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Geometric Relations Between Rigid Bodies (Part 1)

Semantics for Standardization

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This tutorial explicitly states the semantics of all coordinate-invariant properties and operations, and, more importantly, all the choices that are made in coordinate representations of these geometric relations. This results in a set of concrete suggestions for standardizing terminology and notation, allowing programmers to write fully unambiguous software interfaces, including automatic checks for semantic correctness of all geometric operations on rigid-body coordinate representations. A concrete proposal for community-driven standardization via the Robot Engineering Task Force [4] is accepted as a Robotics Request for Comment.

A main characteristic of robotics is that it involves three-dimensional motion of rigid bodies (manipulated objects, robot links, or mobile bases). Rigid bodies are essential primitives in the modeling of robotic devices, tasks, and perception. Basic geometric relations between rigid bodies include relative position, orientation, pose, linear velocity, angular velocity, and twist. Hence, robot programmers and application developers have to deal with time-dependent geometric relations between rigid bodies. Therefore, there is a strong need for a systematic terminology for these geometric relations. In this regard, [11] and [12] enhance the terminology in the area of mechanism and machine science, and contain definitions that are of interest to define geometric relations between rigid bodies (such as rigid body, relative motion, displacement, velocity, and angular velocity). To express geometric relations and perform mathematical operations on them (e.g., composition of relative motion, time differentiation, or integration), robot programmers have to choose coordinate representations with which to perform the corresponding numerical operations. Despite their being used for about 50 years already in robotics, the geometric properties of rigid-body operations, and their coordinate representations, have never been standardized, which has led to a proliferation of

Digital Object Identifier 10.1109/MRA.2012.2205652
Date of publication: 8 March 2013

1070-9932/13/$31.00©2013IEEE

mutually incompatible software libraries, in the robot control products of commercial manufacturers as well as in open-source libraries such as Kinematics and Dynamics Library (KDL) [10], Robot Operating System (ROS) [14], Robotics Library (RL) [1], and so forth. All geometric relations and their coordinate representations entail a number of choices or assumptions, which are often made implicitly, but which are necessary to consider in view of 1) understanding the physical meaning of the numerical values that constitute the coordinate representation of a geometric relation and 2) performing physically meaningful mathematical operations on these numerical values. None of the above libraries provide support for explicitly stating the above assumptions.

For example, in the calculation of the relative twist [3] (linear and angular velocities) between two robot end-effectors out of the end-effector twists of both robots with respect to the world, we have to consider the choices made in the coordinate representations of both end-effector twists (see "How to Use Semantics for Geometric Relations Between Rigid Bodies in Robotics"). Similarly, when time-integrating the twist of a robot end-effector to obtain the end-effector pose as a function of time, we have to consider the choices made in the coordinate representation of the twist (see "How to Use Semantics for Integrating Velocities Between Rigid Bodies in Robotics"). Failing to make these choices explicit and to document them often leads to errors that may be difficult to trace, especially during integration of code from different origins. "Common Errors in Geometric Rigid-Body Relations Calculations in Robotics" lists some common errors that could be prevented by revealing the semantics underlying the geometric rigid-body relations.

In this article, we describe the full semantics underlying the rigid-body geometric relations of position, orientation, pose, linear velocity, angular velocity, and twist, including all the choices to be made when specifying these geometric relations. This clear definition of the semantics serves as a proposal for standardization, forcing researchers and application developers to reveal all the hidden assumptions in their geometric rigid-body relations. In particular, it also supports the development of software for geometric operations that includes semantic checks. This will avoid common errors and hence reduce application (and, especially, system integration) development time considerably. We make concrete suggestions for semantic interfaces for geometric operation software libraries. We are not aware of prior work revealing the semantics underlying the rigid-body geometric relations and of software offering semantic-based checks for calculations involving geometric rigid-body relations.

Background

A rigid body is an idealization of a solid body of infinite or finite size in which deformation is neglected. This document often abbreviates rigid body to body and denotes it by a curly letter such as $\mathcal{A}$. A body in three-dimensional space has six degrees of freedom: three degrees of freedom in translation and three in rotation. The subspace of all body displacements that involve only changes in the orientation is often denoted by SO(3) (the special orthogonal group in three-dimensional space). It forms a group under the operation of composition of relative displacements. The space of all body displacements, including translations, is denoted by SE(3) (the special Euclidean group in three-dimensional space). A general six-dimensional displacement between two bodies is called a (relative) pose: it contains both the position and orientation. Remark that the position, orientation, and pose of a body are not absolute concepts, since they imply a second body with respect to which they are defined. Hence, only the relative position, orientation, and pose between two bodies are relevant geometric relations.

A general six-dimensional velocity between two bodies is called a (relative) twist: it contains both the angular and the linear velocity. Similar to the position, orientation, and pose, the linear velocity, angular velocity, and twist of a body are not absolute concepts, since they imply a second body with respect to which they are defined. Hence, only the relative linear velocity, angular velocity, and twist between two bodies are relevant geometric relations.

When doing actual calculations with the geometric relations between rigid bodies (see the "Geometric Relations" section), one has to use the coordinate representation of the geometric relations and therefore has to choose a coordinate frame in which the coordinates are expressed to obtain numerical values for the geometric relations. Since the geometric relation between two rigid bodies should only include information on the relative motion of the two rigid bodies, the coordinate frame is considered instantaneously fixed to the reference body, i.e., the body with respect to which the body motion is described.

Semantics

Geometric Primitives

The geometric relations between bodies are described using a set of geometric primitives:

- A (spatial) point is the primitive to represent the position of a body. Points have neither volume, area, length, nor any other higher dimensional analog. This document denotes points by the symbols a, b, ...
- A vector is the geometric primitive that connects a point a to a point b. It has a magnitude (the distance between the two points) and a direction (from a to b). To express the magnitude of a vector, a (length) scale must be chosen. [In order not to overload the notation, this article assumes that all coordinate representations use the same scales (linear, angular, and time) and units (i.e., SI units).]
- An orientation frame represents an orientation, by means of three orthonormal vectors indicating the frame's X-axis X, Y-axis Y, and Z-axis Z. This document denotes orientation frames by the symbols $[a]$, $[b]$, ...
- A (displacement) frame represents position and orientation of a bod y, by means of an orientation frame and a point (which is the orientation frame's origin). This document denotes frames by the symbols $\{a\}$, $\{b\}$, ...

How to Use Semantics for Geometric Relations Between Rigid Bodies in Robotics

We illustrate how the proposed semantics for geometric relations between rigid bodies can be used in robotics. To this end, we use the example illustrated in the Figure S1, in which two robots cooperate for spray painting a cylindrical object. The first robot holds the cylindrical object, while the other robot holds the spray gun.

To complete the painting task, the robot programmer has to determine the joint angles of the second robot holding the spray gun such that a predefined pose between the spray gun and cylindrical object is obtained (the joint angles of the first robot holding the cylindrical object are given).

In the first step, the rigid bodies and the frames attached to them are identified:

- $\{b_1\}$ attached to the base $\mathcal{B}_1$ of the first robot,
- $\{e_1\}$ attached to the end-effector $\mathcal{E}_1$ of the first robot,
- $\{o_1\}$ attached to cylindrical object O_1,
- $\{b_2\}$ attached to the base $\mathcal{B}_2$ of the second robot,
- $\{e_2\}$ attached to the end-effector $\mathcal{E}_2$ of the second robot, and
- $\{o_2\}$ attached to the spray gun O_2.

In our example the following poses are available:

- PoseCoord $(\{e_1\}|\mathcal{E}_1, \{b_1\}|\mathcal{B}_1, [b_1])$ determined by the forward position kinematics of the first robot.
- PoseCoord $(\{b_2\}|B_2, \{b_1\}|\mathcal{B}_1, [b_1])$ determined by the mounting of the robots.
- PoseCoord $(\{o_1\}|O_1, \{e_1\}|\mathcal{E}_1, [e_1])$ determined by the mounting of the cylindrical object on the first robot end-effector.
- PoseCoord $(\{o_2\}|O_2, \{e_2\}|\mathcal{E}_2, [e_2])$ determined by the mounting of the spray gun on the second robot end-effector.
- PoseCoord $(\{o_2\}|O_2, \{o_1\}|O_1, [o_1])$ determined by the desired spray-painting pose.

To find the joint angles of the second robot the robot programmer has to find PoseCoord $(\{e_2\}|\mathcal{E}_2, \{b_2\}|\mathcal{B}_2, [b_2])$, and subsequently use the inverse kinematics of the second robot. Semantically, PoseCoord $(\{e_2\}|\mathcal{E}_2, \{b_2\}|\mathcal{B}_2, [b_2])$ can be obtained as:

$$\text{PoseCoord}\,(\{e_2\}|\mathcal{E}_2, \{o_2\}|O_2, [o_2]) = \text{inverse2}\,(\text{PoseCoord}\,(\{o_2\}|O_2, \{e_2\}|\mathcal{E}_2, [e_2])), \quad (1)$$

$$\text{PoseCoord}\,(\{e_2\}|\mathcal{E}_2, \{o_1\}|O_1, [o_1]) = \text{compose}\,(\text{PoseCoord}\,(\{e_2\}|\mathcal{E}_2, \{o_2\}|O_2, [o_2]), \text{PoseCoord}\,(\{o_2\}|O_2, \{o_1\}|O_1, [o_1])), \quad (2)$$

$$\text{PoseCoord}\,(\{e_2\}|\mathcal{E}_2, \{e_1\}|\mathcal{E}_1, [e_1]) = \text{compose}\,(\text{PoseCoord}\,(\{e_2\}|\mathcal{E}_2, \{o_1\}|O_1, [o_1]), \text{PoseCoord}\,(\{o_1\}|O_1, \{e_1\}|\mathcal{E}_1, [e_1])), \quad (3)$$

$$\text{PoseCoord}\,(\{e_2\}|\mathcal{E}_2, \{b_1\}|\mathcal{B}_1, [b_1]) = \text{compose}\,(\text{PoseCoord}\,(\{e_2\}|\mathcal{E}_2, \{e_1\}|\mathcal{E}_1, [e_1]), \text{PoseCoord}\,(\{e_1\}|\mathcal{E}_1, \{b_1\}|\mathcal{B}_1, [b_1])), \quad (4)$$

$$\text{PoseCoord}\,(\{b_1\}|\mathcal{B}_1, \{b_2\}|\mathcal{B}_2, [b_2]) = \text{inverse2}\,(\text{PoseCoord}\,(\{b_2\}|\mathcal{B}_2, \{b_1\}|\mathcal{B}_1, [b_1])), \text{ and} \quad (5)$$

$$\text{PoseCoord}\,(\{e_2\}|\mathcal{E}_2, \{b_2\}|\mathcal{B}_2, [b_2]) = \text{compose}\,(\text{PoseCoord}\,(\{e_2\}|\mathcal{E}_2, \{b_1\}|\mathcal{B}_1, [b_1]), \text{PoseCoord}\,(\{b_1\}|\mathcal{B}_1, \{b_2\}|\mathcal{B}_2, [b_2])). \quad (6)$$

Remark that the operators compose (,) and inverse2() are introduced in the supplemental material (Section B); while the meaning of compose (,) is straightforward, inverse2() is used when both the forward and the inverted geometric relation

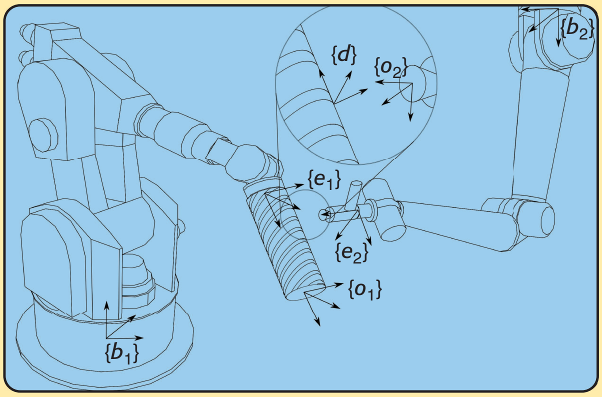


Figure S1. Two robots cooperate for spray-painting a cylindrical object.

have to be expressed in the reference orientation frame. Every equation above can be checked semantically (see the "Semantic Operations" section) for instance by checking if the frame and body of one pose in the composition are equal to the reference frame and reference body, respectively of the other pose in the composition; and if the coordinate frame of every pose in the composition is equal to its reference orientation frame. This way, errors regarding the logic of composition and inversion are prevented (and can be supported by software checks).

Next, to do the actual calculations, the robot programmer can choose a *particular coordinate representation* (see the "Coordinate Representations" section) for the pose, for instance homogeneous transformation matrices. Once a particular coordinate representation is chosen, the semantic constraints imposed by the coordinate representation (see the "Semantic Constraints Imposed by Coordinate Representations" section) can be checked and the semantic operations (such as inversion or composition) can be translated into particular operations (such as matrix inversions or matrix multiplications). The homogeneous transformation matrix coordinate representation, for instance, imposes that the point and the orientation frame belong to a single frame, the reference point and the reference orientation frame belong to a single frame, and the coordinate frame equals the reference orientation frame; the inverse operator is replaced by a matrix inverse and the pose composition is replaced by matrix multiplications (where the multiplication order can be derived from the semantics of the poses). For the example the semantic operations (1)–(6) can be rewritten with homogeneous transformation matrices as:

$${}^{\{e_2\}|\mathcal{E}_2}_{\{o_2\}|O_2}\boldsymbol{T} = \left({}^{\{o_2\}|O_2}_{\{e_2\}|\mathcal{E}_2}\boldsymbol{T}\right)^{-1}, \quad (7)$$

$${}^{\{e_2\}|\mathcal{E}_2}_{\{o_1\}|O_1}\boldsymbol{T} = {}^{\{o_2\}|O_2}_{\{o_1\}|O_1}\boldsymbol{T}\;{}^{\{e_2\}|\mathcal{E}_2}_{\{o_2\}|O_2}\boldsymbol{T}, \quad (8)$$

$${}^{\{e_2\}|\mathcal{E}_2}_{\{e_1\}|\mathcal{E}_1}\boldsymbol{T} = {}^{\{o_1\}|O_1}_{\{e_1\}|\mathcal{E}_1}\boldsymbol{T}\;{}^{\{e_2\}|\mathcal{E}_2}_{\{o_1\}|O_1}\boldsymbol{T}, \quad (9)$$

$${}^{\{e_2\}|\mathcal{E}_2}_{\{b_1\}|\mathcal{B}_1}\boldsymbol{T} = {}^{\{e_1\}|\mathcal{E}_1}_{\{b_1\}|\mathcal{B}_1}\boldsymbol{T}\;{}^{\{e_2\}|\mathcal{E}_2}_{\{e_1\}|\mathcal{E}_1}\boldsymbol{T}, \quad (10)$$

$${}^{\{b_1\}|\mathcal{B}_1}_{\{b_2\}|\mathcal{B}_2}\boldsymbol{T} = \left({}^{\{b_2\}|\mathcal{B}_2}_{\{b_1\}|\mathcal{B}_1}\boldsymbol{T}\right)^{-1}, \text{ and} \quad (11)$$

$${}^{\{e_2\}|\mathcal{E}_2}_{\{b_2\}|\mathcal{B}_2}\boldsymbol{T} = {}^{\{b_1\}|\mathcal{B}_1}_{\{b_2\}|\mathcal{B}_2}\boldsymbol{T}\;{}^{\{e_2\}|\mathcal{E}_2}_{\{b_1\}|\mathcal{B}_1}\boldsymbol{T}. \quad (12)$$

Each of these geometric primitives can be fixed to a body, which means that the geometric primitive coincides with the body not only instantaneously but also over time. For the point a and the body C, for instance, this is written as $a|C$. Figure 1 presents the geometric primitives body, point, vector, orientation frame, and frame graphically. To help the reader, we will consistently use the following naming for the geometric primitives to represent the geometric relation of a body C

How to Use Semantics for Integrating Velocities Between Rigid Bodies in Robotics

Here we illustrate how the proposed semantics can be used for integrating velocities between rigid bodies in robotics. To this end, we will use the example of Figure S1 in which two robots cooperate for spray-painting a cylindrical object. Imagine the robot programmer wants to determine the pose of the first cylinder $\mathcal{O}_1$ with respect to the robot base $\mathcal{B}_1$ after a certain time t, i.e., PoseCoord $(\{o'_1\}|\mathcal{O}'_1, \{b_1\}|\mathcal{B}_1, [b_1])$ when either

1) TwistCoord $(o_1|\mathcal{O}_1, \mathcal{B}_1, [b_1]) = C^{te}$ (pose twist is constant);
2) TwistCoord $(o_1|\mathcal{O}_1, \mathcal{B}_1, [o_1]) = C^{te}$ (body-fixed twist is constant); and
3) TwistCoord $(b_1|\mathcal{O}_1, \mathcal{B}_1, [b_1]) = C^{te}$ (screw twist is constant).

Since integration imposes the semantic constraint that the reference point and the coordinate frame have to belong to the same frame, integration of TwistCoord $(o_1|\mathcal{O}_1, \mathcal{B}_1, [b_1]) = C^{te}$ (first case) is semantically not allowed. For the other two cases integration results in

2) PoseCoord $(\{o'_1\}|\mathcal{O}'_1, \{o_1\}|\mathcal{B}_1, [o_1])$ = integrate (TwistCoord $(o_1|\mathcal{O}_1, \mathcal{B}_1, [o_1])$, t), and
3) PoseCoord $(\{b'_1\}|\mathcal{O}'_1, \{b_1\}|\mathcal{B}_1, [b_1])$ = integrate (TwistCoord $(b_1|\mathcal{O}_1, \mathcal{B}_1, [b_1])$, t)

To obtain the new pose of the cylinder PoseCoord $(\{o'_1\}|\mathcal{O}'_1, \{b_1\}|\mathcal{B}, [b_1])$ the following compositions should be performed:

2) PoseCoord $(\{o'_1\}|\mathcal{O}'_1, \{b_1\}|\mathcal{B}_1, [b_1])$ = compose (integrate (TwistCoord $(o_1|\mathcal{O}_1, \mathcal{B}, [o_1])$, t), PoseCoord $(\{o_1\}|\mathcal{B}_1, \{b_1\}|\mathcal{B}, [b_1])$) = compose (integrate (TwistCoord$(o_1|\mathcal{O}_1, \mathcal{B}_1, [o_1], t)$, PoseCoord $(\{o_1\}|\mathcal{O}_1, \{b_1\}|\mathcal{B}_1, [b_1])$), and
3) PoseCoord $(\{o'_1\}|\mathcal{O}'_1, \{b_1\}|\mathcal{B}_1, [b_1])$ = compose (integrate (TwistCoord $(b_1|\mathcal{O}_1, \mathcal{B}_1, [b_1])$, t), PoseCoord $(\{o'_1\}|\mathcal{O}'_1, \{b'_1\}|\mathcal{O}'_1, [b'_1])$ = compose (integrate (TwistCoord $(b_1|\mathcal{O}_1, \mathcal{B}_1, [b_1], t)$, PoseCoord $(\{o_1\}|\mathcal{O}_1, \{b_1\}|\mathcal{B}_1 [b_1])$).

Next, to do the actual calculations, the robot programmer can choose a particular coordinate representation (see the "Coordinate Representations" section) for the pose and the twist, for instance homogeneous transformation matrices for the pose (see the "Pose Coordinate Presentations" section) and six-dimensional vectors for the twist (see the "Twist Coordinate Representations" section). Once the coordinate representations are chosen, the semantic constraints imposed by the coordinate representation (see the "Semantic Constraints Imposed by Coordinate Representations" section) can be checked and the semantic operations (such as inversion or composition) can be translated into particular operations (such as matrix inversions or matrix multiplications). The homogeneous transformation matrix, for instance, imposes that the point and the orientation frame belong to a single frame, the reference point and the reference orientation frame belong to a single frame, and the coordinate frame equals the reference orientation frame; the integration operator is replaced by the matrix exponential and the pose composition is replaced by matrix multiplications (where the multiplication order can be derived from the semantics of the poses). For example, the semantic operations can be rewritten with homogeneous transformation matrices and six-dimensional vector twists as

2) $^{\{o'_1\}|\mathcal{O}'_1}_{\{b_1\}|\mathcal{B}_1}\boldsymbol{T} = {}^{\{o_1\}|\mathcal{O}_1}_{\{b_1\}|\mathcal{B}_1}\boldsymbol{T} \exp\left({}^{o_1|\mathcal{O}_1}_{[o_1]}\boldsymbol{t}_{\mathcal{B}}\ \ t\right)$ and

3) $^{\{o'_1\}|\mathcal{O}'_1}_{\{b_1\}|\mathcal{B}_1}\boldsymbol{T} = \exp\left({}^{b_1|\mathcal{O}_1}_{[b_1]}\boldsymbol{t}_{\mathcal{B}}\, t\right) {}^{\{o_1\}|\mathcal{O}_1}_{\{b_1\}|\mathcal{B}_1}\boldsymbol{T}$,

with the exponential of the six-dimensional vector twist as defined in [7]. The above example shows that using the semantics, the correct integration and order of multiplication in the compositions with the poses resulting from the integrations are automatically obtained, preventing common errors.

with respect to body $\mathcal{D}$ in this document: $e|\mathcal{C}$, $[a]|\mathcal{C}$, $\{g\}|\mathcal{C}$, $f|\mathcal{D}$, $[b]|\mathcal{D}$, and $\{h\}|\mathcal{D}$.

Geometric Relations

This section introduces the semantics for the following geometric relations between rigid bodies: position, orientation, pose, linear velocity, angular velocity, and twist. To this end, the following systematic procedure is used for every geometric relation. First, the minimal but complete set of geometric primitives (see the "Geometric Primitives" section) needed to unambiguously define the geometric relation is specified. Second, from this minimal but complete set of geometric primitives, a semantic representation of the geometric relation is proposed. Third, the coordinate semantics for coordinate representations of the geometric relation are specified. Table 1 summarizes the minimal but complete set of geometric primitives and the (coordinate) semantics for the geometric relations between rigid bodies handled in this section. Finally, some common and practically convenient semantic choices, i.e., choices of particular geometric primitives, are discussed.

Position

The relative position of body $\mathcal{C}$ with respect to body $\mathcal{D}$ can be represented by the position of a point fixed to body $\mathcal{C}$ with respect to a point fixed to body $\mathcal{D}$. Therefore, semantically the relative position between body $\mathcal{C}$ and body $\mathcal{D}$ is indicated as Position $(e|\mathcal{C}, f|\mathcal{D})$, where point e is fixed to body $\mathcal{C}$ and point f is fixed to body $\mathcal{D}$.

The minimal but complete set of geometric primitives for the position is therefore the body whose position is given (the body), a point on the body (the point), the body with respect to which the position is given (the reference body), and the point on the reference body (the reference point). Coordinate representations require an orientation frame $[r]$, considered instantaneously fixed to the reference body, to express the coordinates in (the coordinate frame), this can be indicated as follows: PositionCoord $(e|\mathcal{C}, f|\mathcal{D}, [r])$.

In practice, it is often convenient to use origins of frames on the bodies (for instance frame $\{g\}$ on body $\mathcal{C}$ and frame $\{h\}$ on body $\mathcal{D}$) to represent relative position of these bodies, i.e., Position $(g|\mathcal{C}, h|\mathcal{D})$ and to use one of these frames as the coordinate frame, i.e., PositionCoord $(g|\mathcal{C}, h|\mathcal{D}, [g])$, or PositionCoord $(g|\mathcal{C}, h|\mathcal{D}, [h])$.

Orientation

The relative orientation of body $\mathcal{C}$ with respect to body $\mathcal{D}$ can be represented by the orientation of an orientation frame fixed to body $\mathcal{C}$ with respect to an orientation frame

Common Errors in Geometric Rigid-Body Relations Calculations in Robotics

Here, we list some common errors in calculations with geometric relations between rigid bodies in robotics that could be prevented by using the semantics proposed in this article.

1) *Logic errors in geometric relation calculations*: A lot of logic errors can occur during geometric relation calculations. For instance, the inverse of the position of point e fixed to body $\mathcal{C}$ with respect to point f fixed to body $\mathcal{D}$ (Position ($e|\mathcal{C}$, $f|\mathcal{D}$)) (the semantics of the geometric relation is introduced in the "Semantics" section, while Table 1 gives an overview of the semantics) is the relative position of point f fixed to body $\mathcal{D}$ with respect to point e fixed to body $\mathcal{C}$ (Position ($f|\mathcal{D}, e|\mathcal{C}$)), while the inverse of the linear velocity of point e fixed to body $\mathcal{C}$ with respect to body $\mathcal{D}$ (LinearVelocity ($e|\mathcal{C}$, $\mathcal{D}$)) is the linear velocity of point e fixed to body $\mathcal{D}$ with respect to body $\mathcal{C}$ (LinearVelocity ($e|\mathcal{D}, \mathcal{C}$)). When using the semantic representation proposed in this article, the semantics of the inverse geometric relation can be automatically derived from the forward geometric relation, preventing logic errors. A second example emerges when composing the relations involving three rigid bodies: to get the geometric relation of body $\mathcal{C}$ with respect to body $\mathcal{D}$, one can compose the geometric relation between $\mathcal{C}$ and a third body $\mathcal{E}$ with the geometric relation between body $\mathcal{E}$ and body $\mathcal{D}$ (and not the geometric relation between body $\mathcal{D}$ and body $\mathcal{E}$ for instance). Such a logic constraint can be checked easily by including, for instance, the body and reference body in the semantic representation of the geometric relations.
2) *Composition of twists with different point*: Composing twists requires a common point (i.e., the twists have to express the linear velocity of the same point on the body). By including the point of the twist in the semantic representation, this constraint can be checked explicitly.
3) *Composition of geometric relations expressed in different coordinate frames*: Composing geometric relations using coordinate representations such as position vectors, linear and angular velocity vectors, and six-dimensional vector twists requires that the coordinates are expressed in the same coordinate frame. By including the coordinate frame in the coordinate semantic representation of the geometric relations, this constraint can be checked explicitly.
4) *Composition of poses and orientation coordinate representations in wrong order*: The rotation matrix and homogeneous transformation matrix coordinate representations can be composed using simple multiplication. Since matrix multiplication is, however, not commutative, a common error is to use a wrong multiplication order in the composition. The correct multiplication order can, however, be directly derived when including the bodies, frames, and points in the coordinate semantic representation of the geometric relations.
5) *Integration of twists when point and coordinate frame do not belong to same frame*: A twist can only be integrated when it expresses the linear velocity of the origin of the coordinate frame the twist is expressed in. When including the point and the coordinate frame in the coordinate semantic representation of the twist, this constraint can be checked explicitly.

fixed to body $\mathcal{D}$. Therefore, semantically the relative orientation between bodies $\mathcal{C}$ and $\mathcal{D}$ is indicated as Orientation ($[a]|\mathcal{C}$, $[b]|\mathcal{D}$), where orientation frame $[a]$ is fixed to body $\mathcal{C}$ and orientation frame $[b]$ is fixed to body $\mathcal{D}$.

The minimal but complete set of geometric primitives for the orientation is therefore the body whose orientation is given (the body), an orientation frame on the body (the orientation frame), the body with respect to which the orientation is given (the reference body), and the orientation frame on the reference body (the reference orientation frame).

Coordinate representations require an orientation frame $[r]$, considered instantaneously fixed to the reference body, to express the coordinates in (coordinate frame), this can be indicated as follows: OrientationCoord ($[a]|\mathcal{C}$, $[b]|\mathcal{D}$, $[r]$).

In practice, it is often convenient to use orientation frames of frames on the bodies to represent their relative orientation, i.e., Orientation ($[g]|\mathcal{C}$, $[h]|\mathcal{D}$), and to use one of these frames as an orientation frame to express the coordinates in, i.e., OrientationCoord ($[g]|\mathcal{C}$, $[h]|\mathcal{D}$, $[g]$) or OrientationCoord ($[g]|\mathcal{C}$, $[h]|\mathcal{D}$, $[h]$).

Most orientation coordinate representations (see the "Orientation Coordinate Representations" section) implicitly assume the latter, i.e., that the coordinate frame equals the reference orientation frame.

Pose

The relative pose of body $\mathcal{C}$ with respect to body $\mathcal{D}$ can be represented by the position of a point fixed to body $\mathcal{C}$ with respect to a point fixed to body $\mathcal{D}$, and the orientation of an orientation frame fixed to body $\mathcal{C}$ with respect to an orientation frame fixed to body D. Therefore, semantically the relative pose between body $\mathcal{C}$ and body $\mathcal{D}$ is indicated as Pose ((e,$[a]$)$|\mathcal{C}$, (f,$[b]$)$|\mathcal{D}$), where orientation frame $[a]$ and point e are fixed to body $\mathcal{C}$ and orientation frame $[b]$ and point f are fixed to body $\mathcal{D}$.

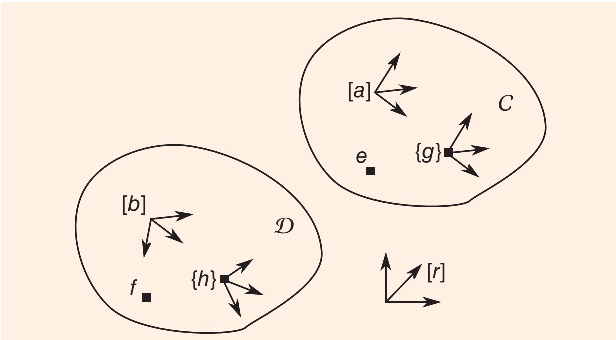
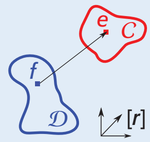
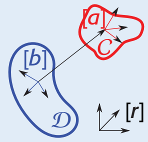
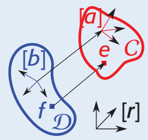
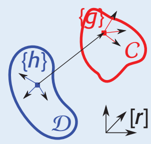
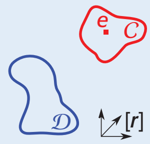
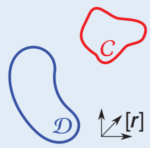
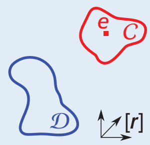


Figure 1. The geometric relation between rigid bodies is described using a set of geometric primitives: points, vectors, orientation frames, and frames. The geometric primitives that are useful to define the position, orientation, pose, linear velocity, angular velocity, and twist of body $\mathcal{C}$ with respect to body $\mathcal{D}$ are shown: an orientation frame $[a]$, a point e, and frame $\{g\}$ fixed to body $\mathcal{C}$, an orientation frame $[b]$, a point f, and frame $\{h\}$ fixed to body $\mathcal{D}$, and a coordinate frame [r], considered instantaneously fixed to body $\mathcal{D}$, in which the coordinates are expressed.

Table 1. Minimal semantics and coordinate semantics (expressed in coordinate frame [*r*]).

Geometric Relation	(Coordinate) Semantics	Geometric Primitives	Graphical Representation
Position	Position $(e\,\vert\,\mathcal{C}, f\,\vert\,\mathcal{D})$ PositionCoord $(e\,\vert\,\mathcal{C}, f\,\vert\,\mathcal{D}, [r])$	Point *e* Body $\mathcal{C}$ Reference point *f* Reference body $\mathcal{D}$ Coordinate frame [*r*]	
Orientation	Orientation $([a]\,\vert\,\mathcal{C}, [b]\,\vert\,\mathcal{D})$ OrientationCoord $([a]\,\vert\,\mathcal{C}, [b]\,\vert\,\mathcal{D}, [r])$	Orientation frame [*a*] Body $\mathcal{C}$ Reference orientation frame [*b*] Reference body $\mathcal{D}$ Coordinate frame [*r*]	
Pose	Pose $((e, [a])\,\vert\,\mathcal{C}, f, ([b])\,\vert\,\mathcal{D})$ PoseCoord $((e, [a])\,\vert\,\mathcal{C}, (f, [b])\,\vert\,\mathcal{D}, [r])$	Point *e* Orientation frame [*a*] Body $\mathcal{C}$ Reference point *f* Reference orientation frame [*b*] Reference body $\mathcal{D}$ Coordinate frame [*r*]	
	Pose $(\{g\}\,\vert\,\mathcal{C}, \{h\}\,\vert\,\mathcal{D})$ PoseCoord $(\{g\}\,\vert\,\mathcal{C}, \{h\}\,\vert\,\mathcal{D}, [r])$	Frame {*g*} Body $\mathcal{C}$ Frame {*h*} Reference body $\mathcal{D}$ Coordinate frame [*r*]	
Linear velocity	LinearVelocity $(e\,\vert\,\mathcal{C}, \mathcal{D})$ LinearVelocityCoord $(e\,\vert\,\mathcal{C}, \mathcal{D}, [r])$	Point *e* Body $\mathcal{C}$ Reference body $\mathcal{D}$ Coordinate frame [*r*]	
Angular velocity	AngularVelocity $(\mathcal{C}, \mathcal{D})$ AngularVelocityCoord $(\mathcal{C}, \mathcal{D}, [r])$	Body $\mathcal{C}$ Reference body $\mathcal{D}$ Coordinate frame [*r*]	
Twist	Twist $(e\,\vert\,\mathcal{C}, \mathcal{D})$ TwistCoord $(e\,\vert\,\mathcal{C}, \mathcal{D}, [r])$	Point *e* Body $\mathcal{C}$ Reference body $\mathcal{D}$ Coordinate frame [*r*]	

The table includes the minimal but complete set of geometric primitives for the position, orientation, pose, linear velocity, angular velocity, and twist of body $\mathcal{C}$ with point *e*, orientation frame [*a*], and frame {*g*} with respect to $\mathcal{D}$ with point *f*, orientation frame [*b*], and frame {*h*}, including a graphical representation.

The minimal but complete set of geometric primitives for the pose is therefore the body whose pose is given (the body), a point on the body (the point), an orientation frame on the body (the orientation frame), the body with respect to which the pose is given (the reference body), the point on the reference body (the reference point), and the orientation frame on the reference body (the reference orientation frame).

In practice, it is often convenient to use frames on the bodies to represent their pose. In this case, the pose of the body is determined by the origin of the frame, and the orientation is determined by the orientation of the frame. In this case, the relative pose between body $\mathcal{C}$ and body $\mathcal{D}$ is indicated as Pose $(\{g\}\,\vert\,\mathcal{C}, \{h\}\,\vert\,\mathcal{D})$, where frame {*g*} is fixed to body $\mathcal{C}$ and frame {*h*} is fixed to body $\mathcal{D}$.

Coordinate representations require an orientation frame [*r*], considered instantaneously fixed to the reference body, to express the coordinates in (the coordinate frame), this can be indicated as follows: PoseCoord $((e, [a])\,\vert\,\mathcal{C}, (f, [b])\,\vert\,\mathcal{D}, [r])$.

In practice, it is often convenient to use the orientation frames of the *frames* on the bodies to express the coordinates in, i.e., PoseCoord $(\{g\}\,\vert\,\mathcal{C}, \{h\}\,\vert\,\mathcal{D}, [g])$ or PoseCoord $(\{g\}\,\vert\,\mathcal{C}, \{h\}\,\vert\,\mathcal{D}, [h])$. Most pose coordinate representations (see the "Pose Coordinate Representations" section) implicitly assume the latter, i.e., that the coordinate frame equals the reference orientation frame.

Linear Velocity

The relative linear velocity of body $\mathcal{C}$ with respect to body $\mathcal{D}$ can be represented by the linear velocity of a point *e* on body

$\mathcal{C}$ with respect to any other point on body $\mathcal{D}$ (remark that we consider the linear velocity part of the six-dimensional motion and not only the special case of a pure translational motion). Remark that the linear velocity of two bodies is the same regardless of which point is chosen on the *reference* body, it is, however, not the same for all points on the moving body (see Section A.1 in the supplemental material available with the article). Therefore, semantically the relative linear velocity between body $\mathcal{C}$ and body $\mathcal{D}$ is indicated as LinearVelocityCoord ($e|\mathcal{C}, \mathcal{D}$), where point e is fixed to body $\mathcal{C}$.

The minimal but complete set of geometric primitives for the linear velocity is, therefore, the body whose linear velocity is given (the body), a point on the body (the point), and the body with respect to which the linear velocity is given (the reference body).

Remark that for a pure translational motion, the linear velocity is the same for every point on the body, i.e., $\forall e_1|\mathcal{C}, e_2|\mathcal{C}$: LinearVelocity ($e_1|\mathcal{C}, \mathcal{D}$) = LinearVelocity ($e_2|\mathcal{C}, \mathcal{D}$).

Coordinate representations require an orientation frame $[r]$, considered instantaneously fixed to the reference body, to express the coordinates in (the coordinate frame), this can be indicated as follows: LinearVelocityCoord ($e|\mathcal{C}, \mathcal{D}, [r]$).

In practice, it is often convenient to use the origin of the frames on the bodies to represent the relative linear velocity, and to use one of the orientation frames on the bodies to express the coordinates in, i.e., LinearVelocityCoord ($g|\mathcal{C}, \mathcal{D}, [g]$), LinearVelocityCoord ($g|\mathcal{C}, \mathcal{D}, [h]$), or LinearVelocityCoord ($h|\mathcal{C}, \mathcal{D}, [h]$).

Rotational Velocity

The relative angular velocity of body $\mathcal{C}$ with respect to body $\mathcal{D}$ can be represented by the angular velocity of any orientation frame on body $\mathcal{C}$ with respect to any another orientation frame on body $\mathcal{D}$ (see Section A.2 in the supplemental material). Therefore, semantically the relative angular velocity between body $\mathcal{C}$ and body $\mathcal{D}$ is indicated as AngularVelocity ($\mathcal{C}, \mathcal{D}$).

The minimal but complete set of geometric primitives for the angular velocity is therefore the body whose angular velocity is given (the body) and the body with respect to which the angular velocity is given (the *reference body*).

Coordinate representations require an orientation frame $[r]$, considered instantaneously fixed to the reference body, to express the coordinates in (the *coordinate frame*), this can be indicated as follows: AngularVelocityCoord ($\mathcal{C}, \mathcal{D}, [r]$). In practice, it is often convenient to use the orientation frames of the frames on the bodies to represent their relative angular velocity and to use one of these frames as an orientation frame to express the coordinates in, i.e., AngularVelocityCoord ($\mathcal{C}, \mathcal{D}, [g]$) or AngularVelocityCoord ($\mathcal{C}, \mathcal{D}, [h]$).

Twist

The relative twist of body $\mathcal{C}$ with respect to body $\mathcal{D}$ can be represented by the angular velocity of body $\mathcal{C}$ with respect to body $\mathcal{D}$, and the linear velocity of a point e on body $\mathcal{C}$ with respect to body $\mathcal{D}$. Therefore, semantically the relative twist between body $\mathcal{C}$ and body $\mathcal{D}$ is indicated as Twist ($e|\mathcal{C}, \mathcal{D}$) and it contains the information on both the linear velocity LinearVelocity ($e|\mathcal{C}, \mathcal{D}$) and the angular velocity, AngularVelocity ($\mathcal{C}, \mathcal{D}$).

The minimal but complete set of geometric primitives for the twist is therefore the body whose twist is given (the *body*), a point on the body (the *point*), and the body with respect to which the twist is given (the *reference body*).

In practice, it is often convenient to use the origins and the orientation *frames* of the frames on the bodies to represent their relative twist, i.e., Twist ($g|\mathcal{C}, \mathcal{D}$).

In practice, two particular choices of points and orientation frames on the bodies are commonly used. 1) For the pose twist of body $\mathcal{C}$ with respect to body $\mathcal{D}$, the origin of frame $\{g\}$ on body $\mathcal{C}$ is used as the reference point of body $\mathcal{C}$. A pose twist is therefore indicated as Twist ($g|\mathcal{C}, \mathcal{D}$). 2) For the screw twist of body $\mathcal{C}$ with respect to body $\mathcal{D}$ the origin of frame $\{h\}$ on body $\mathcal{C}$ is used as the reference point of body $\mathcal{C}$, i.e., the point fixed to body $\mathcal{C}$ that instantaneously coincides with the origin of frame $\{h\}$ fixed to body $\mathcal{D}$ is used. A screw twist is therefore indicated as Twist ($h|\mathcal{C}, \mathcal{D}$).

Coordinate representations require an orientation frame $[r]$, considered instantaneously fixed to the reference body, to express the coordinates in (the *coordinate frame*), this can be indicated by TwistCoord ($e|\mathcal{C}, \mathcal{D}, [r]$). In practice, it is often convenient to use the origins and orientation frames of the frames on the bodies to represent their relative twist and to use one of these frames as an orientation frame to express the coordinates in, i.e., TwistCoord ($g|\mathcal{C}, \mathcal{D}, [g]$), TwistCoord ($h|\mathcal{C}, \mathcal{D}, [h]$), or TwistCoord ($g|\mathcal{C}, \mathcal{D}, [h]$).

When a pose twist Twist ($g|\mathcal{C}, \mathcal{D}$) is expressed in the orientation frame $[g]$ defined on body $\mathcal{C}$, now considered instantaneously fixed to the reference body, it is often called a body-fixed twist and indicated as TwistCoord ($g|\mathcal{C}, \mathcal{D}, [g]$).

Semantic Operations

On the geometric relations defined in the "Geometric Relations" section, semantic operations that compose the geometric relations or that change the point, orientation frame, reference point, reference orientation frame, or coordinate frame of the geometric relation can be applied.

These semantic operations themselves impose constraints on the geometric relation they are applied to and on the operation arguments (which are themselves geometric relations) of the operator. Section B in the supplemental material provides an example to clarify the constraints imposed by semantic operations, while Tables II and III in the same section give an overview of semantic operations that can be applied to geometric relations, and list the constraints imposed by the operations.

Coordinate Representations

When doing actual calculations with the geometric relations between rigid bodies ("Geometric Relations" section), one has to use the coordinate representation of the geometric relations. By adding an orientation frame $[r]$, considered instantaneously fixed to the reference body, in which the coordinates are expressed (the coordinate frame), numerical values for the geometric relations are obtained. Furthermore, both the linear and angular time scales have to be specified. (In this document, it is assumed that the linear and angular time scales are the same for all the geometric relations, and the scales are omitted to simplify the notation. The scales should additionally be identified in the semantic information of every coordinate representation; the standardized system of SI units is the natural candidate for this semantic information.)

This section shows how particular coordinate representations for the geometric relations presented in the "Semantics" section can impose constraints on the semantics. Furthermore, it lists some commonly used coordinate representations together with their semantic description and reveals the semantic constraints imposed by them. Other coordinate representations for the same geometric relationships can be connected to the semantic description, and furthermore, these coordinate representations can all be transformed into the coordinate representations presented below. Although we provide a symbol for each of the coordinate representations, this symbol is less important than the name we attach to the geometric relation in the semantics. In the supplemental material, we also provide a table containing additional commonly used coordinate representations, their properties, and corresponding semantic representation (Table 4).

Background

As mentioned in the "Geometric Primitives" section, points, orientation frames, and frames are attached to bodies to express their relative position, orientation, pose, linear velocity, angular velocity, and twist. This document assumes that frames are orthogonal and right-handed. Often, nonminimal coordinate representations (i.e., with more parameters than the number of physical degrees of freedom) are used to model the properties of rigid-body motion. The tradeoffs between such nonminimal and minimal representations are improved properties with respect to numerical stability and unambiguity in the representation, but extra cost because of the need to carry along a number of constraints between the numbers in the nonminimal representation.

Semantic Constraints Imposed by Coordinate Representations

Particular coordinate representations can make additional (and often hidden) assumptions on the choice of the point, orientation frame, reference point, reference orientation frame, or coordinate frame of the geometric relation. Therefore, the coordinate representation itself can impose constraints on the geometric relations.

Position Coordinate Representations

To express the relative position of two bodies $\mathcal{C}$ and $\mathcal{D}$, one can choose points on the bodies and express their relative position using a position vector [13].

The position vector ${}_{[r]}\boldsymbol{p}^{f|\mathcal{D},e|\mathcal{C}}$ points from the point f fixed to body $\mathcal{D}$ to the point e fixed to body $\mathcal{C}$, and is expressed in $[r]$, i.e.,

$$ {}_{[r]}\boldsymbol{p}^{f|\mathcal{D},e|\mathcal{C}} \sim \text{PositionCoord}\,(e|\mathcal{C}, f|\mathcal{D}, [r]). $$

The position vector does not impose any semantic constraints. The position vector is a three-dimensional vector $(x \;\; y \;\; z)^T$ containing the coordinates in $[r]$ of the vector pointing from the point f fixed to body $\mathcal{D}$ to the point e fixed to body $\mathcal{C}$.

Orientation Coordinate Representations

Euler Axis-Angle and Rotation Vector

From Euler's rotation theorem, it is known that the relative orientation between two bodies can be expressed as a single rotation about some axis with a certain angle. The axis can be represented as a three-dimensional unit vector and the angle by a scalar [13].

The Euler axis-angle coordinate representation ${}_{[r]}\boldsymbol{e}^{[b]|\mathcal{D},[a]|\mathcal{C}}$ represents the axis and angle of rotation needed to rotate from the orientation frame $[b]$ fixed to body $\mathcal{D}$ to the orientation frame $[a]$ fixed to body $\mathcal{C}$ and is expressed in $[r]$, i.e.,

$$ {}_{[r]}\boldsymbol{e}^{[b]|\mathcal{D},[a]|\mathcal{C}} \sim \text{OrientationCoord}\,([a]|\mathcal{C}, [b]|D, [r]). $$

The Euler axis-angle coordinate representation is a fourdimensional vector $(\boldsymbol{e} \;\; \theta)^T$ containing the coordinates of the unit vector $\boldsymbol{e}$ along the rotation axis in $[r]$ and the rotation angle θ. The Euler axis-angle coordinate representation does not impose any semantic constraints.

The rotation vector coordinate representation ${}_{[r]}\boldsymbol{r}^{[b]|\mathcal{D},[a]|\mathcal{C}}$ represents the axis and angle of rotation needed to rotate from the orientation frame $[b]$ fixed to body $\mathcal{D}$ to the orientation frame a fixed to body $\mathcal{C}$ and is expressed in $[r]$, i.e.,

$$ {}_{[r]}\boldsymbol{r}^{[b]|\mathcal{D},[a]|\mathcal{C}} \sim \text{OrientationCoord}\,([a]|\mathcal{C}, [b]|\mathcal{D}, [r]). $$

The rotation vector representation is a three-dimensional vector $(x \;\; y \;\; z)^T$ containing the coordinates of a vector $\boldsymbol{e}$ along the rotation axis in $[r]$ whose norm is equal to the rotation angle θ. The rotation vector representation does not impose any semantic constraints.

Rotation Matrix

The 3×3 rotation matrix ${}^{[a]|\mathcal{C}}_{[b]|\mathcal{D}}\boldsymbol{R}$ is among the most commonly used coordinate representations of relative orientation [13]. Other names for the rotation matrix are orientation matrix or matrix of direction cosines.

The columns of the 3×3 rotation matrix ${}^{[a]|\mathcal{C}}_{[b]|\mathcal{D}}\boldsymbol{R}$ contain the components of the unit vectors ${}^{a}\boldsymbol{e}_X$, ${}^{a}\boldsymbol{e}_Y$, and ${}^{a}\boldsymbol{e}_Z$ along the axes of orientation frame $[b]$, i.e., expressed in the orientation frame $[b]$:

$$ {}^{[a]|\mathcal{C}}_{[b]|\mathcal{D}}\boldsymbol{R} = (R_{ij}) = \left({}^{a}_{[b]}\boldsymbol{e}_X \quad {}^{a}_{[b]}\boldsymbol{e}_Y \quad {}^{a}_{[b]}\boldsymbol{e}_Z\right), $$

where ${}^{a}_{[b]}\boldsymbol{e}_X$ is notation for the coordinates of the unit vector ${}^{a}\boldsymbol{e}_X$ in the coordinate frame $[b]$. The rotation matrix representation implicitly assumes that the relative orientation of $[a]$ fixed to body $\mathcal{C}$ with respect to $[b]$ fixed to body $\mathcal{D}$ is expressed in orientation frame $[b]$, i.e.,

$$ {}^{[a]|\mathcal{C}}_{[b]|\mathcal{D}}\boldsymbol{R} \sim \text{OrientationCoord}\,([a]|\mathcal{C}, [b]\,|\,\mathcal{D}, [b]), $$

thereby imposing a semantic constraint.

Pose Coordinate Representations

Homogeneous Transformation Matrix

The 4×4 homogeneous transformation matrix ${}^{\{g\}|\mathcal{C}}_{\{h\}|\mathcal{D}}\boldsymbol{T}$ is among the most commonly used coordinate representations of relative pose [13]. Other names for the homogeneous transformation matrix are pose matrix or homogeneous transform.

The homogeneous transformation matrix ${}^{\{g\}|\mathcal{C}}_{\{h\}|\mathcal{D}}\boldsymbol{T}$ combines 1) the position vector ${}_{[h]}\boldsymbol{p}^{h|\mathcal{D}, g|\mathcal{C}}$ of the origin of $\{g\}$ fixed to body $\mathcal{C}$ with respect to the origin of $\{h\}$ fixed to body $\mathcal{D}$ and expressed in $[h]$, plus 2) the rotation matrix ${}^{[g]|\mathcal{C}}_{[h]|\mathcal{D}}\boldsymbol{R}$ of the orientation of $\{g\}$ with respect to orientation of $\{h\}$ into a 4×4 matrix as

$$ {}^{\{g\}|\mathcal{C}}_{\{h\}|\mathcal{D}}\boldsymbol{T} \triangleq \begin{bmatrix} {}^{[g]|\mathcal{C}}_{[h]|\mathcal{D}}\boldsymbol{R} & {}_{[h]}\boldsymbol{p}^{h|\mathcal{D}, g|\mathcal{C}} \\ 0_{1\times 3} & 1 \end{bmatrix}. \tag{13} $$

The homogeneous transformation matrix representation implicitly assumes that the origin and the orientation of the same frame on the body and reference body are used, and that the relative pose is expressed in the orientation frame fixed to the reference body, i.e.,

$$ {}^{\{g\}|\mathcal{C}}_{\{h\}|\mathcal{D}}\boldsymbol{T} \sim \text{PoseCoord}(\{g\}|\,\mathcal{C}, \{h\}\,|\mathcal{D}, [h]), $$

thereby imposing three semantic constraints.

Screw Axis

The screw axis (SA) of an Euclidean displacement in threedimensional space is a line that is simultaneously the rotation axis (see the "Euler Axis-Angle and Rotation Vector" section) and the line along which the translation occurs. Other names for the SA are helical axis or twist axis. Chasles' theorem [5], [13] states that each rigid-body displacement in three-dimensional space has a screw axis, and that this displacement can be decomposed into a rotation about and a translation along this screw axis.

The screw axis coordinate representation consists of a set of Plücker coordinates [8], [9], [13] $(\boldsymbol{d} \quad \boldsymbol{m})^T$ that are used to locate the screw axis in space (consisting of two three-dimensional vectors $\boldsymbol{d}$ and $\boldsymbol{m}$ that identify the direction and location of the axis, respectively), the rotation angle θ, and the displacement d: $(\boldsymbol{d} \;\; \boldsymbol{m} \;\; \theta \;\; d)^T$.

The SA coordinate representation implicitly assumes that the origin and the orientation of the same frame on the body and reference body are used, i.e.,

$$ {}^{\{g\}|\mathcal{C}}_{[r]}\boldsymbol{SA}_{\{h\}|\mathcal{D}} \sim \text{PoseCoord}\,(\{g\}\,|\mathcal{C}, \{h\}\,|\mathcal{D}, [\,r\,]), $$

thereby imposing two semantic constraints.

Linear Velocity Coordinate Representations

To express the relative linear velocity of two bodies $\mathcal{C}$ and $\mathcal{D}$, one can choose a point on body $\mathcal{C}$ and express its linear velocity with respect $\mathcal{D}$ using a three-dimensional vector [6].

The linear velocity vector, ${}_{[r]}\dot{\boldsymbol{p}}^{\mathcal{D},e|\mathcal{C}}$, is the derivative of the position vector, ${}_{[r]}\boldsymbol{p}^{\forall f|\mathcal{D},e|\mathcal{C}}$, pointing from any point f fixed to body $\mathcal{D}$ to the point e fixed to body $\mathcal{C}$ and expressed in $[r]$, i.e.,

$$ {}_{[r]}\dot{\boldsymbol{p}}^{\mathcal{D},e|\mathcal{C}} \sim \text{LinearVelocityCoord}\,(e\,|\mathcal{C}, \mathcal{D}, [\,r\,]). $$

The linear velocity vector does not impose any semantic constraints. The linear velocity vector is a three-dimensional vector $(x \;\; y \;\; z)^T$ containing the coordinates in $[r]$ of the derivative of the vector pointing from any point fixed to body $\mathcal{D}$ to the point e fixed to body $\mathcal{C}$.

Angular Velocity Coordinate Representations

Angular Velocity Vector

The relative angular velocity orientation between two bodies can be expressed using a single angular velocity about some axis with a certain angular velocity [6].

The angular velocity vector coordinate representation ${}^{\mathcal{C}}_{[r]}\omega_{\mathcal{D}}$ represents the rotation axis and angular velocity at which any orientation frame fixed to body $\mathcal{C}$ rotates relative to any orientation frame fixed to body $\mathcal{D}$, and is expressed in $[r]$, i.e.,

$$ {}^{\mathcal{C}}_{[r]}\boldsymbol{\omega}_{\mathcal{D}} \sim \text{AngularVelocityCoord}\,(\mathcal{C}, \mathcal{D}, [r]). $$

The angular velocity vector representation is a three-dimensional vector $(\omega_x \;\; \omega_y \;\; \omega_z)^T$ containing the coordinates of a vector $\boldsymbol{\omega}$ along the rotation axis in $[r]$ whose norm is equal to the angular velocity ω. The angular velocity vector representation does not impose any semantic constraints.

Rotation Matrix Time Derivative

The rotation matrix time derivative, ${}^{\mathcal{C}}_{[b]|\mathcal{D}}\dot{\boldsymbol{R}}$, is the derivative of the rotation matrix, ${}^{[\forall a]|\mathcal{C}}_{[b]|\mathcal{D}}\boldsymbol{R}$, expressing the relative orientation of orientation frame $[a]$ fixed to body $\mathcal{C}$ with respect to any orientation frame fixed to body $\mathcal{D}$ and expressed in orientation frame $[b]$, i.e.,

$$ {}^{\mathcal{C}}_{[b]|\mathcal{D}}\dot{\boldsymbol{R}} \sim \text{AngularVelocityCoord}\,(\mathcal{C}, \mathcal{D}, [\,b\,]). $$

The rotation matrix time derivative imposes the same semantic constraint as the rotation matrix, i.e., that the coordinate frame is equal to the orientation frame fixed to body $\mathcal{D}$.

Twist Coordinate Representations

Six-Dimensional Vector Twist

A six-dimensional vector twist ${}^{e|C}_{[r]}\mathbf{t}_{\mathcal{D}}$ [6] combines 1) the angular velocity vector ${}^{C}_{[r]}\boldsymbol{\omega}_{\mathcal{D}}$ of body C with respect to body $\mathcal{D}$ expressed in $[r]$, plus 2) the linear velocity vector ${}_{[r]}\dot{\boldsymbol{p}}^{\mathcal{D},\, e|C}$ of a point e fixed to body C with respect to body $\mathcal{D}$ expressed in $[r]$ into a six-dimensional vector as

$$ {}^{e|C}_{[r]}\mathbf{t}_{\mathcal{D}} = \begin{pmatrix} {}^{C}_{[r]}\boldsymbol{\omega}_{\mathcal{D}} \\ {}_{[r]}\dot{\boldsymbol{p}}^{\mathcal{D},\, e|C} \end{pmatrix}. \tag{14} $$

The six-dimensional vector twist semantically corresponds to

$$ {}^{e|C}_{[r]}\mathbf{t}_{\mathcal{D}} \sim \text{TwistCoord}\,(\,e\,|C, \mathcal{D}, [\,r\,]), $$

and does not impose semantic constraints.

Instantaneous Screw Axis

The instantaneous screw axis (ISA) of a twist is a line around which the angular velocity occurs and the line along which the linear velocity occurs, and can be seen as a limit case of the screw axis (see the "Screw Axis" section) for infinitesimal displacements.

The instantaneous screw axis coordinate representation consists of a set of Plücker coordinates [8], [9], [13] $(\boldsymbol{d}\ \ \boldsymbol{m})^T$ that are used to locate the instantaneous screw axis in space (consisting of two three-dimensional vectors $\boldsymbol{d}$ and $\boldsymbol{m}$ that identify the direction and location of the axis, respectively), the angular velocity ω, and the linear velocity v, $(\boldsymbol{d}\ \ \boldsymbol{m}\ \ \omega\ \ v)^T$.

The ISA coordinate representation expresses the linear velocity of a point s on the screw axis, i.e.,

$$ {}^{C}_{[r]}\mathbf{ISA}_{\mathcal{D}} \sim \text{TwistCoord}\,(\,s\,|C, \mathcal{D}, [\,r\,]), $$

thereby imposing a semantic constraint.

Forces, Torques, and Wrenches

Screw theory [2], [3], the algebra and calculus of pairs of vectors that arise in the kinematics and dynamics of rigid bodies, shows the parallel between wrenches, consisting of the torque and force vectors, and twists, consisting of linear and angular velocity vectors.

The parallelism between linear, angular velocity, and twist, on the one hand, and torque, force, and wrench, on the other hand, is directly reflected in the semantic representation (see supplemental material Section D, Table V) and the coordinate representations.

Discussion and Conclusion

In this article, we described the complete semantics underlying the rigid-body geometric relations of position, orientation, pose, linear velocity, angular velocity, and twist, including all the choices to be made when specifying these geometric relations. This clear definition of the semantics serves as a proposal for standardization, forcing researchers and application developers to reveal all the hidden assumptions in their geometric rigid-body relations. This article illustrated the usefulness of the proposed semantics using several examples from robotics.

The proposed semantics allows to develop software for geometric operations that include semantic checks. This will avoid commonly made errors and hence reduce application (and, especially, system integration) development time considerably. We make concrete suggestions for semantic interfaces for geometric operation software libraries.

Multimedia

To download the supplemental material mentioned in this tutorial, see it on IEEE *Xplore* and click on the multimedia icon.

Acknowledgment

The authors gratefully acknowledge the financial support by KU Leuven's Concerted Research Action GOA/2005/010 and GOA/2010/011, the KU Leuven-BOF PFV/10/002 Center-of-Excellence Optimization in Engineering (OPTEC), and the European FP7 projects 2008-ICT-230902-ROSETTA and 2008-ICT-231940-BRICS. Tinne De Laet is a postdoctoral fellow of the Fund for Scientific Research–Flanders (F.W.O.) in Belgium.

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