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LETTER

A generalization of the four qubits single insertion error-correcting code

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Abstract The first quantum insertion code is the four qubits code given by Hagiwara in 2021. Since then, several quantum insertion codes have been proposed, but they are all unrelated to Hagiwara's four qubits code. This article provides a new class of quantum single insertion codes that include the Hagiwara code. Furthermore, this construction gives a new decoder for insertion errors in the Hagiwara codes.

Keywords: quantum error-correction, insertion/deletion code, single insertion error

Classification: Fundamental theories for communications

1. Introduction

Insertion/deletion error-correcting codes in classical theory were first given by Levenshtein in 1966 [1], while those in quantum theory have a relatively short history. In fact, the first quantum error-correcting code was proposed by Shor in 1995 [2], which could correct errors represented by unitary matrices.

More than 20 years later, the first quantum deletion error-correcting code was proposed by Nakayama in 2020 [3]. Several examples of quantum deletion codes have been given so far since then [4, 5]. The first quantum insertion error-correcting code is the four qubits code given by Hagiwara in 2021 [6]. This code was already known as a single deletion error-correcting code [4], and its construction is simple and basic. However, the decoding process for insertion errors in the Hagiwara code is nontrivial and technical, and was expected to be difficult to generalize. Recently, a class of quantum single insertion codes was given [7], but the Hagiwara code is not included in that class and is still a unique example.

This article gives a class of quantum single insertion codes that are not yet known and presents their decoders. Our class includes the Hagiwara code, and we have successfully constructed a new decoder in Hagiwara's four qubits insertion code.

2. Quantum single insertion

Fix n as a positive integer and $[n] := \{1, 2, \dots, n\}$. Suppose that there are n particles $p_1, p_2, \dots, p_n$ whose quantum states are level 2. These n particles can be represented by a 2^n-by-2^n density matrix. The set of all density matrices of order 2^n is denoted by $S(\mathbb{C}^{2\otimes n})$. We also use a complex vector $|\psi\rangle \in \mathbb{C}^{2\otimes n}$ for representing a pure state $|\psi\rangle\langle\psi| \in S(\mathbb{C}^{2\otimes n})$. The mapping of n particles $p_1, p_2, \dots, p_n$ to $n+1$ particles $p_1, p_2, \dots, p_{i-1}, q, p_i, \dots, p_n$ with a single particle q is called a quantum single insertion. Note that the position of the particle q is unknown. According to a recent report [7], if the n quantum states before insertion are pure states, the single insertion error is expressed as the map $I^n_{i,\sigma} : S(\mathbb{C}^{2\otimes n}) \to S(\mathbb{C}^{2\otimes (n+1)})$ defined as follows:

Suppose that the quantum state before insertion is expressed as a density matrix $\rho = \sum_{\boldsymbol{x},\boldsymbol{y} \in \{0,1\}^n} m_{\boldsymbol{x},\boldsymbol{y}} |\boldsymbol{x}\rangle\langle\boldsymbol{y}| \in S(\mathbb{C}^{2\otimes n})$ with $m_{\boldsymbol{x},\boldsymbol{y}} \in \mathbb{C}$. For an integer $i \in [n+1]$ and a quantum state $\sigma \in S(\mathbb{C}^2)$, define the map $I^n_{i,\sigma} : S(\mathbb{C}^{2\otimes n}) \to S(\mathbb{C}^{2\otimes (n+1)})$ as

$$I^n_{i,\sigma}(\rho) := \sum_{\boldsymbol{x},\boldsymbol{y} \in \{0,1\}^n} m_{\boldsymbol{x},\boldsymbol{y}} |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_{i-1}\rangle\langle y_{i-1}| \otimes \sigma \otimes |x_i\rangle\langle y_i| \otimes \cdots \otimes |x_n\rangle\langle y_n|.$$

Here we denote $\boldsymbol{x} = x_1 x_2 \dots x_n$ and $\boldsymbol{y} = y_1 y_2 \dots y_n$. We call the map $I^n_{i,\sigma}$ a single insertion for quantum state σ with the insertion position i. This article proposes a new class of quantum codes that can correct any single insertion.

3. Code construction

In this article, we suppose that mutually disjoint non-empty sets $A, B \subset \{0, 1, \dots, n\}$ satisfy the following two conditions:

1. $w \in W \implies n - w \in W$, for any $W \in \{A, B\}$.
2. $|w_1 - w_2| \leq 1 \implies w_1 = w_2$, for any $w_1, w_2 \in A \cup B$.

The encoder $\mathrm{Enc}_{A,B} : S(\mathbb{C}^2) \to S(\mathbb{C}^{2\otimes n})$ is defined as

$$\mathrm{Enc}_{A,B}(\alpha|0\rangle + \beta|1\rangle) = \alpha|\tilde{D}_A\rangle + \beta|\tilde{D}_B\rangle, \qquad (1)$$

with $\alpha, \beta \in \mathbb{C}$. Here, $\{|0\rangle, |1\rangle\} \subset \mathbb{C}^2$ is an orthonormal basis of a two-dimensional Hilbert space with inner product $\langle , \rangle$, and

$$|\tilde{D}_W\rangle := \frac{1}{\sqrt{\sum_{w \in W} \binom{n}{w}}} \sum_{w \in W} \sum_{\substack{\boldsymbol{x} \in \{0,1\}^n \\ \mathrm{wt}(\boldsymbol{x}) = w}} |\boldsymbol{x}\rangle$$

for $W \in \{A, B\}$. Here $\mathrm{wt}(\boldsymbol{x})$ is Hamming weight of the classical bit sequence $\boldsymbol{x} \in \{0,1\}^n$. Note that the state $|\tilde{D}_W\rangle$ is not just a Dicke state. Set $Q_{A,B}$ as the image of the encoder $\mathrm{Enc}_{A,B}$. Note that the code $Q_{A,B}$, defined above, is already known as a single deletion error-correcting code [5]. This article claims that the code $Q_{A,B}$ is also single insertion error-correctable.

4. Error-correction

This section describes the decoding process of the codes

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$Q_{A,B}$ for single insertions. The proofs of Lemmas given in this section are shown in Appendix at the end of this article.

4.1 Measurement

In general, if we perform a measurement described by measurement operators $\mathcal{M} = \{M_m\}$ that satisfies the completeness relation $\sum_m M_m^\dagger M_m = I$ under a state ρ, the probability to get an outcome m is $p(m) = \mathrm{Tr}(M_m^\dagger M_m \rho)$ and the after measurement state is $M_m \rho M_m^\dagger / p(m)$. Here, I denotes an identity matrix and $\mathrm{Tr}(M)$ denotes the sum of the diagonal elements of a square matrix M. In the following, we define the measurement operator $\mathcal{M}$ used in the decoder of the code $Q_{A,B}$ for single insertions.

First, vectors needed for the construction of the measurement operators are defined. For $W \in \{A, B\}$, $i \in [n+1]$ and $b \in \{0,1\}$, define a vector $|v_{i,b}^W\rangle \in \mathbb{C}^{2\otimes(n+1)}$ as

$$|v_{i,b}^W\rangle := \frac{1}{\sqrt{\sum_{w\in W}\binom{n}{w}}} \sum_{w\in W} \sum_{\substack{\boldsymbol{x}\in\{0,1\}^{n+1}\\ \mathrm{wt}(\boldsymbol{x})=w+b\\ x_i=b}} |\boldsymbol{x}\rangle.$$

Lemma 1 *For sets $W, W' \in \{A, B\}$, integers $i, i' \in [n+1]$ and bits $b, b' \in \{0,1\}$,*

$$\langle v_{i,b}^W | v_{i',b'}^{W'}\rangle = \begin{cases} 1 & if\ W = W', b = b' and\ i = i', \\ 1/2 & if\ W = W', b = b' and\ i \neq i', \\ 0 & otherwise. \end{cases} \tag{2}$$

Next, using $|v_{i,b}^W\rangle$, define unit vectors $|u_{i,b}^W\rangle$ that are orthogonal to each other. For $W \in \{A, B\}$, $i \in [n+1]$ and $b \in \{0,1\}$, define a vector $|u_{i,b}^W\rangle \in \mathbb{C}^{2\otimes(n+1)}$ as

$$|u_{i,b}^W\rangle := \sqrt{\frac{2i}{i+1}}\left(|v_{i,b}^W\rangle - \frac{1}{i}\sum_{j\in[i-1]}|v_{j,b}^W\rangle\right). \tag{3}$$

Note that $|u_{1,b}^W\rangle = |v_{1,b}^W\rangle$ in Eq. (3).

Lemma 2 *For sets $W, W' \in \{A, B\}$, integers $i, i' \in [n+1]$ and bits $b, b' \in \{0,1\}$,*

$$\langle u_{i,b}^W | u_{i',b'}^{W'}\rangle = \begin{cases} 1 & if\ W = W', b = b' and\ i = i', \\ 0 & otherwise. \end{cases}$$

Define $M_{i,b} := |u_{i,b}^A\rangle\langle u_{i,b}^A| + |u_{i,b}^B\rangle\langle u_{i,b}^B|$ for $i \in [n+1]$ and $b \in \{0,1\}$. For the space $V \subset \mathbb{C}^{2\otimes(n+1)}$ with the basis $\{|u_{i,b}^W\rangle \mid W \in \{A,B\}, i \in [n+1], b \in \{0,1\}\}$, choose a basis $\{|e_i\rangle\}$ of its orthogonal complementary space $V^\perp$ and define $M_\emptyset := \sum_i |e_i\rangle\langle e_i|$. Then $\mathcal{M} := \{M_\emptyset\} \cup \{M_{i,b} \mid i \in [n+1], b \in \{0,1\}\}$ is a set of measurement operators because the completeness relation

$$M_\emptyset^\dagger M_\emptyset + \sum_{\substack{i\in[n+1]\\ b\in\{0,1\}}} M_{i,b}^\dagger M_{i,b} = I$$

is satisfied. Here, I is the identity matrix of size 2^{n+1}.

4.2 Recovery operator

For any $i \in [n+1]$ and any $b \in \{0,1\}$, we can choose a size 2^{n+1} unitary matrix $U_{i,b}$ such that $U_{i,b}|u_{i,b}^A\rangle = |0\ldots00\rangle$ and $U_{i,b}|u_{i,b}^B\rangle = |0\ldots01\rangle$. We call the unitary matrix $U_{i,b}$ a recovery operator.

4.3 Error-correction process

Let $\sigma = \sum_{b_0,b_1\in\{0,1\}} c_{b_0b_1}|b_0\rangle\langle b_1| \in S(\mathbb{C}^2)$ denote the state of the particle to be inserted, where $c_{00} + c_{11} = 1$. For the state expressed in Eq. (1), the state after a single insertion $I_{i,\sigma}^n$ is

$$\begin{aligned} &I_{i,\sigma}^n(\alpha|\tilde{D}_A\rangle + \beta|\tilde{D}_B\rangle) \\ &= \sum_{b_0,b_1\in\{0,1\}} c_{b_0b_1}(\alpha|v_{i,b_0}^A\rangle + \beta|v_{i,b_0}^B\rangle)(\bar{\alpha}\langle v_{i,b_1}^A| + \bar{\beta}\langle v_{i,b_1}^B|). \end{aligned}$$

For this state, the probability to get the outcome (j,b) when performing the measurement $\mathcal{M}$ is $c_{bb}|\langle u_{j,b}|v_{i,b}\rangle|^2$ since

$$\begin{aligned} &p(j,b) \\ &= \mathrm{Tr}\left(M_{j,b}^\dagger M_{j,b} I_{i,\sigma}^n(\alpha|\tilde{D}_A\rangle + \beta|\tilde{D}_B\rangle)\right) \\ &= \sum_k \langle k|M_{j,b}^\dagger M_{j,b} \sum_{b_0,b_1\in\{0,1\}} c_{b_0b_1}(\alpha|v_{i,b_0}^A\rangle + \beta|v_{i,b_0}^B\rangle) \\ &\quad (\bar{\alpha}\langle v_{i,b_1}^A| + \bar{\beta}\langle v_{i,b_1}^B|)|k\rangle \\ &= \sum_{b_0,b_1\in\{0,1\}} c_{b_0b_1} \sum_k (\bar{\alpha}\langle v_{i,b_1}^A| + \bar{\beta}\langle v_{i,b_1}^B|)|k\rangle\langle k|M_{j,b}^\dagger \\ &\quad M_{j,b}(\alpha|v_{i,b_0}^A\rangle + \beta|v_{i,b_0}^B\rangle) \\ &= \sum_{b_0,b_1\in\{0,1\}} c_{b_0b_1}(\bar{\alpha}\langle v_{i,b_1}^A| + \bar{\beta}\langle v_{i,b_1}^B|)M_{j,b}^\dagger \\ &\quad M_{j,b}(\alpha|v_{i,b_0}^A\rangle + \beta|v_{i,b_0}^B\rangle) \\ &= c_{bb}(\bar{\alpha}\langle v_{i,b}^A|u_{j,b}^A\rangle\langle u_{j,b}^A| + \bar{\beta}\langle v_{i,b}^B|u_{j,b}^B\rangle\langle u_{j,b}^B|) \\ &\quad (\alpha|u_{j,b}^A\rangle\langle u_{j,b}^A|v_{i,b}^A\rangle + \beta|u_{j,b}^B\rangle\langle u_{j,b}^B|v_{i,b}^B\rangle) \\ &= c_{bb}|\langle u_{j,b}|v_{i,b}\rangle|^2(\bar{\alpha}\langle u_{j,b}^A| + \bar{\beta}\langle u_{j,b}^B|)(\alpha|u_{j,b}^A\rangle + \beta|u_{j,b}^B\rangle) \\ &= c_{bb}|\langle u_{j,b}|v_{i,b}\rangle|^2, \end{aligned}$$

and the after measurement state is $\alpha|u_{j,b}^A\rangle + \beta|u_{j,b}^B\rangle \in S(\mathbb{C}^{2\otimes(n+1)})$ since

$$\begin{aligned} &M_{j,b} I_{i,\sigma}^n(\alpha|\tilde{D}_A\rangle + \beta|\tilde{D}_B\rangle)M_{j,b}^\dagger \\ &= \sum_{b_0,b_1\in\{0,1\}} c_{b_0b_1} M_{j,b}(\alpha|v_{i,b_0}^A\rangle + \beta|v_{i,b_0}^B\rangle) \\ &\quad (\bar{\alpha}\langle v_{i,b_1}^A| + \bar{\beta}\langle v_{i,b_1}^B|)M_{j,b}^\dagger \\ &= c_{bb}(\alpha|u_{j,b}^A\rangle\langle u_{j,b}^A|v_{i,b}^A\rangle + \beta|u_{j,b}^B\rangle\langle u_{j,b}^B|v_{i,b}^B\rangle) \\ &\quad (\bar{\alpha}\langle v_{i,b}^A|u_{j,b}^A\rangle\langle u_{j,b}^A| + \bar{\beta}\langle v_{i,b}^B|u_{j,b}^B\rangle\langle u_{j,b}^B|) \\ &= c_{bb}|\langle u_{j,b}|v_{i,b}\rangle|^2(\alpha|u_{j,b}^A\rangle + \beta|u_{j,b}^B\rangle)(\bar{\alpha}\langle u_{j,b}^A| + \bar{\beta}\langle u_{j,b}^B|) \\ &= p(j,b)(\alpha|u_{j,b}^A\rangle + \beta|u_{j,b}^B\rangle)(\bar{\alpha}\langle u_{j,b}^A| + \bar{\beta}\langle u_{j,b}^B|). \end{aligned}$$

From Eq. (2) and Eq. (3), we get $\langle u_{j,b}^A|v_{i,b}^A\rangle = \langle u_{j,b}^B|v_{i,b}^B\rangle$, and we denoted this value as $\langle u_{j,b}|v_{i,b}\rangle$ in the above. On the other hand, from the definition of $M_\emptyset$, the probability of obtaining outcome $\emptyset$ is 0.

After obtaining the outcome (j,b), we apply the recovery operator $U_{j,b}$, which gives

$$U_{j,b}(\alpha|u_{j,b}^A\rangle + \beta|u_{j,b}^B\rangle) = \alpha|0\ldots00\rangle + \beta|0\ldots01\rangle.$$

Finally, the original state $\alpha|0\rangle + \beta|1\rangle \in S(\mathbb{C}^2)$ can be obtained by deleting the 1st through the nth particles and error-correction is completed. Note that this error-correction process does not depend on the insertion position i and the quantum state $\sigma \in S(\mathbb{C}^2)$, and error-correction is possible no matter what state is inserted at any position.

5. Examples

5.1 Decoder for the 4 qubit insertion code

The sets A and B such that n is the smallest are $A = \{0, 4\}$ and $B = \{2\}$. In this case

$$|\tilde{D}_A\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$
$$|\tilde{D}_B\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle),$$

Table I Values of $|\langle u_{j,b}|v_{i,b}\rangle|^2$ for $i, j \in [5]$ and $b \in \{0, 1\}$

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$(j, b) = (1, b)$	2/2	1/4	1/4	1/4	1/4
$(j, b) = (2, b)$	0	3/4	1/12	1/12	1/12
$(j, b) = (3, b)$	0	0	4/6	1/24	1/24
$(j, b) = (4, b)$	0	0	0	5/8	1/40
$(j, b) = (5, b)$	0	0	0	0	6/10

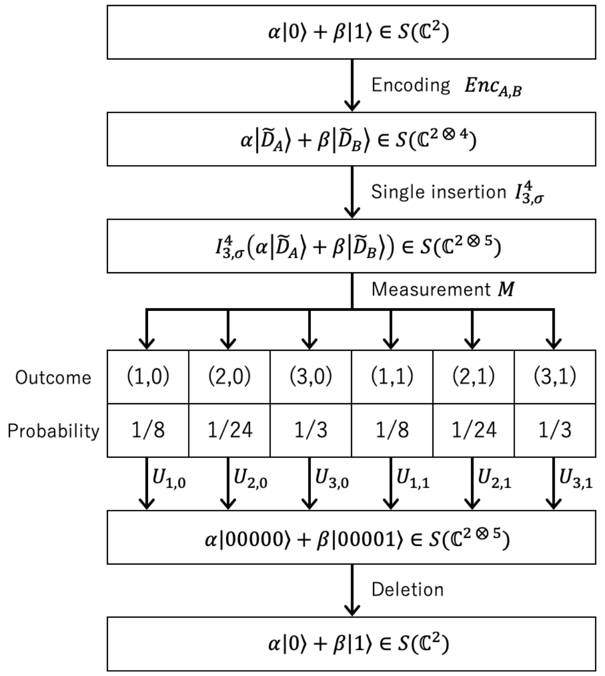

Fig. 1 Error-correction process for the 4-qubit insertion code when $\sigma = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \in S(\mathbb{C}^2)$ is inserted in the 3rd position

Table II All codes with length 8 or less

Length	(A, B)		
4	$(\{0, 4\}, \{2\})$.		
5			
6	$(\{0, 6\}, \{3\})$,	$(\{0, 6\}, \{2, 4\})$.	
7	$(\{0, 7\}, \{2, 5\})$.		
8	$(\{0, 8\}, \{4\})$,	$(\{0, 8\}, \{3, 5\})$,	$(\{0, 8\}, \{2, 6\})$,
	$(\{1, 7\}, \{4\})$,	$(\{1, 7\}, \{3, 5\})$,	$(\{2, 6\}, \{4\})$,
	$(\{0, 4, 8\}, \{2, 6\})$,	$(\{0, 2, 6, 8\}, \{4\})$.	

where the code $Q_{A,B}$ is the 4 qubit insertion code given by Hagiwara [6]. However, the decoder given in this article is different from the one first given by Hagiwara.

Table I shows the values of $|\langle u_{j,b}|v_{i,b}\rangle|^2$ for integers $i, j \in [5]$ and a bit $b \in \{0, 1\}$. For the state after encoding, if $\sigma = \sum_{b_0, b_1 \in \{0,1\}} |b_0\rangle\langle b_1|$ is inserted at the ith position, the probability of obtaining the outcome (j, b) by applying the measurement $\mathcal{M}$ is $c_{bb}|\langle u_{j,b}|v_{i,b}\rangle|^2$. Therefore, it is possible to calculate that probability using Table I.

For example, if $\sigma = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \in S(\mathbb{C}^2)$ is inserted in the 3rd position, the probabilities of obtaining outcomes $(1, b)$, $(2, b)$, $(3, b)$, $(4, b)$, $(5, b)$ are $1/8$, $1/24$, $1/3$, 0, 0, respectively. The decoding process after obtaining each outcome is shown in Fig. 1.

5.2 All codes with length 8 or less in our construction

The list of all codes with length 8 or less is shown in Table II. In particular, there are no codes with length 5 in our construction. Of course, codes with lengths greater than 8 can be constructed in the same way.

6. Conclusion

This article gave the first class of quantum single insertion error-correcting codes which includes Hagiwara's four qubits insertion code. From that construction, a new decoder for the Hagiwara code was given.

Already known decoders for quantum deletion codes and quantum insertion codes often use projective measurements. In this study, we presented a decoder using a generalized measurement described by measurement operators. Of course, keeping in mind the axioms of quantum mechanics, generalized measurements can be described using projective measurements. However, for the decoder presented in this paper, generalized measurements are simpler to describe. Based on this idea, we would like to construct new codes and clarify the relationship between insertion codes and deletion codes.

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Appendix A. Proofs of lemmas

A.1 Proof of Lem. 1

If $W \neq W'$ or $b \neq b'$, then $\langle v^W_{i,b} | v^{W'}_{i',b'} \rangle = 0$ from the 2nd condition about sets A and B. If $W = W'$, $b = b'$ and $i = i'$, then $\langle v^W_{i,b} | v^{W'}_{i',b'} \rangle = 1$ by direct calculation.

In the case of $W = W'$, $b = b'$ and $i \neq i'$, we have

$$\langle v^W_{i,b} | v^{W'}_{i',b'} \rangle = \begin{cases} \dfrac{1}{\sum_{w\in W} \binom{n}{w}} \displaystyle\sum_{w\in W} \binom{n-1}{w} & if\ b = 0, \\ \dfrac{1}{\sum_{w\in W} \binom{n}{w}} \displaystyle\sum_{w\in W} \binom{n-1}{w-1} & if\ b = 1. \end{cases} \tag{A.1}$$

Note that $\binom{n-1}{w} := 0$ for $n = w$ and $\binom{n-1}{w-1} := 0$ for $w = 0$. We have $\sum_{w\in W} \binom{n-1}{w} = \sum_{w\in W} \binom{n-1}{w-1}$ from the 1st condition about sets A and B. In addition, since $\binom{n}{w} = \binom{n-1}{w} + \binom{n-1}{w-1}$, we obtain

$$\sum_{w\in W} \binom{n-1}{w} = \sum_{w\in W} \binom{n-1}{w-1} = \frac{1}{2} \sum_{w\in W} \binom{n}{w}.$$

Therefore, from Eq. (A.1), $\langle v^W_{i,b} | v^{W'}_{i',b'} \rangle = 1/2$ is shown. □

A.2 Proof of Lem. 2

If $W \neq W'$ or $b \neq b'$, then $\langle u^W_{i,b} | u^{W'}_{i',b'} \rangle = 0$ by Lem. 1. If $W = W'$, $b = b'$ and $i = i'$, then we have

$$\begin{aligned} \langle u^W_{i,b} | u^W_{i,b} \rangle =& \frac{2i}{i+1} \Bigg\{ \langle v^W_{i,b} | v^W_{i,b} \rangle - \frac{1}{i} \sum_{j\in[i-1]} \langle v^W_{i,b} | v^W_{j,b} \rangle \\ & - \frac{1}{i} \sum_{j\in[i-1]} \langle v^W_{j,b} | v^W_{i,b} \rangle + \frac{1}{i^2} \sum_{j,k\in[i-1]} \langle v^W_{j,b} | v^W_{k,b} \rangle \Bigg\} \\ =& 1 \end{aligned}$$

by Lem. 1.

In the case of $W = W'$, $b = b'$ and $i \neq i'$, we prove by induction on $i \in [n+1]$. Fix $k \in [n]$ and assume that $\langle u^W_{i,b} | u^W_{i',b} \rangle = 0$ holds for different positive integers $i, i' \in [k]$. Then, it is sufficient to prove $\langle u^W_{k+1,b} | u^W_{i,b} \rangle = 0$ for any $i \in [k]$.

It can be checked by direct calculation that

$$|u^W_{k+1,b}\rangle = \sqrt{\frac{2(k+1)}{k+2}} (|v^W_{k+1,b}\rangle - |v^W_{k,b}\rangle) + \sqrt{\frac{k}{k+2}} |u^W_{k,b}\rangle$$

holds. Therefore,

$$\begin{aligned} & \langle u^W_{k+1,b} | u^W_{i,b} \rangle \\ =& \sqrt{\frac{2(k+1)}{k+2} \cdot \frac{2i}{i+1}} \left(\langle v^W_{k+1,b} | - \langle v^W_{k,b} | \right) \\ & \left(|v^W_{i,b}\rangle - \frac{1}{i} \sum_{j\in[i-1]} |v^W_{j,b}\rangle \right) + \sqrt{\frac{k}{k+2}} \langle u^W_{k,b} | u^W_{i,b} \rangle \\ =& \begin{cases} \sqrt{\frac{2(k+1)}{k+2} \cdot \frac{2i}{i+1}} \cdot 0 + \sqrt{\frac{k}{k+2}} \cdot 0 & if\ 1 \le i < k \\ \sqrt{\frac{2(k+1)}{k+2} \cdot \frac{2k}{k+1}} \cdot \left(-\frac{1}{2}\right) + \sqrt{\frac{k}{k+2}} \cdot 1 & if\ i = k \end{cases} \\ =& 0 \end{aligned}$$

by Lem. 1. From the above, it is proved that $\langle u^W_{i,b} | u^W_{i',b} \rangle = 0$ for all different positive integers $i, i' \in [n+1]$. □