A} := \mathcal{E}^{M F_A \to A}(|j\rangle\langle j|^{M_c} \otimes \Phi_{2^q}^{M_q R_q} \otimes \Phi_{2^e}^{F_A F_B}). \quad (162)$$

We relabel R_c and $R_q F_B$ by S_c and S_r, respectively, in which case we have $d_S \leq 2^{c+q+e}$. Noting that $\rho_j^{R_q F_B}$ is the maximally mixed state, we complete the proof of Theorem 4. ∎

VII. APPLICATION TO A MEMORYLESS CHANNEL

In this section, we consider an asymptotic limit of infinitely many uses of a memoryless channel. We consider a scenario of transmitting classical and quantum messages with the assistance of shared entanglement. First, we review a complete characterization of the achievable rate region, which was obtained by Hsieh and Wilde [14]. Second, we present another characterization that is obtained from the direct and converse bounds for the one-shot scenario presented in Section III (Theorem 3 and 4). A complete proof of the latter will be provided in Section VIII. Despite their seeming inconsistency, the two characterizations are equivalent. A detailed proof for this equivalence will be provided in Appendix D. In Section IX, we will apply the achievable rate region obtained in this section to special cases such as the classical capacity, the quantum capacity and the entanglement assisted classical/quantum capacities. We will see that the known results for each case are recovered from our result.

The definition of a code for the channel $\mathcal{N}$ in the scenario of many uses of the channel follows from Definition 1 by the correspondence $(c, q, e) \to (nC, nQ, nE)$ and $\mathcal{N} \to \mathcal{N}^{\otimes n}$, where n is the number of the uses of the channel. The three-dimensional achievable rate region is defined as follows:

Definition 10: A rate triplet (C, Q, E) is achievable if, for any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, there exists an (nC, nQ, nE, δ) code for $(\mathcal{N}^{A \to B})^{\otimes n}$. The closure of the set of all achievable rate triplets is called the achievable rate region, and is denoted by $\mathcal{C}(\mathcal{N})$.

A. Hsieh-Wilde Theorem

Hsieh and wilde [14] analyzed the coding problem described above and obtained a complete characterization of the achievable rate region as follows:

Definition 11: Let $\mathcal{N}^{A \to B}$ be a quantum channel. Consider finite dimensional quantum systems S_c and S_r, the former of which is equipped with a fixed orthonormal basis $\{|j\rangle\}_{j=1}^{d_{S_c}}$. We denote $S_c S_r$ by S. Consider a state ρ^{SA} in the form of

$$\rho^{SA} = \sum_{j=1}^{d_{S_c}} p_j |j\rangle\langle j|^{S_c} \otimes \rho_j^{S_r A}, \quad (163)$$

where $\{p_j, \rho_j\}_{j=1}^{d_{S_c}}$ is an ensemble of quantum states on $S_r A$. Let $\Lambda(\mathcal{N}, \rho) \in \mathbb{R}^3$ be the set of all triplets (C, Q, E) that satisfy

$$C + 2Q \leq I(S : B)_{\mathcal{N}(\rho)}, \quad (164)$$
$$C + Q - E \leq H(S_c)_\rho - H(S|B)_{\mathcal{N}(\rho)}, \quad (165)$$
$$Q - E \leq -H(S_r|BS_c)_{\mathcal{N}(\rho)}, \quad (166)$$

$$C, Q, E \geq 0, \quad (167)$$

and define

$$\Lambda_p(\mathcal{N}) := \bigcup_{S_c, S_r, \rho} \Lambda(\mathcal{N}, \rho). \quad (168)$$

Here, the union is taken over all S_c, S_r and ρ that is decomposed as (163), for which we assume that ρ_j is a pure state on $S_r A$ for each j. The set $\Lambda_p(\mathcal{N}^{\otimes n})$ is defined for any $n \in \mathbb{N}$ along the same line. The regularized version of Λ_p is defined by

$$\Lambda_p^\infty(\mathcal{N}) := \bigcup_{n=1}^\infty \frac{1}{n} \Lambda_p(\mathcal{N}^{\otimes n}). \quad (169)$$

Theorem 12: (Theorem 1 in [14]) For any quantum channel $\mathcal{N}^{A \to B}$, it holds that $\mathcal{C}(\mathcal{N}) = \overline{\Lambda_p^\infty(\mathcal{N})}$. Here, the overline represents the closure of the set.

It was proved in [14] that, in the special cases where one or two of C, Q or E vanish, the above theorem recovers the known results cases such as the Holevo-Schumache-Westmoreland theorem for the classical capacity [3], [4], the Lloyd-Shor-Devetak theorem for the quantum capacity [5], [6], [7], the entanglement-assisted classical [8], [9], [10] and quantum [11], [12] capacity theorems. Note that, due to the chain rule of the quantum entropies, the R.H.S. of (172) is equal to $I(S_c : B)_{\mathcal{N}(\rho)} - H(S_r|BS_c)_{\mathcal{N}(\rho)}$.

B. Alternative Expression

We present an alternative expression for the achievable rate region that is obtained from the one-shot result presented in Section III.

Definition 13: Let $\mathcal{N}^{A \to B}$ be a quantum channel. Consider finite dimensional quantum systems S_c and S_r. Let S be $S_c S_r$, and $\{|j\rangle\}_{j=1}^{d_{S_c}}$ be a fixed orthonormal basis of S_c. Consider a state ρ^{SA} in the form of

$$\rho^{SA} = \sum_{j=1}^{d_{S_c}} p_j |j\rangle\langle j|^{S_c} \otimes \rho_j^{S_r A}, \quad (170)$$

where $\{p_j, \rho_j\}_{j=1}^{d_{S_c}}$ is an ensemble of quantum states on $S_r A$. Let $\Theta(\mathcal{N}, \rho) \subset \mathbb{R}^3$ be the set of all triplets (C, Q, E) that satisfy

$$Q + E \leq H(S_r|S_c)_\rho, \quad (171)$$
$$C + Q - E \leq H(S_c)_\rho - H(S|B)_{\mathcal{N}(\rho)}, \quad (172)$$
$$Q - E \leq -H(S_r|BS_c)_{\mathcal{N}(\rho)}, \quad (173)$$
$$C, Q, E \geq 0, \quad (174)$$

and define

$$\Theta_\pi(\mathcal{N}) := \bigcup_{S_c, S_r, \rho} \Theta(\mathcal{N}, \rho). \quad (175)$$

Here, the union is taken over all finite dimensional systems S_c, S_r and all states ρ that is decomposed as (170), for which we assume that ρ^S is the full-rank maximally mixed state on

S. The set $\Theta_\pi(\mathcal{N}^{\otimes n})$ is defined for any $n \in \mathbb{N}$ along the same line. The regularized version of Θ_π is defined by

$$\Theta_\pi^\infty(\mathcal{N}) := \bigcup_{n=1}^\infty \frac{1}{n} \Theta_\pi(\mathcal{N}^{\otimes n}). \tag{176}$$

Theorem 14: For any quantum channel $\mathcal{N}^{A \to B}$, it holds that $\mathcal{C}(\mathcal{N}) = \overline{\Theta_\pi^\infty(\mathcal{N})}$. Here, the overline represents the closure of the set.

A proof of Theorem 14 will be provided in Section VIII.

In Section IX, we will apply Theorem 14 to the special cases where one or two of C, Q or E vanish. We prove that the known results for those special cases such as the Holevo-Schumache-Westmoreland theorem for the classical capacity [3], [4], the Lloyd-Shor-Devetak theorem for the quantum capacity [5], [6], [7] and the entanglement-assisted capacity theorems [8], [9], [10], [11], [12] are recovered from Theorem 14, in the same way as those are obtained from Theorem 12.

C. Equivalence of the Two Expressions

The region defined by Definition 13 may look different from the one defined by Definition 11. But the two regions are actually the same, i.e., it holds that

$$\overline{\Theta_\pi^\infty(\mathcal{N})} = \overline{\Lambda_p^\infty(\mathcal{N})}. \tag{177}$$

The point is that we take the union over S_c, S_r and ρ both in (175) and in (168). Taking the union eliminate the seeming mismatch between the conditions (164) and (171).

To be more precise, we introduce two alternative expressions for the achievable rate region. We define

$$\Theta(\mathcal{N}) := \bigcup_{S_c, S_r, \rho} \Theta(\mathcal{N}, \rho), \tag{178}$$

where $\Theta(\mathcal{N}, \rho) \in \mathbb{R}^3$ is given by Definition 13. Here, the union is taken over all S and ρ that is in the form of (170). We do not require the condition that ρ^S is the full-rank maximally mixed state on S. We also define

$$\Lambda(\mathcal{N}) := \bigcup_{S_c, S_r, \rho} \Lambda(\mathcal{N}, \rho), \tag{179}$$

where $\Lambda(\mathcal{N}, \rho)$ is defined in Definition 11. The union is taken over all S and ρ that is decomposed as (163), but we do not require that $\{\rho_j^{S_r A}\}_j$ are pure states. Note that the only difference between $\Theta(\mathcal{N}, \rho)$ and $\Lambda(\mathcal{N}, \rho)$ is in one of the inequality conditions for (C, Q, E). That is, we require $Q + E \leq H(S_r|S_c)_\rho$ for $\Theta(\mathcal{N}, \rho)$ and $C + 2Q \leq I(S : B)_{\mathcal{N}(\rho)}$ for $\Lambda(\mathcal{N}, \rho)$ (see Inequalities (171) and (164)). The two sets are regularized into

$$\Theta^\infty(\mathcal{N}) := \bigcup_{n=1}^\infty \frac{1}{n} \Theta(\mathcal{N}^{\otimes n}), \tag{180}$$

$$\Lambda^\infty(\mathcal{N}) := \bigcup_{n=1}^\infty \frac{1}{n} \Lambda(\mathcal{N}^{\otimes n}). \tag{181}$$

In appenxid Appendix D, we prove the following proposition that implies (177):

Proposition 15: For any quantum channel $\mathcal{N}$, it holds that

$$\overline{\Theta_\pi^\infty(\mathcal{N})} = \overline{\Theta^\infty(\mathcal{N})} = \overline{\Lambda^\infty(\mathcal{N})} = \overline{\Lambda_p^\infty(\mathcal{N})}. \tag{182}$$

The first equality in (182) implies that the condition that the reduced state of the state ρ on system $S_c S_r$ is the completely mixed state does not impose any restriction on the achievable rate region. I.e., the rate region does not change when we drop this condition. In Section IX, we will invoke this fact to show that the known capacity theorems in special cases are obtained from Theorem 14.

D. Remark On the Non-Optimality of Time-Sharing Strategies

Ref. [13] addressed the task of simultaneously transmitting classical and quantum messages through a noisy quantum channel without the assistance of shared entanglement, i.e. the case where $E = 0$ in Definition 10. A strategy for this task is to use $n\lambda$ copies of the channel to transmit classical messages and the remaining $n(1 - \lambda)$ copies for quantum messages ($0 \leq \lambda \leq 1$), which is often referred to as the time sharing strategy. It achieves the rate pair

$$(C, Q) = (\lambda C(\mathcal{N}), (1 - \lambda) Q(\mathcal{N})), \tag{183}$$

with $C(\mathcal{N})$ and $Q(\mathcal{N})$ being the classical and quantum capacities of the channel. Ref. [13] proved that the time sharing strategy is not optimal in general: For a certain channel $\mathcal{N}$, there exists an achievable rate pair (C, Q) that cannot be represented as (183). In contrast, the time sharing strategy is optimal when the entanglement resource is freely available. This is because $t > 0$ qubits of quantum communication is converted to $2t$ bits of classical communication and vice versa, by means of superdense coding and quantum teleportation.

In the case where a *limited* amount of shared entanglement is available, it is not clear whether or not the time sharing strategy is optimal. This problem may be of independent interest because it would be closely related to the non-additivity of the classical capacity of a quantum channel under a limited amount of entanglement assistance [43]. We leave this problem as an open question.

VIII. PROOF OF THEOREM 14

In this section, we provide a proof for Theorem 14.

A. Proof of the Direct Part

We prove the direct part of Theorem 14, i.e.

$$\mathcal{C}(\mathcal{N}) \supseteq \overline{\Theta_\pi^\infty(\mathcal{N})}. \tag{184}$$

Since $\mathcal{C}(\mathcal{N})$ is a closed set, it suffices to prove that, for any n, a rate triplet (C, Q, E) is achievable if (C, Q, E) is an inner point of $\frac{1}{n} \Theta_\pi(\mathcal{N}^{\otimes n})$. We only consider the case where $n = 1$. It is straightforward to generalize the proof for $n \geq 2$. Fix an arbitrary state ρ that satisfies the condition (170), in addition to the condition that ρ^S is the full-rank maximally mixed state on S. Fix an arbitrary triplet (C, Q, E) that is an inner point of $\Theta(\mathcal{N}, \rho)$. Then, there exists $\nu > 0$ such that

$$Q + E \leq H(S_r|S_c)_\rho, \tag{185}$$

$$C + Q - E \leq H(S_c)_\rho - H(S|B)_{\mathcal{N}(\rho)} - 2\nu, \quad (186)$$
$$Q - E \leq -H(S_r|BS_c)_{\mathcal{N}(\rho)} - 2\nu. \quad (187)$$

Fix arbitrary $\epsilon, \delta > 0$ and choose sufficiently large m. Due to the fully-quantum asymptotic equipartition property ([44]: see also Lemma 22 in Appendix B), it holds that

$$mH(S|B)_{\mathcal{N}(\rho)} \geq H_{\max}^\epsilon(S^m|B^m)_{\mathcal{N}(\rho)^{\otimes m}} - m\nu, \quad (188)$$
$$mH(S_r|BS_c)_{\mathcal{N}(\rho)} \geq H_{\max}^\epsilon(S_r^m|B^m S_c^m)_{\mathcal{N}(\rho)^{\otimes m}} - m\nu. \quad (189)$$

Combining this with (185)-(187), and noting that $H(S_r|S_c)_\rho \leq \log \dim S_r$ and $H(S_c)_\rho = \log d_{S_c}$, we obtain

$$m(Q + E) \leq m \log \dim S_r, \quad (190)$$
$$m(C + Q - E) \leq m \log d_{S_c} - H_{\max}^\epsilon(S^m|B^m)_{\mathcal{N}(\rho)^{\otimes m}} - m\nu, \quad (191)$$
$$m(Q - E) \leq -H_{\max}^\epsilon(S_r^m|B^m S_c^m)_{\mathcal{N}(\rho)^{\otimes m}} - m\nu. \quad (192)$$

We choose m sufficiently large so that we have $-m\nu \leq \log \delta$. Denoting mC, mQ, mE by c, q, e, respectively, it follows that

$$q + e \leq \log \dim S_r^m, \quad (193)$$
$$c + q - e \leq -H_{\max}^\epsilon(S^m|B^m)_{\mathcal{N}^{\otimes m}(\rho^{\otimes m})} + \log d_{S_c}^m + \log \delta, \quad (194)$$
$$q - e \leq -H_{\max}^\epsilon(S_r^m|B^m S_c^m)_{\mathcal{N}^{\otimes m}(\rho^{\otimes m})} + \log \delta. \quad (195)$$

We separately consider the cases $d_{S_c}^m \geq \max\{2^c, 2\}$ and $d_{S_c}^m < \max\{2^c, 2\}$. For the former case, we simply proceed with (194) to obtain

$$c + q - e \leq -H_{\max}^\epsilon(S^m|B^m)_{\mathcal{N}(\rho)^{\otimes m}} + \log(d_{S_c}^m - 1) + \log 2\delta, \quad (196)$$

where we have used $d_{S_c}^m/(d_{S_c}^m - 1) \leq 2$. Combining this with (193) and (195), it follows from Theorem 3 that there exists a (c, q, e, δ') code for the channel $\mathcal{N}^{\otimes m}$, where $\delta' := 2\sqrt{\sqrt{2\delta} + \sqrt{\delta} + 4\epsilon}$. For the latter case, we introduce a system S_c' such that $d_{S_c}^m d_{S_c'} \geq \max\{2^c, 2\}$. Denoting $S_c' S_c^m$ by $\check{S}_c$ and $\hat{S}_c S_r^m = S_c' S^m$ by $\hat{S}$, we define the state

$$\hat{\rho}_m^{\hat{S}A^m} := \frac{1}{d_{S_c'}} \sum_{j'=1}^{d_{S_c'}} |j'\rangle\langle j'|^{S_c'} \otimes \rho^{SA}. \quad (197)$$

Due to the property of the smooth max entropy for product states (see Lemma 24 in Appendix B), we have

$$H_{\max}^\epsilon(S^m|B^m)_{\mathcal{N}^{\otimes m}(\rho^{\otimes m})}$$
$$\geq H_{\max}^\epsilon(\hat{S}|B^m)_{\mathcal{N}^{\otimes m}(\hat{\rho}_m)} - \log d_{\hat{S}_c}, \quad (198)$$
$$H_{\max}^\epsilon(S_r^m|B^m S_c^m)_{\mathcal{N}^{\otimes m}(\rho^{\otimes m})}$$
$$= H_{\max}^\epsilon(S_r^m|B^m \hat{S}_c)_{\mathcal{N}^{\otimes m}(\hat{\rho}_m)}. \quad (199)$$

Thus, Inequalities (194) and (195) yield

$$c + q - e \leq -H_{\max}^\epsilon(\hat{S}|B^m)_{\mathcal{N}^{\otimes m}(\hat{\rho}_m)} + \log(d_{\hat{S}_c} - 1) + \log 2\delta, \quad (200)$$
$$q - e \leq -H_{\max}^\epsilon(S_r^m|B^m \hat{S}_c)_{\mathcal{N}^{\otimes m}(\hat{\rho}_m)} + \log \delta, \quad (201)$$

where we have used $d_{\hat{S}_c}/(d_{\hat{S}_c} - 1) \leq 2$ in the first line. These two inequalities together with (193) imply that there exists a

(c, q, e, δ') code for the channel $\mathcal{N}^{\otimes m}$. Since ϵ and δ can be arbitrarily small in both cases, we complete the proof of the direct part (184). ∎

B. Proof of the Converse Part

The converse part of Theorem 14 is given by

$$\mathcal{C}(\mathcal{N}) \subseteq \overline{\Theta_\pi^\infty(\mathcal{N})}, \quad (202)$$

and is proved as follows. Suppose that a rate triplet (C, Q, E) is an inner point of $\mathcal{C}(\mathcal{N})$. By definition, for any $\delta > 0$ and sufficiently large n, there exist a (nC, nQ, nE, δ) code for the channel $\mathcal{N}^{\otimes n}$. Due to Theorem 4, there exists a quantum system $S \equiv S_c S_r$ and a state in the form of

$$\rho_n^{SA^n} = \frac{1}{d_{S_c}} \sum_{j=1}^{d_{S_c}} |j\rangle\langle j|^{S_c} \otimes \rho_j^{S_r A^n}, \quad (203)$$

satisfying $d_S \leq 2^{n(C+Q+E)}$ and $\rho^S = \pi^S$, such that for any $\iota \in (0, 1]$, it holds that

$$n(Q + E) \leq \log \dim S_r, \quad (204)$$
$$n(C + Q - E) \leq \log d_{S_c} - H_{\max}^\lambda(S|B^n) - \log \iota, \quad (205)$$
$$n(Q - E) \leq -H_{\max}^{\lambda'}(S_r|B^n S_c) - \log \iota. \quad (206)$$

The entropies are for the state $\mathcal{N}^{\otimes n}(\rho_n)$, and the smoothing parameters λ and λ' are given by (38) and (39). Note that we have $\rho^S = \mathcal{N}(\rho)^S$ since $\mathcal{N}$ acts only on A. From the condition $\rho^S = \pi^S$, we have $\log \dim S_r = H(S_r|S_c)$. Using the relation between the smooth max entropy and the von Neumann entropy (Lemma 26 in Appendix B), we also have

$$H_{\max}^\lambda(S|B^n) \geq H(S|B^n) - \eta(\lambda) \log d_S, \quad (207)$$
$$H_{\max}^{\lambda'}(S_r|B^n S_c) \geq H(S_r|B^n S_c) - \eta(\lambda') \log d_S, \quad (208)$$

where η is a function that satisfies $\lim_{x \to 0} \eta(x) = 0$. Combining these inequalities, we arrive at

$$Q + E \leq \frac{1}{n} H(S_r|S_c),$$
$$C + Q - E \leq \frac{1}{n}(\log d_{S_c} - H(S|B^n) + \eta(\lambda) \log d_S - \log \iota),$$
$$Q - E \leq \frac{1}{n}(-H(S_r|B^n S_c) + \eta(\lambda') \log d_S - \log \iota).$$

By taking the limit of $n \to \infty$ and $\iota, \delta \to 0$, we obtain $(C, Q, E) \in \Theta_\pi^\infty(\mathcal{N})$ and complete the proof. ∎

IX. COMPARISON TO PREVIOUS RESULTS IN SPECIAL CASES

In this section, we apply the results presented in Section III and Section VII to the special cases where one or two of C, Q and E is/are equal to zero and compare the results with those in the previous literature. That is, we investigate the classical capacity, the quantum capacity, the entanglement-assisted classical capacity, the entanglement-assisted quantum capacity and the simultaneous capacity for classical and quantum information without the entanglement assistance, and discuss the relation with the existing results. We consider a quantum channel $\mathcal{N}$ with the input system A and the output system B.

For the one-shot scenario, we will invoke Theorem 3, Theorem 4, Corollary 6 and Corollary 7. The smoothing parameters $\epsilon(\delta, \delta')$, $\lambda(\delta, \iota)$ and $\lambda'(\delta, \iota)$ are set to be $\epsilon(\delta, \delta') := \delta^2/16 - \sqrt{\delta'}/4$, (38) and (39), respectively. When we write $\widehat{\sup}$, the supremum is taken over all finite dimensional quantum systems $S \equiv S_c S_r$ and all classical-quantum states $\rho^{S_c S_r A}$ between S_c and S_r such that $\rho^{S_c S_r}$ is the full-rank maximally mixed state. For each of the special cases, we compare our results with several results in the previous literature. We, however, do not provide quantitative comparisons because all the results, including ours, involve complex optimizations, which makes quantitative analyses intractable. All the one-shot results presented or reviewed in this section shall coincide with the known coding theorems in the asymptotic limit of infinitely many uses of a memoryless channel.

For the asymptotic scenario, we invoke Theorem 14 and prove that the known capacity theorems proved in the previous literature are recovered. As mentioned in the discussion below Proposition 15, and as proved in Appendix D-A in detail, the restriction in Definition 13 that the reduced state of ρ on S is the completely mixed state does not impose any restriction on the achievable rate region. Therefore, in the following subsections, we assume that this restriction is removed. It is convenient to note that the R.H.S.s of (171)-(173) are evaluated as

$$H(S_r|S_c)_\rho \geq \max\{0, -H(S_r|BS_c)_\rho\}, \quad (209)$$
$$H(S_c)_\rho - H(S|B)_{\mathcal{N}(\rho)} \geq -H(S_r|BS_c)_{\mathcal{N}(\rho)}. \quad (210)$$

A. Classical Capacity Without Entanglement Assistance

The δ-classical capacity $c(\mathcal{N}, \delta)$ of a channel $\mathcal{N}$ is defined as the supremum of c such that the triplet $(c, q = 0, e = 0)$ is achievable within the error δ for the channel $\mathcal{N}$. Lower and upper bounds on the δ-classical capacity are obtained from Theorem 3 and Theorem 4 by letting $q = e = 0$. We have

$$\widehat{\sup}_{S_c, \rho} \sup_{\delta' \in (0, \delta^4/16]} \left[\log d_{S_c} - H_{\max}^{\epsilon(\delta, \delta')}(S_c|B)_{\mathcal{N}(\rho)} + \log 2\delta' \right]$$
$$\leq c(\mathcal{N}, \delta) \leq$$
$$\widehat{\sup}_{S_c, S_r, \rho} \inf_{\iota \in (0, 1]} \left[\log d_{S_c} - H_{\max}^{\lambda(\delta, \iota)}(S_c S_r|B)_{\mathcal{N}(\rho)} - \log \iota \right], \quad (211)$$

where the map $\mathcal{C}$ is the completely dephasing operation on S with respect to a fixed orthonormal basis.

Dupuis et al. [21] addressed the one-shot classical capacity of a channel $\mathcal{N}'^{A \to B}$ with classical input and quantum output (the classical-quantum channel), namely a channel that satisfies $\mathcal{N}'^{A \to B} = \mathcal{N}'^{A \to B} \circ \mathcal{C}^A$ with $\mathcal{C}$ being the completely dephasing map on A with respect to a fixed orthonormal basis. Based on the 'dequantizing theorem' (Theorem 3.1 in [21]), they obtained a lower bound that holds for any $0 < \delta < 1$:

$$c(\mathcal{N}', \delta) \geq \sup_{\varepsilon \in [0, \delta^2/8)} \left[\log (d_A - 1) - H_{\max}^\varepsilon(A|B)_\omega \right.$$
$$\left. + 2 \log (\delta^2 - 8\varepsilon) - 1 \right], \quad (212)$$

where ω is the Choi-Jamiołkowski state of $\mathcal{N}'$. Renes et al. [16] addressed the same task and obtained both lower and upper bounds (see Theorem 1 in [16]). The channel is modeled

by an input alphabet $\mathcal{X}$ and the set of output quantum states $\{\theta_x\}_{x \in \mathcal{X}}$. They obtained lower and upper bounds that yield

$$\max_{P_X} \left[H_{\min}^{\delta/8}(X)_\vartheta - H_{\max}^{\delta/8}(X|B)_\vartheta + 4 \log \delta - 16 \right]$$
$$\leq c(\mathcal{N}, \delta) \leq \max_{P_X} \left[H_{\min}(X)_\vartheta - H_{\max}^{\sqrt{2\delta}}(X|B)_\vartheta \right] \quad (213)$$

for any $\delta > 0$, where the maximization is taken over all probability distribution P_X on alphabet $\mathcal{X}$ and the entropies are for the state $\vartheta = \sum_{x \in \mathcal{X}} p_x |x\rangle\langle x|^X \otimes \theta_x^B$. From both results, by taking supremum over all sets of output states of the form $\{\mathcal{N}(\rho_x)\}_{x \in \mathcal{X}}$, we obtain one-shot lower bounds that are similar to (212). The upper bounds in (211) and (213) are different in that (211) includes system S_r. We shall see below that this seeming mismatch vanishes in the asymptotic scenario.

Wang et al. [17] addressed the same task in terms of the hypothesis testing approach and obtained lower and upper bounds that yield

$$\sup_{P_X} D_H^{\delta/2}(\vartheta^{XB} \| \vartheta^X \otimes \vartheta^B) + \log \epsilon - 4$$
$$\leq c(\mathcal{N}, \delta) \leq \sup_{P_X} D_H^\delta(\vartheta^{XB} \| \vartheta^X \otimes \vartheta^B). \quad (214)$$

Here, D_H is the hypothesis testing relative entropy and ϑ is defined in the same way as above. The hypothesis testing approach to this problem was further improved by Datta et al. [23] and Matthews et al. [24]. In particular, the result in Ref. [23] for the one-shot scenario recovers the strong converse rate [45], [46] when applied to the asymptotic scenario.

In the asymptotic limit, we obtain from (172) and (176) that

$$C(\mathcal{N}) \leq \lim_{n \to \infty} \frac{1}{n} \sup_{S_c, S_r, \rho} [H(S_c)_\rho - H(S|B^n)_{\mathcal{N}^{\otimes n}(\rho)}] \quad (215)$$
$$= \lim_{n \to \infty} \frac{1}{n} \sup_{S_c, S_r, \rho} [I(S_c : B^n)_{\mathcal{N}^{\otimes n}(\rho)} - H(S_r|B^n S_c)_{\mathcal{N}^{\otimes n}(\rho)}],$$

where the supremum is taken over all finite-dimensional quantum systems S_c, S_r and all classical-quantum states ρ between S_c and $S_r A$. Due to the argument in [1] (see the argument after Theorem 24.2.2 therein), we may, without loss of generality, assume that S_r is a trivial (one-dimensional) system. Hence, if C is below the R.H.S. of (215), both the conditions (171) and (173) are satisfied as well. Therefore the inequality (215) is achievable and we recover the Holevo-Schumacher-Westmoreland theorem [3], [4], which yields

$$C(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} \sup_{S_c, \rho} [I(S_c : B^n)_{\mathcal{N}^{\otimes n}(\rho)}]. \quad (216)$$

B. Quantum Capacity Without Entanglement Assistance

The δ-quantum capacity $q(\mathcal{N}, \delta)$ of a channel $\mathcal{N}$ is defined as the supremum of c such that the triplet $(c = 0, q, e = 0)$ is achievable within the error δ for the channel $\mathcal{N}$. Lower and upper bounds on the δ-quantum capacity are obtained from Theorem 3 and Theorem 4 by letting $c = e = 0$, which yields

$$\widehat{\sup}_{S_r, \rho} \sup_{\delta' \in (0, \delta^4/16]} \left[-H_{\max}^{\epsilon(\delta, \delta')}(S_r|B)_{\mathcal{N}(\rho)} + \log \delta' \right]$$
$$\leq q(\mathcal{N}, \delta) \leq$$

$$\widehat{\sup_{S_r,\rho}} \inf_{\iota\in(0,1]} \left[-H_{\max}^{\lambda'(\delta,\iota)}(S_r|B)_{\mathcal{N}(\rho)} - \log\iota\right]. \quad (217)$$

Datta et al. [22] and Buscemi et al. [18] addressed this task based on the decoupling approach. Ref. [22] defines the one-shot δ-error quantum capacity $q_\delta^{(1)}(\mathcal{N})$ in terms of the entanglement transmission fidelity (see Definition 11 therein) and obtained the following lower and upper bounds that holds for any $0 < \delta \le 1$:

$$\max_{\mathcal{M}\subseteq\mathcal{H}^A} \left[-H_{\max}^\varepsilon(S_r|B)_{\mathcal{N}(\Phi_{\mathcal{M}})}\right] + 2\log\left(\varepsilon + \sqrt{4\sqrt\varepsilon - 4\varepsilon}\right)$$
$$\le q_\delta^{(1)}(\mathcal{N}) \le \max_{\mathcal{M}\subseteq\mathcal{H}^A}\left[-H_{\max}^\delta(S_r|B)_{\mathcal{N}(\Phi_{\mathcal{M}})}\right]. \quad (218)$$

Here, the maximization is taken over all subspaces $\mathcal{M}\subseteq\mathcal{H}^A$, $|\Phi_{\mathcal{M}}\rangle \in \mathcal{M}\otimes\mathcal{H}^{S_r}$ is the maximally entangled state with Schmidt rank $\dim\mathcal{M}$, and $\varepsilon > 0$ is chosen so that $\delta = 2(5(\varepsilon + \sqrt{4\sqrt\varepsilon - 4\varepsilon}))^{1/2} + 2\sqrt\varepsilon$. A comparison of this result with the one in [18] was discussed in Section 3.3 of [22]. Besides the values of the smoothing parameters, the difference between (217) and (218) is in whether the supremum over the states is limited to the maximally entangled state or not. Tomamichel et al. [26] addressed the same task from the hypothesis-testing approach.

In the asymptotic limit, we have

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \sup_{S_c,S_r,\rho} [-H(S_r|B^n S_c)_{\mathcal{N}^{\otimes n}(\rho)}], \quad (219)$$

where the supremum is taken over all finite-dimensional quantum systems S_c, S_r and a classical-quantum state ρ between S_c and $S_r A$. Since the von Neumann entropy conditioned by a classical system is equal to the entropy averaged over the values of the classical system, without loss of generality we can assume that S_c is a trivial (one-dimensional) system. Thus, we recover the Lloyd-Shor-Devetak theorem [5], [6], [7], which yields

$$Q(\mathcal{N}) = \lim_{n\to\infty}\frac{1}{n}\sup_{S_r,\rho}[-H(S_r|B^n)_{\mathcal{N}^{\otimes n}(\rho)}]. \quad (220)$$

One can also obtain (220) directly from (217).

C. Classical Capacity With Free Entanglement Assistance

The entanglement-assisted δ-classical capacity $c_e(\mathcal{N},\delta)$ is defined as the supremum of c such that there exists $e > 0$ and the triplet $(c, q = 0, e)$ is achievable within the error δ for the channel $\mathcal{N}$. Lower and upper bounds on the entanglement-assisted δ-classical capacity is obtained from Corollary 6 and Corollary 7 by letting $q = 0$, which yields

$$\widehat{\sup_{S,\rho}} \sup_{\delta'\in(0,\delta^4/16]}\left[\log d_S - H_{\max}^{\epsilon(\delta,\delta')}(S|B)_{\mathcal{N}(\rho)} + \log\delta'\right]$$
$$\le c_e(\mathcal{N},\delta) \le$$
$$\widehat{\sup_{S,\rho}} \inf_{\iota\in(0,1]}\left[\log d_S - H_{\max}^{\lambda'(\delta,\iota)}(S|B)_{\mathcal{N}(\rho)} - \log\iota\right]. \quad (221)$$

Datta et al. [19] investigated this task and obtained lower and upper bounds based on the decoupling approach. Their proof is based on the fact that the classical capacity is twice as large as the quantum capacity if entanglement resource

is freely available. They defined the one-shot entanglement-assisted classical capacity $c_{\text{ea},\epsilon}^{(1)}(\mathcal{N})$ in terms of the worst-case error probability (see Definition 13 in [19]) and obtained lower and upper bounds thereof. They proved that for any $0 < \varepsilon < 1$ and ε'' such that $\varepsilon = 2\sqrt[4]{2\sqrt{27\varepsilon''} + 27\varepsilon''}$, it holds that

$$\max_{\varrho\in\mathcal{S}(\mathcal{H}^A)}\left[H_{\min}^{\varepsilon''}(S)_{\mathcal{N}(\phi)} - H_{\max}^{\varepsilon''}(S|B)_{\mathcal{N}(\phi)}\right] + 4\log\varepsilon'' - 2$$
$$\le c_{\text{ea},\epsilon}^{(1)}(\mathcal{N}) \le$$
$$\max_{\varrho\in\mathcal{S}(\mathcal{H}^A)}\left[H_{\min}^{4\varepsilon}(S)_{\mathcal{N}(\phi)} - H_{\max}^{8\varepsilon+4\sqrt{2\sqrt\varepsilon}}(S|B)_{\mathcal{N}(\phi)}\right]$$
$$+ \log\frac{1}{2\sqrt2\varepsilon}, \quad (222)$$

where ϕ^{SA} is a purification of $\varrho \in \mathcal{S}(\mathcal{H}^A)$.

Datta et al. also addressed this task from the hypothesis testing approach and obtained a lower bound in terms of the average error probability [25]. Namely, they proved that for any $0 < \delta < 1$ and $0 < \epsilon < \delta/2$, it holds that

$$c_e(\mathcal{N},\delta) \ge D_H^{\delta-2\epsilon}(\mathcal{N}^{A'\to B}(\vartheta^{AA'})\|\mathcal{N}^{A'\to B}(\kappa^{AA'}))$$
$$- \log\frac{1-\delta}{\epsilon^2}, \quad (223)$$

where $\vartheta^{AA'}$ is an arbitrary pure state that can be represented as a superposition of the maximally entangled state in orthogonal subspaces and κ is the probabilistic mixture of the maximally mixed state in those subspaces. Matthews et al. [24] obtained a converse bound from the hypothesis testing approach, which yields

$$c_e(\mathcal{N},\delta) \le \max_{\rho\in\mathcal{S}(\mathcal{H}^{A'})}\min_{\sigma\in\mathcal{S}(\mathcal{H}^B)} D_H^\delta(\mathcal{N}^{A'\to B}(\phi^{AA'})\|\rho^A\otimes\sigma^B), \quad (224)$$

where ϕ is a purification of ρ. These two bounds are more general than ours, in that they do not assume that the resource state is the maximally entangled state. The hypothesis testing approach to the entanglement-assisted quantum capacity has further been developed based on the convex splitting and position based coding [27], [28], [29].

In the asymptotic limit, the conditions (171)-(174) are equivalent to

$$\begin{cases} C \le H(S_c)_\rho - H(S|B)_{\mathcal{N}(\rho)} + E, \\ H(S_r|BS_c)_{\mathcal{N}(\rho)}^+ \le E \le H(S_r|S_c)_\rho, \end{cases} \quad (225)$$

where

$$H(S_r|BS_c)_{\mathcal{N}(\rho)}^+ := \max\{H(S_r|BS_c)_{\mathcal{N}(\rho)}, 0\}. \quad (226)$$

The two-dimensional achievable rate region in this scenario was originally proved by Shor [10], which is represented by the condition

$$\begin{cases} C \le I(S_cS_r : B)_{\mathcal{N}(\varrho)}, \\ E \ge H(S_r|S_c)_\varrho. \end{cases} \quad (227)$$

The regions (225) and (227) coincide with each other by taking the union over all classical-quantum states ρ and ϱ' between

S_c and $S_r A$. To show this, observe that the nontrivial vertices of the region (225) are given by

$$(C_1, E_1) = (I(S_c S_r : B)_{\mathcal{N}(\rho)}, H(S_r|S_c)_\rho), \qquad (228)$$

$$(C_2, E_2) = (I(S_c : B)_{\mathcal{N}(\rho)}, H(S_r|BS_c)^+_{\mathcal{N}(\rho)}). \qquad (229)$$

In both cases, there exists a state ϱ such that $C_i \leq I(S_c S_r : B)_{\mathcal{N}(\varrho)}$ and $E \geq H(S_r|S_c)_\varrho$. For (C_2, E_2), we may choose $\varrho = |0\rangle\langle 0|^{S_r} \otimes \rho^{S_c A}$. Conversely, let (C^*, E^*) be the nontrivial vertex of the region defined by (227), i.e., let C^* and E^* be equal to the R.H.S.s of (227). It is straightforward to verify that (C^*, E^*) satisfies the condition (225). Thus, by taking the union over all ρ in (225) and by regularizing it, we obtain the two-dimensional achievable rate region proved by Shor [10].

As proved in [10], the single-letter formula for the entanglement-assisted classical capacity by Bennet et al. [8], [9] is obtained from (227). An alternative proof is obtained from Theorem 14 as follows. One can see if a pair (C, E) satisfies the conditions (171)-(173) (which is equivalent to (225) when $Q = 0$), it holds that

$$C \leq I(S_r : B|S_c)_{\mathcal{N}(\rho)}. \qquad (230)$$

Conversely, if the above inequality holds, we have

$$C - H(S_c)_\rho + H(S|B)_{\mathcal{N}(\rho)} \leq H(S_r|S_c)_\rho. \qquad (231)$$

Since the R.H.S. of the above inequality is nonnegative, there exists $E \geq 0$ that satisfies the conditions (171)-(174). Hence, we have

$$C_e(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \sup_{S_c, S_r, \rho} I(S_r : B|S_c)_{\mathcal{N}^{\otimes n}(\rho)}, \qquad (232)$$

where the supremum is taken over all finite-dimensional quantum systems S_c, S_r and a classical-quantum state ρ between S_c and $S_r A^n$. Since the von Neumann entropy conditioned by a classical system is equal to the entropy averaged over the values of the classical system, without loss of generality we can assume that S_c is a trivial (one-dimensional) system. Thus, we have

$$C_e(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \sup_{S_r, \rho} I(S_r : B)_{\mathcal{N}^{\otimes n}(\rho)}. \qquad (233)$$

It is known that the mutual information of quantum channels is additive, and that it is sufficient to take the supremum over all pure states (see e.g. Section 12.4 in [1]). Thus, we arrive at the entanglement-assisted classical capacity theorem by Bennet et al. [8], [9], which yields

$$C_e(\mathcal{N}) = \sup_{|\phi\rangle^{AA'}} I(S_r : B)_{\mathcal{N}(\phi)}. \qquad (234)$$

D. Quantum Capacity With Free Entanglement Assistance

The entanglement-assisted δ-quantum capacity $q_e(\mathcal{N}, \delta)$ is defined as the supremum of q such that there exists $e > 0$ and the triplet $(c = 0, q, e)$ is achievable within the error δ for the channel $\mathcal{N}$. Lower and upper bounds on the entanglement-assisted δ-quantum capacity is obtained from Corollary 6 and Corollary 7 by letting $c = 0$, which yields

$$\frac{1}{2}\widehat{\sup_{S, \rho}} \sup_{\delta' \in (0, \delta^4/16]} \left[\log d_S - H_{\max}^{\epsilon(\delta, \delta')}(S|B)_{\mathcal{N}(\rho)} + \log \delta' \right]$$

$$\leq q_e(\mathcal{N}, \delta) \leq$$

$$\frac{1}{2}\widehat{\sup_{S, \rho}} \inf_{\iota \in (0, 1]} \left[\log d_{S_r} - H_{\max}^{\lambda'(\delta, \iota)}(S|B)_{\mathcal{N}(\rho)} - \log \iota \right]. \qquad (235)$$

A similar result was obtained in Datta et al. [19] based on the decoupling approach. They define the one-shot entanglement-assisted quantum capacity $q_{\mathrm{ea}, \epsilon}^{(1)}$ in terms of the entanglement transmission fidelity (see Definition 6 and 7 therein) and obtained lower and upper bounds thereof. They proved that for any $0 < \varepsilon < 1$ and ε' such that $\varepsilon = 2\sqrt{2\sqrt{27\varepsilon'} + 27\varepsilon'}$, it holds that

$$\max_{\varrho \in \mathcal{S}(\mathcal{H}^A)} \frac{1}{2} \left[H_{\min}^{\varepsilon'}(S)_{\mathcal{N}(\phi)} - H_{\max}^{\varepsilon'}(S|B)_{\mathcal{N}(\phi)} \right] + 2\log\varepsilon'$$

$$\leq q_{\mathrm{ea}, \varepsilon}^{(1)}(\mathcal{N}) \leq$$

$$\max_{\varrho \in \mathcal{S}(\mathcal{H}^A)} \frac{1}{2} \left[H_{\min}^{\varepsilon}(S)_{\mathcal{N}(\phi)} - H_{\max}^{2\varepsilon + 4\sqrt[4]{\varepsilon}}(S|B)_{\mathcal{N}(\phi)} \right] + \log\frac{\sqrt{2}}{\varepsilon},$$
$$(236)$$

where ϕ^{SA} is a purification of $\varrho \in \mathcal{S}(\mathcal{H}^A)$. They also proved that similar bounds hold for the entanglement-assisted quantum capacity defined in terms of the minimum output fidelity (see Theorem 11 in [19]).

In the asymptotic limit, one can see that if there exists (Q, E) satisfying the conditions (171) and (173), it holds that

$$\begin{cases} Q & \leq \frac{1}{2} I(S_r : B|S_c)_{\mathcal{N}(\rho)}, \\ Q - E & \leq -H(S_r|BS_c)_{\mathcal{N}(\rho)}. \end{cases} \qquad (237)$$

Conversely, if both of the above inequality hold, there exists $\Delta \geq 0$ such that

$$Q + E \leq H(S_r|S_c)_{\mathcal{N}(\rho)} + \Delta, \qquad (238)$$

$$Q - E \leq -H(S_r|BS_c)_{\mathcal{N}(\rho)} - \Delta. \qquad (239)$$

Indeed, letting $\Delta = Q + E - H(S_r|S_c)_{\mathcal{N}(\rho)}$ yields

$$-H(S_r|BS_c)_{\mathcal{N}(\rho)} - \Delta = I(S_r : B|S_c)_{\mathcal{N}(\rho)} - Q - E$$
$$\geq Q - E, \qquad (240)$$

where the second line follows from the first condition in (237). For any $m \in \mathbb{N}$, let S_m be an $2^{\lfloor m\Delta \rfloor}$-dimensional quantum system, π_m be the maximally mixed state thereon, and $\varrho_m \equiv \pi_m \otimes \rho$. Denoting $S_m S_r^m \equiv \hat{S}_r$, we obtain

$$Q + E \leq \frac{1}{m}[H(\hat{S}_r|S_c^m)_{\mathcal{N}^{\otimes m}(\varrho_m)} + 1], \qquad (241)$$

$$Q - E \leq -\frac{1}{m} H(\hat{S}_r|B^n S_c^n)_{\mathcal{N}^{\otimes n}(\varrho_m)}. \qquad (242)$$

Taking the limit of m to infinity, we observe that the pair (Q, E) belongs to the region defined by Definition 13 for $C = 0$. Thus, we obtain the two-dimensional rate region obtained by Devetak et al. [12] (Theorem 7.7 in [12]). I.e., the region obtained from the set of all pairs (Q, E) satisfying

$$\begin{cases} Q & \leq \frac{1}{2} I(A : B)_{\mathcal{N}(\phi)}, \\ Q - E & \leq -H(A|B)_{\mathcal{N}(\phi)}, \end{cases} \qquad (243)$$

by taking the union over all pure states $|\phi\rangle^{AA'}$ and by regularizing it. Note that the region defined in [12] is convex because of regularization. Hence, the conditioning by a classical system S_c does not change the rate region.

E. Simultaneous Capacity of Classical and Quantum Information Without Entanglement Assistance

We consider the one-shot capacity region for simultaneously transmitting classical and quantum messages through the channel without entanglement assistance. The δ-simultaneous capacity region of a channel $\mathcal{N}$ is defined as the set of all pairs $(c, q) \in \mathbb{R}_{\geq}^2$ such that the triplet $(c, q, e = 0)$ is achievable within the error δ for the channel $\mathcal{N}$. We denote the achievable rate region by $\Gamma_\delta(\mathcal{N})$, and assume that $\delta \in (0, 2]$. For an arbitrary system $S \equiv S_c S_r$ and an arbitrary state ρ^{SA} that is diagonal in S_c with respect to a fixed orthonormal basis, let $\Gamma_{\delta, \delta', \epsilon}^{\text{in}}(\mathcal{N}, \rho)$ be the set of all pairs $(c, q) \in \mathbb{R}_{\geq}^2$ that satisfy

$$\begin{cases} c + q & \leq -H_{\max}^{\epsilon(\delta, \delta')}(S|B)_{\mathcal{N}(\rho)} + \log(d_{S_c} - 1) + \log \delta', \\ q & \leq -H_{\max}^{\epsilon(\delta, \delta')}(S_r|BS_c)_{\mathcal{N}(\rho)} + \log \delta'(1 - 2\epsilon(\delta, \delta')) \end{cases}$$

and let $\Gamma_{\delta, \iota}^{\text{out}}(\mathcal{N}, \rho)$ be the one that satisfy

$$\begin{cases} c + q & \leq -H_{\max}^{\lambda(\delta, \iota)}(S|B)_{\mathcal{N}(\rho)} + \log d_{S_c} - \log \iota, \\ q & \leq -H_{\max}^{\lambda'(\delta, \iota)}(S_r|BS_c)_{\mathcal{N}(\rho)} - \log \iota. \end{cases} \quad (244)$$

It follows from Theorem 3 and Theorem 4 that

$$\bigcup_{S, \rho} \bigcup_{\delta'} \Gamma_{\delta, \delta'}^{\text{in}}(\mathcal{N}, \rho) \subseteq \Gamma_\delta(\mathcal{N}) \subseteq \bigcup_{S, \rho} \bigcap_{\iota} \Gamma_{\delta, \iota}^{\text{out}}(\mathcal{N}, \rho). \quad (245)$$

Here, the union over δ' and the intersection over ι are taken in the intervals $\delta' \in (0, \delta^4/16]$ and $\iota \in (0, 1]$, respectively. The union over ρ is taken over all classical-quantum states ρ between S_c and $S_r A$ such that ρ^S is the full-rank maximally mixed.

The same communication task has been analyzed by Salek et al. [20] for the one-shot scenario based on the hypothesis-testing approach. They first proved the inner and outer bounds on the two-dimensional capacity region for simultaneously transmitting public and private messages (see Theorem 2 and 3 therein). Then they apply the argument by Devetak et al. [7], which states that the private classical capacity of a quantum channel is equal to the quantum capacity. The outer bound is given as the union, over all states ρ, of the set of rate pairs (c, q) satisfying

$$\begin{cases} c & \leq I_H^\delta(X : B)_\varrho \\ q & \leq I_H^{\sqrt{\delta}}(Y : B|X)_\varrho - I_{\max}^{\sqrt{2\delta'}}(Y : E|X)_\varrho \end{cases}. \quad (246)$$

Here, I_H and $I_{\max}$ are the hypothesis testing mutual information and the max mutual information, and δ and δ' are the error parameters for the classical message and the quantum message, respectively. The mutual informations are for the state of the form $\varrho = \sum_{x,y} p(x, y)|x\rangle\langle x| \otimes |y\rangle\langle y| \otimes \mathcal{N}^{A \to B, E}(\rho_{x,y})$, where $\mathcal{N}^{A \to E}$ is the complementary channel of $\mathcal{N}^{A \to B}$. The inner bound is given in a similar form, with the additional terms on the R.H.S.s of the two inequality that depends only on the smoothing parameters and the errors.

One may think that our bound (245) is not consistent with those obtained (246). The condition (246) defines a rectangle, while (244) defines a trapezoid. Note, however, that the actual bounds are obtained by taking the union over states ρ in both cases, which resolves this seeming mismatch.

In the asymptotic limit, we obtain from the conditions (171)-(173) that

$$\begin{cases} C + Q & \leq H(S_c)_\rho - H(S|B)_{\mathcal{N}(\rho)}, \\ Q & \leq -H(S_r|BS_c)_{\mathcal{N}(\rho)}, \end{cases} \quad (247)$$

which is equivalent to the conditions obtained by Hsieh and Wilde in [14] (see Theorem 5). Originally, Devetak and Shor [13] addressed this problem and obtained the achievable rate region which is characterized by the following conditions:

$$\begin{cases} C & \leq I(S_c : B)_{\mathcal{N}(\rho)}, \\ Q & \leq -H(S_r|BS_c)_{\mathcal{N}(\rho)}. \end{cases} \quad (248)$$

As discussed in [14], by taking the union over all state ρ and by regularizing it, the above regions coincide with each other.

X. CONCLUSION

In this paper, we analyzed the task of simultaneously transmitting classical and quantum messages through a noisy quantum channel assisted by a limited amount of shared entanglement in the one-shot scenario. We have derived direct and converse bounds for the one-shot achievable rate region. To our knowledge, this is the first time that trade-off relations that apply simultaneously to all the three resources of classical communication, quantum communication, and shared entanglement has been obtained in the one-shot channel coding scenario. We then applied the one-shot result to the asymptotic scenario of infinitely many uses of a memoryless channel. We obtained a complete characterization of the triple resource trade-offs, which coincides with the prior result by Hsieh and Wilde [14]. Numerical calculations of the achievable rate region for specific qubit channels are given in [47]. In [48], we analyzed quantum state redistribution for a classical-quantum hybrid source in the one-shot scenario in terms of randomized partial decoupling. To investigate relations between these two tasks is left as future work.

To investigate quantum channel coding problems in the one-shot scenario, an approach from hypothesis testing (see e.g. [15], [17], [20], [23], [24], [25], [26], [49]) is complementary to the one based on decoupling. The hypothesis-testing approach has an advantage over the decoupling approach in that it recovers the optimal asymptotic results not only in the i.i.d. setting but also in the non-i.i.d. (information spectrum) setting. This approach has been widely used for quantum channel coding problems e.g. to investigate the strong converse bounds and the second-order asymptotics. Along this line, the concepts of convex splitting [50] and position-based coding [28] have recently proved useful in various settings of one-shot quantum channel coding [20], [27], [28], [29], [51], [52], [53], [54], [55], [56], [57], [58]. One future direction is to apply the hypothesis-testing approach to the problem addressed in this paper and obtain triple-resource trade-off bounds that are asymptotically tight in the non-i.i.d. setting.

APPENDIX A
TECHNICAL LEMMAS

We introduce some technical lemmas that are used in the main text.

Lemma 16: Consider two states ρ and σ in the form of

$$\rho = \frac{1}{K} \sum_{k=1}^{K} |k\rangle\langle k|^X \otimes |k\rangle\langle k|^Y \otimes \rho_k^A, \tag{249}$$

$$\sigma = \frac{1}{K} \sum_{k=1}^{K} |k\rangle\langle k|^X \otimes \left(\sum_{k'=1}^{K} p(k'|k)|k'\rangle\langle k'|^Y \otimes \sigma_{kk'}^A \right), \tag{250}$$

where $\{|k\rangle\}_k$ is an orthonormal basis of $\mathcal{H}^X$ and $\mathcal{H}^Y$, $\{p(k'|k)\}_{k'=1}^K$ is a conditional probability distribution, and ρ_k and $\sigma_{kk'}$ are normalized states on A for each k and k'. Suppose that we have

$$\frac{1}{K} \sum_{k=1}^{K} (1 - p(k|k)) \le \frac{\delta}{3}, \tag{251}$$

$$\frac{1}{K} \sum_{k=1}^{K} \left\| \rho_k - \sum_{k'=1}^{K} p(k'|k)\sigma_{kk'} \right\|_1 \le \frac{\delta}{3} \tag{252}$$

for $\delta > 0$. Then, it holds that

$$\|\rho - \sigma\|_1 \le \delta. \tag{253}$$

Proof: Using the properties of the trace distance (see e.g. Section 9.1 in [1]), we have

$$
\begin{aligned}
&\|\rho - \sigma\|_1 \\
&= \frac{1}{K} \sum_{k=1}^{K} \left\| |k\rangle\langle k|^Y \otimes \rho_k^A - \sum_{k'=1}^{K} p(k'|k)|k'\rangle\langle k'|^Y \otimes \sigma_{kk'}^A \right\|_1 \\
&= \frac{1}{K} \sum_{k=1}^{K} \left\| \rho_k^A - p(k|k)\sigma_{kk}^A \right\|_1 \\
&\quad + \frac{1}{K} \sum_{k=1}^{K} \sum_{k' \ne k} p(k'|k) \left\| \sigma_{kk'}^A \right\|_1
\end{aligned}
\tag{254}
$$

$$
\begin{aligned}
&\le \frac{1}{K} \sum_{k=1}^{K} \left\| \rho_k^A - \sum_{k'=1}^{K} p(k'|k)\sigma_{kk'}^A \right\|_1 \\
&\quad + \frac{2}{K} \sum_{k=1}^{K} \sum_{k' \ne k} p(k'|k) \left\| \sigma_{kk'}^A \right\|_1
\end{aligned}
\tag{255}
$$

$$
= \frac{1}{K} \sum_{k=1}^{K} \left\| \rho_k^A - \sum_{k'=1}^{K} p(k'|k)\sigma_{kk'}^A \right\|_1 + \frac{2}{K} \sum_{k=1}^{K} (1 - p(k|k))
$$

$$\le \delta, \tag{256}$$

where Inequality (255) follows due to the triangle inequality. ∎

Lemma 17: Consider a state ρ on XA and a pure state $|\phi\rangle$ on $XYAB$ that take the forms of

$$\rho = \sum_k p_k |k\rangle\langle k|^X \otimes \varrho_k^A, \tag{257}$$

$$|\phi\rangle = \sum_k \sqrt{p_k} |k\rangle^X |k\rangle^Y |\varphi_k\rangle^{AB}, \tag{258}$$

where $\{p_k\}_k$ is a probability distribution, ϱ_k is a state on A and $|\varphi_k\rangle$ is a pure state on AB for each k. Suppose that

$$\left\| \phi^{XA} - \rho^{XA} \right\|_1 \le \delta. \tag{259}$$

Then, there exists a purification $|\Psi\rangle^{XYAB}$ of ρ that takes the form of

$$|\Psi\rangle = \sum_k \sqrt{p_k} |k\rangle^X |k\rangle^Y |\psi_k\rangle^{AB}, \tag{260}$$

and satisfies $\||\Psi\rangle\langle\Psi| - |\phi\rangle\langle\phi|\|_1 \le 2\sqrt{\delta}$.

Proof: Any purification $|\Psi'\rangle$ of ρ is represented as

$$|\Psi'\rangle = \sum_k \sqrt{p_k} |k\rangle^X |\psi_k'\rangle^{YAB}, \tag{261}$$

where $|\psi_k'\rangle$ is a purification of ϱ_k for each k. It follows that

$$|\langle\Psi'|\phi\rangle| = \left| \sum_k p_k \langle\psi_k'|^{YAB} |k\rangle^Y |\psi_k\rangle^{AB} \right| \tag{262}$$

$$\le \sum_k p_k \left| \langle\psi_k'|^{YAB} |k\rangle^Y |\psi_k\rangle^{AB} \right| \tag{263}$$

$$\le \sum_k p_k \max_{\psi_k''} |\langle\psi_k''|\psi_k\rangle| \tag{264}$$

$$= \sum_k p_k |\langle\psi_k^*|\psi_k\rangle|, \tag{265}$$

where we have defined

$$|\psi_k^*\rangle := \arg \max_{|\psi_k''\rangle} |\langle\psi_k''|\psi_k\rangle|. \tag{266}$$

The maximization in the fourth line is taken over all purifications $|\psi_k''\rangle^{AB}$ of ϱ_k^A. We consider a state

$$|\Psi\rangle = \sum_k \sqrt{p_k} |k\rangle^X |k\rangle^Y |\psi_k^*\rangle^{AB}. \tag{267}$$

Due to (265), it holds that

$$F(\rho^{XA}, \phi^{XA}) = \max_{|\Psi'\rangle} |\langle\Psi'|\phi\rangle| = |\langle\Psi|\phi\rangle| = F(|\Psi\rangle, |\phi\rangle),$$

where F is the fidelity defined by $F(\sigma, \tau) := \|\sqrt{\sigma}\sqrt{\tau}\|_1$. The maximization is taken over all purifications of ρ^{XA} and the first equality follows from Uhlmann's theorem [42]. By using the relation between the trace distance and the fidelity (see e.g. Section 9.2.3 in [2]), we obtain

$$1 - F(\phi^{XA}, \rho^{XA}) \le \left\| \phi^{XA} - \rho^{XA} \right\|_1 \le \delta, \tag{268}$$

$$\||\Psi\rangle\langle\Psi| - |\phi\rangle\langle\phi|\|_1 \le 2\sqrt{1 - F(|\Psi\rangle, |\phi\rangle)}. \tag{269}$$

Combining these all together, we complete the proof. ∎

APPENDIX B
PROPERTIES OF ENTROPIES

In this section, we summarize properties of quantum entropies that are used in the proofs of the main results. Note that the set of positive semidefinite operators, normalized states and subnormalized states are defined by

$$\mathcal{P}(\mathcal{H}) = \{\rho \in \mathrm{Her}(\mathcal{H}) : \rho \ge 0\}, \tag{270}$$

$$\mathcal{S}_=(\mathcal{H}) = \{\rho \in \mathcal{P}(\mathcal{H}) : \mathrm{Tr}[\rho] = 1\}, \tag{271}$$

$$\mathcal{S}_\le(\mathcal{H}) = \{\rho \in \mathcal{P}(\mathcal{H}) : \mathrm{Tr}[\rho] \le 1\}. \tag{272}$$

Lemma 18 (Definition 14, Equality (6) and Lemma 16 in [38]): For any subnormalized pure state $|\psi\rangle$ on system ABC, and for any $\epsilon > 0$, $H_{\max}^\epsilon(A|B)_\psi = -H_{\min}^\epsilon(A|C)_\psi$.

Lemma 19 (Corollary of Theorem 18 in [38]): For any state $\rho^{AB} \in \mathcal{S}_=(\mathcal{H}^{AB})$, any CPTP map $\mathcal{E}^{A \to B}$ and any $\epsilon \geq 0$, it holds that

$$H_{\max}^\epsilon(A|B)_\rho \leq H_{\max}^\epsilon(A|C)_{\mathcal{E}(\rho)}. \qquad (273)$$

Lemma 20 (Corollary of Lemma 20 in [38]): For any $\rho^{AB} \in \mathcal{S}_\leq(\mathcal{H}^{AB})$, it holds that

$$-\log d_A \leq H_{\max}(A|B)_\rho - \log \mathrm{Tr}[\rho^{AB}] \leq \log d_A. \quad (274)$$

Lemma 21 (Lemma A.2 in [35]): For any $\rho^{AB} \in \mathcal{S}_=(\mathcal{H}^{AB})$, $\sigma^{CD} \in \mathcal{S}_=(\mathcal{H}^{CD})$ and any $\epsilon, \epsilon' \geq 0$, it holds that

$$H_{\min}^{\epsilon+\epsilon'}(AC|BD)_{\rho \otimes \sigma} \geq H_{\min}^\epsilon(A|B)_\rho + H_{\min}^{\epsilon'}(C|D)_\sigma. \quad (275)$$

Lemma 22 (Theorem 1 in [44]): For any $\rho \in \mathcal{S}_=(\mathcal{H}^{AB})$ and $0 < \epsilon < 1$, it holds that

$$\lim_{n \to \infty} \frac{1}{n} H_{\max}^\epsilon(A^n|B^n)_{\rho^{\otimes n}} = H(A|B)_\rho. \qquad (276)$$

Lemma 23 (Corollary of Lemma 2 in [44]): For any $\rho \in \mathcal{S}_=(\mathcal{H}^{AB})$, it holds that

$$H_{\max}(A|B)_\rho \geq H(A|B)_\rho. \qquad (277)$$

Lemma 24: For any $\rho^{AB} \in \mathcal{S}_=(\mathcal{H}^{AB})$ and $\xi^C \in \mathcal{S}_=(\mathcal{H}^C)$, it holds that

$$H_{\max}^\epsilon(A|BC)_{\rho \otimes \xi} = H_{\max}^\epsilon(A|B)_\rho, \qquad (278)$$
$$H_{\max}^\epsilon(AC|B)_{\rho \otimes \xi} \leq H_{\max}^\epsilon(A|B)_\rho + \log d_C. \qquad (279)$$

Proof: To prove Equality (278), define an operation $\mathcal{E}_\xi^{B \to BC}$ by $\mathcal{E}_\xi(\tau^B) = \tau^B \otimes \xi^C$. Due to the monotonicity of the smooth max entropy (Lemma 19) under $\mathcal{E}_\xi^{B \to BC}$ and the partial trace operation, it holds that

$$H_{\max}^\epsilon(A|B)_\rho \leq H_{\max}^\epsilon(A|BC)_{\mathcal{E}_\xi(\rho)}$$
$$= H_{\max}^\epsilon(A|BC)_{\rho \otimes \xi} \leq H_{\max}^\epsilon(A|B)_\rho, \qquad (280)$$

which implies (278). To prove (279), note that Lemma 21 and the duality relation (Lemma B) imply, for any $\eta \in \mathcal{S}_=(\mathcal{H}^D)$,

$$H_{\max}^{\epsilon+\epsilon'}(AC|BD)_{\rho \otimes \xi \otimes \eta}$$
$$\leq H_{\max}^\epsilon(A|B)_\rho + H_{\max}^{\epsilon'}(C|D)_{\xi \otimes \eta}. \qquad (281)$$

We particularly choose $\epsilon' = 0$. Due to Inequality (278), the L.H.S. is equal to $H_{\max}^\epsilon(AC|B)_{\rho \otimes \xi}$. From Lemma 20, the second term in the R.H.S. is bounded as $H_{\max}(C|D)_{\xi \otimes \eta} \leq \log d_C$, which completes the proof. ∎

Lemma 25: For any $\rho^{AB} \in \mathcal{S}_=(\mathcal{H}^{AB})$ and $\epsilon \geq 0$, it holds that

$$-\log d_A \leq H_{\max}^\epsilon(A|B)_\rho - \log(1 - 2\epsilon). \qquad (282)$$

Proof: Let $\hat{\rho}^{AB} \in \mathcal{B}^\epsilon(\rho)$ be such that $H_{\max}^\epsilon(A|B)_\rho = H_{\max}(A|B)_{\hat{\rho}}$. Due to Lemma 20, it holds that

$$-\log d_A \leq H_{\max}(A|B)_{\hat{\rho}} - \log \mathrm{Tr}[\hat{\rho}^{AB}]. \qquad (283)$$

Using the triangle inequality for the trace distance, we have

$$\mathrm{Tr}[\hat{\rho}^{AB}] = \|\hat{\rho}^{AB}\|_1 \geq \|\rho^{AB}\|_1 - \|\rho^{AB} - \hat{\rho}^{AB}\|_1 \geq 1 - 2\epsilon,$$

where the last inequality follows from the relation between the trace distance and the purified distance (7). Substituting this to (283), we complete the proof. ∎

Lemma 26: For any $0 \leq \epsilon < 1$ and any state $\rho \in \mathcal{S}_=(\mathcal{H}^{AB})$, it holds that

$$H_{\max}^\epsilon(A|B)_\rho \geq H(A|B)_\rho - \eta(\epsilon) \log d_A, \qquad (284)$$

where η is a function that satisfies $\lim_{x \to 0} \eta(x) = 0$ and is independent of the dimensions of the systems.

Proof: Let $\hat{\rho} \in \mathcal{B}^\epsilon(\rho)$ be a subnormalized state such that $H_{\max}^\epsilon(A|B)_\rho = H_{\max}^\epsilon(A|B)_{\hat{\rho}}$. From Lemma 23, it holds that

$$H_{\max}(A|B)_{\hat{\rho}/\mathrm{Tr}[\hat{\rho}]} \geq H(A|B)_{\hat{\rho}/\mathrm{Tr}[\hat{\rho}]}. \qquad (285)$$

Thus, Inequality (284) follows due to the Alicki-Fannes inequality ([59], see also Inequality (89) in [60]). ∎

Lemma 27: Let $\{\Pi_m\}_{m=1}^M$ be a complete set of orthogonal projectors on a finite dimensional Hilbert space $\mathcal{H}^A$, and let X be a quantum system with a fixed orthonormal basis $\{|m\rangle\}_{m=1}^M$. Consider a map $\mathcal{E}: A \to XA$ defined by

$$\mathcal{E}(\cdot) := \sum_{m=1}^M |m\rangle\langle m|^X \otimes \Pi_m(\cdot)\Pi_m^A. \qquad (286)$$

For any state ρ on system AB, it holds that

$$H(A|B)_\rho \leq H(XA|B)_{\mathcal{E}(\rho)}, \qquad (287)$$
$$H(A|BX)_{\mathcal{E}(\rho)} \leq H(A|B)_\rho + \log M. \qquad (288)$$

Proof: The first inequality follows from the isometric invariance of the conditional quantum entropy and its monotonicity under the completely dephasing operation (see e.g. Corollary 11.9.4 in [1]). Note that the map $\mathcal{E}$ is represented as $\mathcal{E} = \mathcal{C}^X \circ \mathcal{V}^{A \to XA}$, where V is a linear isometry defined by $V := \sum_{m=1}^M |m\rangle^X \otimes \Pi_m^A$ and $\mathcal{C}$ is the completely dephasing operation on X with respect to the basis $\{|m\rangle\}_{m=1}^M$. To prove the second inequality, let X' be a M-dimensional Hilbert space with a fixed orthonormal basis $\{|m\rangle\}_{m=1}^M$. Define a linear isometry $V: \mathcal{H}^A \to \mathcal{H}^X \otimes \mathcal{H}^{X'} \otimes \mathcal{H}^A$ by

$$V := \sum_{m=1}^M |m\rangle^X \otimes |m\rangle^{X'} \otimes \Pi_m^A. \qquad (289)$$

A Stinespring dilation of the map $\mathcal{E}$ is given by $\mathcal{E} = \mathrm{Tr}_{X'} \circ \mathcal{V}$. It holds that

$$H(A|B)_\rho \qquad (290)$$
$$= H(XX'A|B)_{\mathcal{V}(\rho)} \qquad (291)$$
$$\geq H(X|B)_{\mathcal{V}(\rho)} + H(A|BX)_{\mathcal{V}(\rho)}$$
$$\qquad + H(X'|ABX)_{\mathcal{V}(\rho)} \qquad (292)$$
$$\geq H(A|BX)_{\mathcal{V}(\rho)} - \log \dim X \qquad (293)$$
$$= H(A|BX)_{\mathcal{E}(\rho)} - \log M, \qquad (294)$$

where the first line follows from the isometric invariance of the conditional quantum entropy, the second line from the chain rule, the third line due to $H(X|B)_{\mathcal{V}(\rho)} \geq 0$ and $H(X'|ABX)_{\mathcal{V}(\rho)} \geq -\log \dim X$, and the last line from $\mathcal{V}(\rho)^{AB} = \mathcal{E}(\rho)$. ∎

Lemma 28: Let $\{p_x, \rho_x\}_{x \in \mathcal{X}}$ be an ensemble of states on system A, and suppose that there exists a POVM $\{M_x\}_{x \in \mathcal{X}}$ such that

$$\sum_{x \in \mathcal{X}} p_x \mathrm{Tr}[M_x \rho_x] \geq 1 - \epsilon. \qquad (295)$$

Then, for the state

$$\rho^{XA} := \sum_{x \in \mathcal{X}} p_x |x\rangle\langle x|^X \otimes \rho_x^A, \qquad (296)$$

it holds that

$$H(X|A)_\rho \leq \eta(\epsilon) \log |\mathcal{X}|, \qquad (297)$$

where η is a function that satisfies $\lim_{\epsilon \to 0} \eta(\epsilon) = 0$ and is independent of the dimensions of the systems.

Proof: Let $\hat{X}$ be a quantum system with a fixed orthonormal basis $\{|x\rangle\}_{x \in \mathcal{X}}$, and define a CPTP map $\mathcal{M} : A \to \hat{X}$ by $\mathcal{M}(\cdot) := \mathrm{Tr}[M_x(\cdot)]|x\rangle\langle x|^{\hat{X}}$. It follows that

$$\tilde{\rho}^{X\hat{X}} := \mathrm{id}^X \otimes \mathcal{M}(\rho^{XA}) \qquad (298)$$

$$= \sum_{x,x' \in \mathcal{X}} p_x |x\rangle\langle x|^X \otimes p_{x'|x} |x'\rangle\langle x'|^{\hat{X}}, \qquad (299)$$

where $\{p_{x'|x}\}_{x' \in \mathcal{X}}$ is a conditional probability distribution defined by $p_{x'|x} = \mathrm{Tr}[M_{x'}\rho_x]$. Thus, due to the monotonicity of the conditional quantum entropy, we have

$$H(X|A)_\rho \leq H(X|\hat{X})_{\tilde{\rho}}. \qquad (300)$$

With a slight abuse of notation, let $(X, \hat{X})$ be a pair of random variables that takes values in $\mathcal{X} \times \mathcal{X}$ according to a joint probability distribution $P \equiv \{p(x, \hat{x})\}$, where $p(x, \hat{x}) := p_x p_{x'|x}$. Due to the condition (295), it holds that

$$P(X \neq \hat{X}) = \sum_{\substack{x,x' \in \mathcal{X} \\ x \neq x'}} p(x, \hat{x}) = \sum_{x \in \mathcal{X}} p_x(1 - p_{x|x}) \leq \epsilon. \quad (301)$$

Thus, due to Fano's inequality (see e.g. Theorem 2.10.1 in [61]), it follows that

$$H(X|\hat{X})_{\tilde{\rho}} = H(X|\hat{X})_P \leq \eta(\epsilon) \log |\mathcal{X}|. \qquad (302)$$

Combining this with (300), we complete the proof. ∎

APPENDIX C
METHOD OF TYPES AND TYPE SUBSPACES

In this section, we briefly review the definitions and properties of types and type subspaces. For the details, see e.g. Section 13.7 and 14.3 in [1]. The properties of type subspaces presented in this section will be used in Appendix D to prove the equivalence between the achievable rate region for the asymptotic limit given by Theorem 14 in the main text and the one obtained in Ref. [14]

Let $\mathcal{X}$ be a finite alphabet and $n \in \mathbb{N}$. A probability distribution $\{t(x)\}_{x \in \mathcal{X}}$ is called a *type of length* n if $nt(x) \in \mathbb{N}$ for all $x \in \mathcal{X}$. Let $x^n \equiv x_1 \cdots x_n$ be a sequence of variables of length n such that $x_i \in \mathcal{X}$ for each i. The sequence x^n is said to be of type t if

$$\frac{1}{n} N(x|x^n) = t(x) \qquad (303)$$

for all $x \in \mathcal{X}$, where $N(x|x^n)$ is the number of the symbol $x \in \mathcal{X}$ that appears in the sequence x^n. The *type class* corresponding to the type t of length n, which we denote by T_t^n, is the set of all sequences whose type is t. Let $\mathfrak{T}(\mathcal{X}, n)$

be the set of all types of length n on the alphabet $\mathcal{X}$. It holds that

$$T_t^n \cap T_{t'}^n = \emptyset \ (t \neq t'), \quad \mathcal{X}^n = \bigcup_{t \in \mathfrak{T}(\mathcal{X},n)} T_t^n. \qquad (304)$$

The size of $\mathfrak{T}(\mathcal{X}, n)$ is bounded above by a polynomial function of n as

$$|\mathfrak{T}(\mathcal{X}, n)| \leq (n+1)^{|\mathcal{X}|}. \qquad (305)$$

By definition, any two sequences in the same type class are transformed with each other by permuting the elements. That is, for any $t \in \mathfrak{T}(\mathcal{X}, n)$ and $x^n, x'^n \in T_t^n$, there exists a permutation s such that $x'_i = x_{s(i)}$ for any $1 \leq i \leq n$.

Let $\mathcal{H}$ be a Hilbert space with a fixed orthonormal basis $\{|x\rangle\}_{x \in \mathcal{X}}$. For any $n \in \mathbb{N}$, the *type subspace* corresponding to the type t of length n is defined by

$$\mathcal{H}_t^n := \mathrm{span}\{|x^n\rangle : x^n \in T_t^n\} \subseteq \mathcal{H}^{\otimes n}, \qquad (306)$$

where $|x^n\rangle = |x_1\rangle \cdots |x_n\rangle$, and the *type projector* is defined by

$$\Pi_t^n := \sum_{x^n \in T_t^n} |x^n\rangle\langle x^n|. \qquad (307)$$

It follows from (304) that

$$\Pi_t^n \Pi_{t'}^n = 0 \ (t \neq t'), \quad I = \sum_{t \in \mathfrak{T}(\mathcal{X},n)} \Pi_t^n. \qquad (308)$$

Consider a state $\rho \in \mathcal{S}_=(\mathcal{H})$ and suppose that the eigen decomposition of ρ is given by $\rho = \sum_{x \in \mathcal{X}} p_x |x\rangle\langle x|$. By definition, it holds that

$$\Pi_t^n \rho^{\otimes n} \Pi_t^n = \sum_{x^n \in T_t^n} p_{x^n} |x^n\rangle\langle x^n| = q_t \Pi_t^n, \qquad (309)$$

where

$$q_t := \prod_{x \in \mathcal{X}} p_x^{nt(x)}. \qquad (310)$$

The projectors $\{\Pi_t^n\}_{t \in \mathfrak{T}(\mathcal{X},n)}$ are called the *type projectors corresponding to the state* $\rho^{\otimes n}$.

Consider an ensemble $\{p_j, \rho_j\}_{j=1}^J$, where $J \in \mathbb{N}$, $\rho_j \in \mathcal{S}_=(\mathcal{H})$, and fix arbitrary $n \in \mathbb{N}$. For any type t of length n over $[J] := \{1, \cdots, J\}$, define a sequence of length n by

$$\boldsymbol{j}_t := \underbrace{11\cdots1}_{nt(1)}\underbrace{22\cdots2}_{nt(2)}\cdots\underbrace{JJ\cdots J}_{nt(J)}. \qquad (311)$$

For each $j \in [J]$, let $\{\Pi_{\nu_j}\}_{\nu_j \in \mathfrak{T}(\mathcal{X}, nt(j))}$ be the set of type projectors corresponding to the state $\rho_j^{\otimes nt(j)}$. We define a projector on $\mathcal{H}^{\otimes n}$ by

$$\Pi_{\vec{\nu}|\boldsymbol{j}_t} := \Pi_{\nu_1} \otimes \cdots \otimes \Pi_{\nu_J} \qquad (312)$$

for each $\vec{\nu} := \nu_1 \cdots \nu_J \in \bigotimes_{j=1}^J \mathfrak{T}(\mathcal{X}, nt_j)$. Using (305), the number of conditional type projectors is bounded above by

$$\prod_{j=1}^J |\mathfrak{T}(\mathcal{X}, nt(j))| \leq \prod_{j=1}^J (nt(j)+1)^{|\mathcal{X}|} \leq (n+1)^{2\dim\mathcal{H}}. \qquad (313)$$

As mentioned above, for any $t \in \mathfrak{T}(J, n)$ and $\boldsymbol{j} \in T_t^n$, there exists a permutation s such that $j_i = j_{t,s(i)}$ for each $1 \le i \le n$, where $j_{t,s(i)}$ is the $s(i)$-th element of the sequence $\boldsymbol{j}_t$ defined by (311). Let P_s be a unitary that acts on $\mathcal{H}^{\otimes n}$ as

$$P_s(|\varphi_1\rangle \otimes \cdots \otimes |\varphi_n\rangle) = |\varphi_{s(1)}\rangle \otimes \cdots \otimes |\varphi_{s(n)}\rangle. \quad (314)$$

We define the set of *conditional type projectors* $\{\Pi_{\vec{\nu}|\boldsymbol{j}}\}_{\vec{\nu}}$ on $\mathcal{H}^{\otimes n}$ by

$$\Pi_{\vec{\nu}|\boldsymbol{j}} = P_s \Pi_{\vec{\nu}|\boldsymbol{j}_t} P_s^\dagger. \quad (315)$$

By definition, it holds that

$$\mathrm{Tr}[\Pi_{\vec{\nu}|\boldsymbol{j}} \rho_{\boldsymbol{j}}] = \mathrm{Tr}[P_s \Pi_{\vec{\nu}|\boldsymbol{j}_t} P_s^\dagger P_s \rho_{\boldsymbol{j}} P_s^\dagger] = \mathrm{Tr}[\Pi_{\vec{\nu}|\boldsymbol{j}_t} \rho_{\boldsymbol{j}_t}]. \quad (316)$$

Note that the number of conditional type projectors for the sequences $\boldsymbol{j}$ is equal to that for $\boldsymbol{j}'$ if the two sequences belong to the same type class.

APPENDIX D
PROOF OF PROPOSITION 15

In this appendix, we prove Proposition 15 in Section VII. For the simplicity of notations, we denote d_{S_c} by J.

A. Proof of $\overline{\Theta_\pi^\infty(\mathcal{N})} = \overline{\Theta^\infty(\mathcal{N})}$

By definition, it is straightforward to verify that $\overline{\Theta_\pi^\infty(\mathcal{N})} \subseteq \overline{\Theta^\infty(\mathcal{N})}$. Thus, it suffices to prove the converse relation $\overline{\Theta_\pi^\infty(\mathcal{N})} \supseteq \overline{\Theta^\infty(\mathcal{N})}$. We prove this by showing that $\overline{\Theta_\pi^\infty(\mathcal{N})} \supseteq \frac{1}{n}\Theta(\mathcal{N}^{\otimes n}, \rho)$ for any n and any state ρ in the form of

$$\rho^{SA^n} = \sum_{j=1}^J p_j |j\rangle\langle j|^{S_c} \otimes \rho_j^{S_r A^n}, \quad (317)$$

where we do not require that $\rho^S = \pi^S$. We only consider the case where $n = 1$. It is straightforward to generalize the proof for $n \ge 2$.

1) Construction of States: Fix an arbitrary $\epsilon > 0$ and choose sufficiently large m. Let $\mathfrak{T}(J, m)$ be the set of all types of length m over $[J] := \{1, \cdots, J\}$, and $T_t^n \subset [J]^{\times m}$ be the type class corresponding to a type $t \in \mathfrak{T}(J, m)$ (see Appendix C for the definitions and properties of types and type subspaces). For any $t \in \mathfrak{T}(J, m)$ and $\vec{j} \in T_t^n$, let $\{\Pi_{\varsigma|\vec{j}}\}_{\varsigma=1}^{\theta(\vec{j})}$ be the set of conditional type projectors on $(\mathcal{H}^{S_r})^{\otimes m}$ with respect to $\rho_{\vec{j}}^{S_r^m}$. Here, $\theta(\vec{j})$ is the number of the conditional type subspaces. Note that $\theta(\vec{j}) = \theta(\vec{j}')$ if $\vec{j}$ and $\vec{j}'$ belong to the same type class. Thus, we will denote $\theta(\vec{j})$ also as $\theta(t)$ if $\vec{j} \in T_t^n$. Define probability distributions $\{p_t\}_{t \in \mathfrak{T}(J,m)}$ and $\{p_{\varsigma|\vec{j}}\}_{\varsigma=1}^{\theta(\vec{j})}$ for each $\vec{j} \in J^{\times m}$ by

$$p_t := \sum_{\vec{j} \in T_t^n} p_{\vec{j}}, \quad p_{\varsigma|\vec{j}} := \mathrm{Tr}[\Pi_{\varsigma|\vec{j}} \rho_{\vec{j}}^{S_r^m}]. \quad (318)$$

Due to the properties of the conditional type projectors, it holds that $p_{\varsigma|\vec{j}} = p_{\varsigma|\vec{j}'}$ for any $\vec{j}$ and $\vec{j}'$ in the same type set t, which we denote by $p_{\varsigma|t}$. We define a probability distribution $\{p_{\varsigma,t}\}_{\varsigma \in [\theta(t)], t \in \mathfrak{T}(J,m)}$ by

$$p_{\varsigma,t} = p_t \cdot p_{\varsigma|t}. \quad (319)$$

Let Y and Y' be quantum systems with dimensions $|\mathfrak{T}(J, m)|$ and $\theta_* := \max_{t \in \mathfrak{T}(J,m)} |\theta(t)|$, respectively. Consider states

$$\rho_{\varsigma,\vec{j}}^{S_r^m A^m} := p_{\varsigma|\vec{j}}^{-1} \Pi_{\varsigma|\vec{j}}^{S_r^m} \rho_{\vec{j}}^{S_r^m A^m} \Pi_{\varsigma|\vec{j}}^{S_r^m}, \quad (320)$$

$$\rho_{\varsigma,t}^{S^m A^m} := \frac{1}{|T_t^n|} \sum_{\vec{j} \in T_t^n} |\vec{j}\rangle\langle\vec{j}|^{S_c^m} \otimes \rho_{\varsigma,\vec{j}}^{S_r^m A^m} \quad (321)$$

and define

$$\rho_m^{YY'S^m A^m} := \sum_{t \in \mathfrak{T}(J,m)} p_t |t\rangle\langle t|^Y \otimes \frac{1}{|T_t^n|} \sum_{\vec{j} \in T_t^n} |\vec{j}\rangle\langle\vec{j}|^{S_c^m}$$
$$\otimes \sum_{\varsigma=1}^{\theta(\vec{j})} p_{\varsigma|\vec{j}} |\varsigma\rangle\langle\varsigma|^{Y'} \otimes \rho_{\varsigma,\vec{j}}^{S_r^m A^m}. \quad (322)$$

By the definition of the type subspaces, it is straightforward to verify that $\rho_{\varsigma,t}^{S^m}$ has a flat distribution on its support, that is,

$$\rho_{\varsigma,t}^{S^m} = \frac{1}{|T_t^n|} \sum_{\vec{j} \in T_t^n} |\vec{j}\rangle\langle\vec{j}|^{S_c^m} \otimes \frac{\Pi_{\varsigma,\vec{j}}^{S_r^m}}{\mathrm{Tr}[\Pi_{\varsigma,\vec{j}}]}. \quad (323)$$

By using (319) and (321), the state (322) is rewritten into

$$\rho_m^{YY'S^m A^m} = \sum_{t \in \mathfrak{T}(J,m)} \sum_{\varsigma=1}^{\theta(t)} p_{\varsigma,t} |\varsigma, t\rangle\langle\varsigma, t|^{YY'} \otimes \rho_{\varsigma,t}^{S^m A^m}.$$

For the later convenience, we introduce a map $\mathcal{E}_{\vec{j}} : S_r^m \to Y'S_r^m$ by

$$\mathcal{E}_{\vec{j}}(\cdot) = \sum_{\varsigma=1}^{\theta(\vec{j})} |\varsigma\rangle\langle\varsigma|^{Y'} \otimes \Pi_{\varsigma|\vec{j}}^{S_r^m} (\cdot) \Pi_{\varsigma|\vec{j}}^{S_r^m} \quad (324)$$

for each $\vec{j}$, which leads to

$$\sum_{\varsigma=1}^{\theta(\vec{j})} p_{\varsigma|\vec{j}} |\varsigma\rangle\langle\varsigma|^{Y'} \otimes \rho_{\varsigma,\vec{j}}^{S_r^m A^m} = \mathcal{E}_{\vec{j}}(\rho_{\vec{j}}^{S_r^m A^m}). \quad (325)$$

A useful relation which follows from the properties of the conditional type projectors is that for any $\vec{j}$ and $\vec{j}'$ in the same type set, there exits a permutation s such that

$$\rho_{\varsigma,\vec{j}'}^{S_r^m} = \mathcal{P}_s^{S_r^m}(\rho_{\varsigma,\vec{j}}^{S_r^m}) \quad (326)$$

and

$$(\mathcal{N}^{A \to B})^{\otimes m}(\rho_{\varsigma,\vec{j}'}^{S_r^m A^m})$$
$$= \mathcal{P}_s^{S_r^m} \otimes \mathcal{P}_s^{B^m} \circ (\mathcal{N}^{A \to B})^{\otimes m}(\rho_{\varsigma,\vec{j}}^{S_r^m A^m}). \quad (327)$$

These properties will be used in the following subsections to calculate the entropies of the states.

2) Calculation of Entropies: Let us calculate entropies and mutual informations of the state defined above. We use the definition of the state ρ_m given by (322) and the fact that $(\rho^{S_c A})^{\otimes m} = \rho_m^{S_c^m A^m}$. Due to the properties of the quantum mutual information, we have

$$mI(S_c : B)_{\mathcal{N}(\rho)} \quad (328)$$
$$= I(S_c^m : B^m)_{\mathcal{N}^{\otimes m}(\rho^{\otimes m})} \quad (329)$$

$$= I(S_c^m : B^m)_{\mathcal{N}^{\otimes m}(\rho_m)} \tag{330}$$

$$\leq I(YY'S_c^m : B^m)_{\mathcal{N}^{\otimes m}(\rho_m)} \tag{331}$$

$$= I(YY' : B^m)_{\mathcal{N}^{\otimes m}(\rho_m)}$$
$$+ I(S_c^m : B^m | YY')_{\mathcal{N}^{\otimes m}(\rho_m)} \tag{332}$$

$$\leq I(S_c^m : B^m | YY')_{\mathcal{N}^{\otimes m}(\rho_m)} + H(YY') \tag{333}$$

$$\leq \sum_{t \in \mathfrak{T}(J,m)} \sum_{\varsigma=1}^{\theta(t)} p_{\varsigma,t} I(S_c^m : B^m)_{\mathcal{N}^{\otimes m}(\rho_{\varsigma,t})}$$
$$+ \log \theta_* |\mathfrak{T}(J,m)|. \tag{334}$$

We also have

$$m H(S_r | BS_c)_{\mathcal{N}(\rho)} \tag{335}$$

$$= H(S_r^m | B^m S_c^m)_{\mathcal{N}^{\otimes m}(\rho^{\otimes m})} \tag{336}$$

$$= \sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} H(S_r^m | B^m)_{\mathcal{N}^{\otimes m}(\rho_{\vec{j}})} \tag{337}$$

$$\geq \sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} [H(S_r^m | B^m Y')_{\mathcal{N}^{\otimes m} \otimes \mathcal{E}_j(\rho_{\vec{j}})} - \log \theta(\vec{j})] \tag{338}$$

$$\geq \sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} H(S_r^m | B^m Y')_{\mathcal{N}^{\otimes m} \otimes \mathcal{E}_j(\rho_{\vec{j}})} - \log \theta_*, \tag{339}$$

where (338) follows from Lemma 27 in Appendix B. Using (325) and (321), the first term in (339) is calculated as

$$\sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} H(S_r^m | B^m Y')_{\mathcal{N}^{\otimes m} \otimes \mathcal{E}_j(\rho_{\vec{j}})} \tag{340}$$

$$= \sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} \sum_{\varsigma=1}^{\theta(\vec{j})} p_{\varsigma|\vec{j}} H(S_r^m | B^m)_{\mathcal{N}^{\otimes m}(\rho_{\varsigma,\vec{j}})} \tag{341}$$

$$= \sum_{t \in \mathfrak{T}(J,m)} \sum_{\vec{j} \in T_t^n} \frac{p_t}{|T_t^n|} \sum_{\varsigma=1}^{\theta(t)} p_{\varsigma|t} H(S_r^m | B^m)_{\mathcal{N}^{\otimes m}(\rho_{\varsigma,\vec{j}})} \tag{342}$$

$$= \sum_{t \in \mathfrak{T}(J,m)} \sum_{\varsigma=1}^{\theta(t)} p_{\varsigma,t} \cdot \frac{1}{|T_t^n|} \sum_{\vec{j} \in T_t^n} H(S_r^m | B^m)_{\mathcal{N}^{\otimes m}(\rho_{\varsigma,\vec{j}})} \tag{343}$$

$$= \sum_{t \in \mathfrak{T}(J,m)} \sum_{\varsigma=1}^{\theta(t)} p_{\varsigma,t} \cdot H(S_r^m | B^m S_c^m)_{\mathcal{N}^{\otimes m}(\rho_{\varsigma,t})}. \tag{344}$$

Here, the last line follows from the fact that the entropies of the state $\mathcal{N}^{\otimes m}(\rho_{\varsigma,\vec{j}})$ depends only on ς and the type of $\vec{j}$, because of the local unitary equivalence (327). Similarly, we have

$$m H(S_r | S_c)_\rho = H(S_r^m | S_c^m)_{\rho^{\otimes m}} = \sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} H(S_r^m)_{\rho_{\vec{j}}}$$

$$\leq \sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} H(S_r^m Y')_{\mathcal{E}_{\vec{j}}(\rho_{\vec{j}})} \tag{345}$$

$$= \sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} [H(Y')_{\mathcal{E}_{\vec{j}}(\rho_{\vec{j}})} + H(S_r^m | Y')_{\mathcal{E}_{\vec{j}}(\rho_{\vec{j}})}] \tag{346}$$

$$\leq \sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} [\log \theta(\vec{j}) + H(S_r^m | Y')_{\mathcal{E}_{\vec{j}}(\rho_{\vec{j}})}] \tag{347}$$

$$\leq \sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} H(S_r^m | Y')_{\mathcal{E}_{\vec{j}}(\rho_{\vec{j}})} + \log \theta_* \tag{348}$$

$$= \sum_{t \in \mathfrak{T}(J,m)} \sum_{\varsigma=1}^{\theta(t)} p_{\varsigma,t} \cdot H(S_r^m | S_c^m)_{\rho_{\varsigma,t}} + \log \theta_*. \tag{349}$$

The second line follows from Lemma 27 in Appendix B, and the last line from the similar argument as in (344), for which we use the local unitary equivalence (326). The cardinalities of the type sets $\mathfrak{T}(J,m)$ and θ_* are bounded from above by

$$|\mathfrak{T}(J,m)| \leq (m+1)^J, \quad \theta_* \leq (m+1)^{2d_A}. \tag{350}$$

Consider an arbitrary inner point (C, Q, E) of $\Theta(\mathcal{N}, \rho)$ and choose sufficiently large m. Combining Inequalities (334), (339), (344), (349) and (350) with the conditions (171)-(173), it follows that

$$Q + E \leq \frac{1}{m} \sum_{\varsigma,t} p_{\varsigma,t} H(S_r^m | S_c^m)_{\rho_{\varsigma,t}}, \tag{351}$$

$$C + Q - E \leq \frac{1}{m} \sum_{\varsigma,t} p_{\varsigma,t} [H(S_c^m)_{\rho_{\varsigma,t}}$$
$$- H(S^m | B^m)_{\mathcal{N}^{\otimes m}(\rho_{\varsigma,t})}], \tag{352}$$

$$Q - E \leq -\frac{1}{m} \sum_{\varsigma,t} p_{\varsigma,t} H(S_r^m | B^m S_c^m)_{\rho_{\varsigma,t}}. \tag{353}$$

Here, we have used the fact that the chain rule of quantum enrtopies implies

$$I(S_c : B)_{\mathcal{N}(\rho)} - H(S_r | BS_c)_{\mathcal{N}(\rho)}$$
$$= H(S_c)_\rho - H(S | B)_{\mathcal{N}(\rho)} \tag{354}$$

and

$$I(S_c^m : B^m)_{\mathcal{N}^{\otimes m}(\rho_{\varsigma,t})} - H(S_r^m | B^m S_c^m)_{\mathcal{N}^{\otimes m}(\rho_{\varsigma,t})}$$
$$= H(S_c^m)_{\rho_{\varsigma,t}} - H(S^m | B^m)_{\mathcal{N}^{\otimes m}(\rho_{\varsigma,t})}. \tag{355}$$

Thus, we arrive at

$$(C, Q, E) \in \sum_{\varsigma,t} p_{\varsigma,t} \cdot \frac{1}{m} \Theta_\pi(\mathcal{N}^{\otimes m}, \rho_{\varsigma,t}) \tag{356}$$

$$\subset \text{conv} \frac{1}{m} \Theta_\pi(\mathcal{N}^{\otimes m}) \tag{357}$$

$$\subset \text{conv} \Theta_\pi^\infty(\mathcal{N}) \subseteq \overline{\Theta_\pi^\infty(\mathcal{N})}, \tag{358}$$

where the last line follows from the convexity of $\overline{\Theta_\pi^\infty(\mathcal{N})}$ (see the next subsection). This completes the proof of $\frac{1}{n} \Theta(\mathcal{N}^{\otimes n}, \rho) \subseteq \overline{\Theta_\pi^\infty(\mathcal{N})}$ for $n = 1$. The proofs for $n \geq 2$ are obtained along the same line. ∎

3) Proof of $\text{conv} \Theta_\pi^\infty(\mathcal{N}) \subseteq \overline{\Theta_\pi^\infty(\mathcal{N})}$: We prove that the convex hull of $\Theta_\pi^\infty(\mathcal{N})$ is a subset of $\overline{\Theta_\pi^\infty(\mathcal{N})}$. Fix arbitrary $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 + \lambda_2 = 1$, and suppose that $(C^{(i)}, Q^{(i)}, E^{(i)}) \in \Theta_\pi^\infty(\mathcal{N})$ for $i = 1, 2$. By definition, for any sufficiently large n, there exist quantum systems $S^{(i)} \equiv S_c^{(i)} S_r^{(i)}$ and a quantum state ρ_i on $S^{(i)} C^{n\lambda_i}$ such that $\rho_i^{S^{(i)}}$ is the maximally mixed state, and it holds that

$$n_i(Q^{(i)} + E^{(i)}) \leq H(S_r^{(i)} | S_c^{(i)})_{\rho_i}, \tag{359}$$

$$n_i(C^{(i)} + Q^{(i)} - E^{(i)}) \leq H(S_c^{(i)})_{\rho_i}$$
$$- H(S^{(i)} | B^{n\lambda_i})_{\mathcal{N}^{\otimes n_i}(\rho_i)}, \tag{360}$$

$$n_i(Q^{(i)} - E^{(i)}) \leq -H(S_r^{(i)} | B^{n\lambda_i} S_c^{(i)})_{\mathcal{N}^{\otimes n_i}(\rho_i)} \tag{361}$$
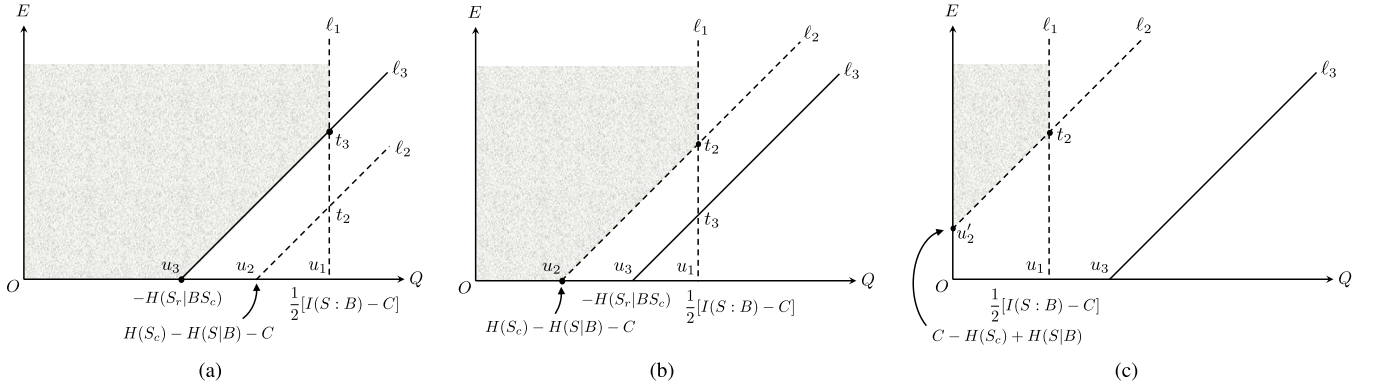
Fig. 7. The two dimensional regions of (Q, E) satisfying inequalities (164)-(166) in the case of $-H(S_r|BS_c)_{\mathcal{N}(\rho)} > 0$ are depicted. The figures (a), (b) and (c) are for $0 \leq C \leq I(S_c : B)$, $I(S_c : B) \leq C \leq I(S_c : B) - H(S_r|BS_c)$ and $I(S_c : B) - H(S_r|BS_c) \leq C \leq I(S : B)$, respectively. Lines ℓ_1, ℓ_2 and ℓ_3 represent the boundaries represented by inequalities (164)-(166). The points of intersection of the three lines and the axes are denoted by u_1, u_2, u_3, t_2, t'_2 and t_3. At $C = 0$, both u_1 and u_2 are on the right than u_3. Thus u_3 and t_3 are vertices of the region that yield P_1^+ and P_2, respectively. The two points u_1 and u_2 approaches to the origin as C increases. At $C = I(S_c : B)$, the point u_2 coincides u_3, in which case the points $u_2 = u_3$ and $t_2 = t_3$ are the vertices P_3^+ and P_4. In $C = H(S_c) - H(S|B)$, the point u_2 meets the origin, and yields P_5. Finally, the point u_1 coincides the origin at $C = I(S : B)$, in which case the point of $u'_2 = t_2$ yields P_6.
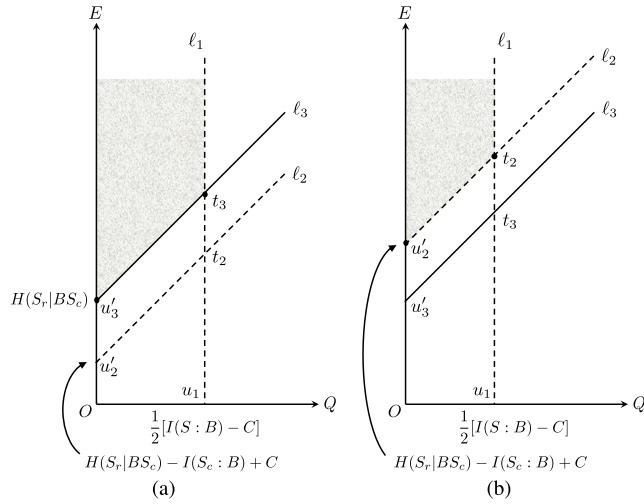

Fig. 8. The two dimensional regions of (Q, E) satisfying inequalities (164)-(166) in the case of $-H(S_r|BS_c)_{\mathcal{N}(\rho)} < 0$ are depicted. The figure (a) is for $0 \leq C \leq I(S_c : B)$, and (b) is for $I(S_c : B) \leq C \leq I(S : B)$. Lines ℓ_1, ℓ_2 and ℓ_3 represent the boundaries represented by inequalities (164)-(166), respectively, and the crossing points are denoted by u_1, u'_2, u'_3, t_2 and t_3. At $C = 0$, the point u'_2 is below u'_3, in which case u'_3 and t_3 are vertices of the region that yield P_1^- and P_2, respectively. The point u'_2 approaches u'_3 as C increases, and coincides it at $C = I(S_c : B)$. In this case, the points $u'_2 = u'_3$ and $t_2 = t_3$ correspond to the vertices P_3^- and P_4. At $C = I(S : B)$, the point u_1 coincides the origin, where the point $u'_2 = t_2$ yields P_6.

for $i = 1, 2$, where $n_1 := \lfloor n\lambda_1 \rfloor$ and $n_2 := \lceil n\lambda_i \rceil$. Define $S_c \equiv S_c^{(1)} S_c^{(2)}$, $S_r \equiv S_r^{(1)} S_r^{(2)}$, $S \equiv S_c S_r$ and consider a quantum state $\bar{\rho}$ on SA^n defined by

$$\bar{\rho}^{SA^n} := \rho_1^{S^{(1)}A^{n_1}} \otimes \rho_2^{S^{(2)}A^{n_2}}. \tag{362}$$

It is straightforward to verify that the state is diagonal on S_c with respect to a fixed basis, and that $\bar{\rho}^S$ is the full-rank maximally mixed state on S. Due to the additivity of the conditional quantum entropy, we have

$$H(S_r|S_c)_{\bar{\rho}} = H(S_r^{(1)}|S_c^{(1)})_{\rho_1} + H(S_r^{(2)}|S_c^{(2)})_{\rho_2} \tag{363}$$

and so forth. Define

$$(C_n, Q_n, E_n) \equiv \sum_{i=1,2} n_i (C^{(i)}, Q^{(i)}, E^{(i)}). \tag{364}$$

It follows from (359)-(361) that

$$Q_n + E_n \leq H(S_r|S_c)_{\bar{\rho}} \tag{365}$$

$$C_n + Q_n - E_n \leq H(S_c)_{\bar{\rho}} - H(S|B^n)_{\mathcal{N}^{\otimes n}(\bar{\rho})} \tag{366}$$

$$Q_n - E_n \leq -H(S_r|B^n S_c)_{\mathcal{N}^{\otimes n}(\bar{\rho})}, \tag{367}$$

which implies $(C_n, Q_n, E_n) \in \Theta_\pi(\mathcal{N}^{\otimes n})$. Furthermore, it is straightforward to verify that

$$\lim_{n \to \infty} \frac{1}{n} (C_n, Q_n, E_n) = (\bar{C}, \bar{Q}, \bar{E}) \tag{368}$$

$$:= \sum_{i=1,2} \lambda_i (C^{(i)}, Q^{(i)}, E^{(i)}). \tag{369}$$

This implies $(\bar{C}, \bar{Q}, \bar{E}) \in \overline{\Theta_\pi^\infty(\mathcal{N})}$, and completes the proof. ■

B. Proof of $\overline{\Theta^\infty(\mathcal{N})} = \overline{\Lambda^\infty(\mathcal{N})}$

Suppose that a triplet (C, Q, E) belongs to $\Theta(\mathcal{N}, \rho)$, which is defined by Inequalities (171)-(174). Noting that $\rho^S = \mathcal{N}(\rho)^S$, Inequalities (171) and (172) implies (164). Thus, we have $\Theta(\mathcal{N}, \rho) \subseteq \Lambda(\mathcal{N}, \rho)$, which leads to $\overline{\Theta^\infty(\mathcal{N})} \subseteq \overline{\Lambda^\infty(\mathcal{N})}$.

To prove the converse relation $\overline{\Theta^\infty(\mathcal{N})} \supseteq \overline{\Lambda^\infty(\mathcal{N})}$, we show that $\Theta^\infty(\mathcal{N}) \supseteq \Lambda(\mathcal{N}, \rho)$ for any state ρ. Note that $\Lambda(\mathcal{N}, \rho)$ is a convex polytope such that $(C, Q, E + \Delta E) \in \Lambda(\mathcal{N}, \rho)$ for any $(C, Q, E) \in \Lambda(\mathcal{N}, \rho)$ and any $\Delta E > 0$ (see Definition 11). Thus, it suffices to prove that (i) all vertices of $\Lambda(\mathcal{N}, \rho)$ belongs to $\Theta^\infty(\mathcal{N})$, and that (ii) if $(C, Q, E) \in \Theta^\infty(\mathcal{N})$, then $(C, Q, E + \Delta E) \in \Theta^\infty(\mathcal{N})$ for any $\Delta E > 0$.

1) Vertices of $\Lambda(\mathcal{N}, \rho)$: Consider the following points in $\mathbb{R}^3$, where all entropies and mutual informations are for the state $\mathcal{N}(\rho)$:

$$P_0 := (0, 0, 0)$$

$$P_1^+ := (0, -H(S_r|BS_c), 0)$$

$P_1^- := (0, \, 0, \, H(S_r|BS_c))$

$P_2 := \left(0, \, \dfrac{1}{2}I(S:B), \dfrac{1}{2}I(S:B) + H(S_r|BS_c)\right)$

$P_3^+ := (I(S_c:B), \, -H(S_r|BS_c), \, 0)$

$P_3^- := (I(S_c:B), \, 0, \, H(S_r|BS_c))$

$P_4 := \left(I(S_c:B), \dfrac{1}{2}I(S_r:D|S_c), \dfrac{1}{2}I(S_r:D|S_c) + H(S_r|BS_c)\right)$

$P_5 := (H(S_c) - H(S|B), \, 0, \, 0)$

$P_6 := (I(S:B), \, 0, \, H(S_r|S_c))$

The vertices of $\Lambda(\mathcal{N}, \rho)$ are $P_0, P_1^+, P_2, P_3^+, P_4, P_5, P_6$ in the case of $-H(S_r|BS_c)_{\mathcal{N}(\rho)} > 0$ and $P_1^-, P_2, P_3^-, P_4, P_6$ when $-H(S_r|BS_c)_{\mathcal{N}(\rho)} < 0$ (Figures 7 and 8: see also Section VI in [14]). Note that, by the chain rule of the mutual information, it holds that

$$H(S_c)_\rho - H(S|B)_{\mathcal{N}(\rho)}$$
$$= I(S_c:B)_{\mathcal{N}(\rho)} - H(S_r|BS_c)_{\mathcal{N}(\rho)} \quad (370)$$
$$= I(S:B)_{\mathcal{N}(\rho)} - H(S_r|S_c)_{\mathcal{N}(\rho)}. \quad (371)$$

By a simple calculation, it is straightforward to verify that all of the above points except P_2 belong to $\Theta(\mathcal{N}, \rho)$, and consequently to $\Theta^\infty(\mathcal{N})$.

2) Proof of $P_2 \in \Theta^\infty(\mathcal{N})$: Consider the point P_2 represented by the coordinate (C_2, Q_2, E_2), where

$$C_2 = 0, \quad Q_2 = \dfrac{1}{2}I(S:B)_{\mathcal{N}(\rho)}, \quad (372)$$

$$E_2 = \dfrac{1}{2}I(S:B)_{\mathcal{N}(\rho)} + H(S_r|S_cB)_{\mathcal{N}(\rho)}. \quad (373)$$

A simple calculation yields

$$Q_2 + E_2 = H(S_r|S_c)_{\mathcal{N}(\rho)} + I(S_c:B)_{\mathcal{N}(\rho)}, \quad (374)$$
$$C_2 + Q_2 - E_2 = -H(S_r|S_cB)_{\mathcal{N}(\rho)}, \quad (375)$$
$$Q_2 - E_2 = -H(S_r|S_cB)_{\mathcal{N}(\rho)}. \quad (376)$$

Fix arbitrary $\epsilon, \delta > 0$ and choose sufficiently large n. Due to the data compression theorem for classical information source with quantum side information (Theorem 1 in [62]), there exist a countable set $\mathcal{Y}_{n,\delta}$ satisfying

$$|\mathcal{Y}_{n,\delta}| \leq 2^{n(H(S_c|B)_{\mathcal{N}(\rho)}+\delta)}, \quad (377)$$

a function $f : [J]^{\times n} \to \mathcal{Y}_{n,\delta}$ and for each $y \in \mathcal{Y}_{n,\delta}$, there exists a measurement $\{M_{\vec{j}}^y\}_{\vec{j} \in [J]^{\times n}}$ on B^n that satisfies

$$\sum_{\vec{j} \in [J]^{\times n}} p_{\vec{j}} \mathrm{Tr} \left[M_{\vec{j}}^y (\mathcal{N}^{A \to B})^{\otimes n}(\rho_{\vec{j}}^{A^n}) \right] \geq 1 - \epsilon. \quad (378)$$

We introduce a $|\mathcal{Y}_{n,\delta}|$-dimensional quantum system Y and define a state

$$\rho_n^{YS^nA^n} := \sum_{\vec{j} \in [J]^{\times n}} p_{\vec{j}} |f(\vec{j})\rangle\langle f(\vec{j})|^Y \otimes |\vec{j}\rangle\langle\vec{j}|^{S_c^n} \otimes \rho_{\vec{j}}^{S_r^n A^n}.$$

We denote the system $S_c^n S_r^n$ by $\hat{S}_r$. It is straightforward to verify that

$$\rho_n^{YS^nA^n} = (\rho^{SA})^{\otimes n}. \quad (379)$$

Using the properties of quantum entropies and (379), we have

$$H(\hat{S}_r|Y)_{\rho_n}$$
$$= H(S_c^n S_r^n|Y)_{\rho_n} \quad (380)$$
$$= H(S_c^n Y)_{\rho_n} - H(Y)_{\rho_n} + H(S_r^n|S_c^n Y)_{\rho_n} \quad (381)$$
$$= H(S_c^n)_{\rho_n} - H(Y)_{\rho_n} + H(S_r^n|S_c^n)_{\rho_n} \quad (382)$$
$$\geq H(S_c^n)_{\rho_n} - |\mathcal{Y}_{n,\delta}| + H(S_r^n|S_c^n)_{\rho_n} \quad (383)$$
$$= nH(S_c)_\rho - |\mathcal{Y}_{n,\delta}| + nH(S_r|S_c)_\rho \quad (384)$$
$$\geq nI(S_c:B)_{\mathcal{N}(\rho)} + nH(S_r|S_c)_\rho - n\delta, \quad (385)$$

where η is a function that satisfies $\lim_{\epsilon \to 0} \eta(\epsilon) = 0$ and is independent of the dimensions of the systems. From (378), (379) and Lemma 28 in Appendix B, we also have

$$H(\hat{S}_r|YB^n)_{\mathcal{N}^{\otimes n}(\rho_n)}$$
$$= H(S_c^n S_r^n|YB^n)_{\mathcal{N}^{\otimes n}(\rho_n)} \quad (386)$$
$$= H(S_c^n|YB^n)_{\mathcal{N}^{\otimes n}(\rho_n)} + H(S_r^n|S_c^n YB^n)_{\mathcal{N}^{\otimes n}(\rho_n)} \quad (387)$$
$$= H(S_c^n|YB^n)_{\mathcal{N}^{\otimes n}(\rho_n)} + H(S_r^n|S_c^n B^n)_{\mathcal{N}^{\otimes n}(\rho_n)} \quad (388)$$
$$= H(S_c^n|YB^n)_{\mathcal{N}^{\otimes n}(\rho_n)} + nH(S_r|S_cB)_{\mathcal{N}(\rho)} \quad (389)$$
$$\leq nH(S_r|S_cB)_{\mathcal{N}(\rho)} + 2n\eta(\epsilon) \log d_{S_c}. \quad (390)$$

In addition, a simple calculation using the chain rule yields

$$H(Y)_{\rho_n} - H(Y\hat{S}_r|B^n)_{\mathcal{N}^{\otimes n}(\rho_n)}$$
$$= -H(\hat{S}_r|YB^n)_{\mathcal{N}^{\otimes n}(\rho_n)} + I(Y:B^n)_{\mathcal{N}^{\otimes n}(\rho_n)} \quad (391)$$
$$\geq -H(\hat{S}_r|YB^n)_{\mathcal{N}^{\otimes n}(\rho_n)}. \quad (392)$$

Combining these relations with (374)-(376), we arrive at

$$Q_2 + E_2 \leq \dfrac{1}{n}H(\hat{S}_r|Y)_{\mathcal{N}^{\otimes n}(\rho_n)} + \delta, \quad (393)$$

$$C_2 + Q_2 - E_2 \leq \dfrac{1}{n}[H(Y)_{\rho_n} - H(Y\hat{S}_r|B^n)_{\mathcal{N}^{\otimes n}(\rho_n)}]$$
$$+ 2\eta(\epsilon) \log d_{S_c}, \quad (394)$$

$$Q_2 - E_2 \leq -\dfrac{1}{n}H(\hat{S}_r|YB^n)_{\mathcal{N}^{\otimes n}(\rho_n)}$$
$$+ 2\eta(\epsilon) \log d_{S_c}. \quad (395)$$

Since these relations hold for any small $\epsilon, \delta > 0$ and sufficiently large n, we obtain $P_2 \in \Theta^\infty(\mathcal{N})$. ∎

3) Proof of $(C, Q, E + \Delta E) \in \Theta^\infty(\mathcal{N})$ for Any $\Delta E > 0$ and $(C, Q, E) \in \Theta^\infty(\mathcal{N})$: We complete the proof of $\Theta^\infty(\mathcal{N}) \supseteq \Lambda(\mathcal{N}, \rho)$ by showing that, if $(C, Q, E) \in \Theta^\infty(\mathcal{N})$, then $(C, Q, E + \Delta E) \in \Theta^\infty(\mathcal{N})$ for any $\Delta E > 0$. It suffices to prove that for any $n \in \mathbb{N}$ and $\Delta E > 0$, it holds that $(C, Q, E + \Delta E) \in \Theta^\infty(\mathcal{N})$ if $(C, Q, E) \in \dfrac{1}{n}\Theta(\mathcal{N}^{\otimes n})$. We only consider the case where $n = 1$. It is straightforward to generalize the proof for $n \geq 2$.

Consider a triplet $(C, Q, E) \in \Theta(\mathcal{N})$, and fix arbitrary $\Delta E > 0$ and $m \in \mathbb{N}$. By definition, there exist finite dimensional quantum systems S_c, S_r and a state in the form of

$$\rho^{SA} = \sum_{j=1}^J p_j |j\rangle\langle j|^{S_c} \otimes \rho_j^{S_r A}, \quad (396)$$

such that

$$Q + E \leq H(S_r|S_c)_\rho, \quad (397)$$

$$C + Q - E \leq H(S_c)_\rho - H(S|B)_{\mathcal{N}(\rho)}, \tag{398}$$

$$Q - E \leq -H(S_r|BS_c)_{\mathcal{N}(\rho)}. \tag{399}$$

Define $\Delta E_m := \lfloor m \Delta E \rfloor$, and let S_r' be a quantum system with dimension $2^{\Delta E_m}$. Consider a state

$$\rho_m^{S_c^m S_r' A^m} = (\rho^{SA})^{\otimes m} \otimes \pi^{S_r'}, \tag{400}$$

where π is the full-rank maximally mixed state on S_r'. Relabelling S_c^m by $\hat{S}_c$, $S_r^m S_r'$ by $\hat{S}_r$ and $\hat{S}_c \hat{S}_r$ by $\hat{S}$, the above state is represented as

$$\rho_m^{\hat{S} A^m} := \sum_{\vec{j} \in [J]^{\times m}} p_{\vec{j}} |\vec{j}\rangle\langle\vec{j}|^{\hat{S}_c} \otimes \hat{\rho}_{\vec{j}}^{\hat{S}_r A^m}, \tag{401}$$

where

$$\hat{\rho}_{\vec{j}}^{\hat{S}_r A^m} := \rho_{\vec{j}}^{S_r^m A^m} \otimes \pi^{S_r'} \tag{402}$$

and

$$p_{\vec{j}} := p_{j_1} \cdots p_{j_m}, \quad \rho_{\vec{j}} := \rho_{j_1} \otimes \cdots \otimes \rho_{j_m} \tag{403}$$

for $\vec{j} = j_1 \cdots j_m$. Noting that

$$H(\hat{S}_r|\hat{S}_c)_{\rho_m} = m H(S_r|S_c)_\rho + \Delta E_m \tag{404}$$

and so forth, it follows from (397)-(399) that

$$m(Q + E) + \Delta E_m \leq H(\hat{S}_r|\hat{S}_c)_{\rho_m}, \tag{405}$$

$$m(C + Q - E) - \Delta E_m \leq H(\hat{S}_c)_\rho \\ - H(\hat{S}|B^m)_{\mathcal{N}^{\otimes m}(\rho_m)}, \tag{406}$$

$$m(Q - E) - \Delta E_m \leq -H(\hat{S}_r|B^m \hat{S}_c)_{\mathcal{N}^{\otimes m}(\rho_m)}. \tag{407}$$

This implies $(C, Q, E + \Delta E_m/m) \in \Theta^\infty(\mathcal{N})$. Noting that $\lim_{m \to \infty} (\Delta E_m/m) = \Delta E$, this implies $(C, Q, E + \Delta E) \in \Theta^\infty(\mathcal{N})$ and completes the proof. ■

C. Proof of $\overline{\Lambda^\infty(\mathcal{N})} = \overline{\Lambda_p^\infty(\mathcal{N})}$

It is straightforward to verify that $\overline{\Lambda^\infty(\mathcal{N})} \supseteq \overline{\Lambda_p^\infty(\mathcal{N})}$. Thus, we prove $\overline{\Lambda^\infty(\mathcal{N})} \subseteq \overline{\Lambda_p^\infty(\mathcal{N})}$ by showing that $\Lambda(\mathcal{N}^{\otimes n}) \subseteq \Lambda_p(\mathcal{N}^{\otimes n})$ for any n. We only consider the case $n = 1$. It is straightforward to generalize the proof for $n \geq 2$.

Fix an arbitrary state ρ in the form of (163), and suppose that $(C, Q, E) \in \Lambda(\mathcal{N}, \rho)$. For each j, let $\{q_{k|j}, |\phi_{j,k}\rangle\}_k$ be an ensemble of pure states on $S_r A$ such that $\rho_j = \sum_k q_{k|j} |\phi_{j,k}\rangle\langle\phi_{j,k}|$. We denote $p_j q_{k|j}$ by p_{jk}. Let Y be a finite dimensional quantum system with a fixed orthonormal basis $\{|k\rangle\}_k$, and define a state $\tilde{\rho}^{YSA}$ by

$$\tilde{\rho}^{YSA} := \sum_{j=1}^J \sum_k p_{jk} |k\rangle\langle k|^Y \otimes |j\rangle\langle j|^{S_c} \otimes |\phi_{j,k}\rangle\langle\phi_{j,k}|^{S_r A}.$$

It is straightforward to verify that $\tilde{\rho}^{SA} = \rho^{SA}$. Denoting YS_c by $\hat{S}_c$ and $\hat{S}_c S_r$ by $\hat{S}$, the data processing inequality yields

$$I(S : B)_{\mathcal{N}(\rho)} \leq I(\hat{S} : B)_{\mathcal{N}(\tilde{\rho})}, \tag{408}$$

$$-H(S_r|BS_c)_{\mathcal{N}(\rho)} \leq -H(S_r|B\hat{S}_c)_{\mathcal{N}(\tilde{\rho})}, \tag{409}$$

in addition to

$$H(S_c)_\rho - H(S|B)_{\mathcal{N}(\rho)}$$

$$= I(S_c : B)_{\mathcal{N}(\rho)} - H(S_r|BS_c)_{\mathcal{N}(\rho)} \tag{410}$$

$$\leq I(\hat{S}_c : B)_{\mathcal{N}(\tilde{\rho})} - H(S_r|B\hat{S}_c)_{\mathcal{N}(\tilde{\rho})} \tag{411}$$

$$= H(\hat{S}_c)_{\tilde{\rho}} - H(\hat{S}|B)_{\mathcal{N}(\tilde{\rho})}. \tag{412}$$

Combining these inequalities with (164)-(166), we have $(C, Q, E) \in \Lambda(\mathcal{N}, \tilde{\rho})$, which implies $\Lambda(\mathcal{N}, \rho) \subseteq \Lambda(\mathcal{N}, \tilde{\rho})$. By taking the union over all ρ, we arrive at $\Lambda(\mathcal{N}) \subseteq \Lambda_p(\mathcal{N})$ and complete the proof. ■

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Eyuri Wakakuwa received the Ph.D. degree from The University of Tokyo in 2015. He was a Post-Doctoral Fellow at The University of Electro-Communications, Japan, in 2017, an Assistant Professor at The University of Tokyo, Japan, from then to 2018, and a JSPS Post-Doctoral Research Fellow at The University of Electro-Communications from 2018 to 2021. He is currently an Assistant Professor at The University of Tokyo. His research interests include quantum information theory and the foundation of quantum mechanics.

Yoshifumi Nakata received the M.S. and Ph.D. degrees in physics from The University of Tokyo, Tokyo, Japan, in 2008 and 2013, respectively. He worked as a Post-Doctoral Fellowship for Research Abroad with the Japan Society of the Promotion (JSPS), Leibniz University of Hannover, from 2013 to 2015, a Research Fellow with JSPS, The University of Tokyo, from 2015 to 2017, a Post-Doctoral Fellow at The University of Tokyo from 2017 to 2018, a Research Assistant Professor at Kyoto University from 2019 to 2020, and an Assistant Professor at The University of Tokyo from 2020 to 2022. He was appointed as a Research Associate Professor at the Yukawa Institute for Theoretical Physics, Kyoto University, in 2022. His research interests cover a range of theoretical topics in quantum information and physics. He currently focuses on investigating many-body quantum physics by using the methods developed in quantum information theory.