Equations (3)–(5), (8), and either (6) or (7), since $V_{\tilde{x}}(t,\cdot,0)=V_{\tilde{x}}^T(t,0,\cdot)$, form the optimal filter which can be processed by specifying the initial conditions $\hat{x}(t_0,\sigma)$ and $V_{\tilde{x}}(t_0,\sigma,\alpha)$, $0\leqslant\sigma$, $\alpha\leqslant\alpha_N$.

If, on the other hand, Liang's (10) and (11) are applied to the system defined by (1) and (2), one gets

$$\frac{d\hat{x}(t,0)}{dt}=\sum_{i=0}^{N}F_i(t-\alpha_i)\hat{x}(t,\alpha_i)+K(t,0,t)[z(t)-H(t)\hat{x}(t,0)] \quad (12)$$

$$\frac{dV_{\tilde{x}}(t,0,0)}{dt}=\sum_{i=0}^{N}F_i(t-\alpha_i)V_{\tilde{x}}(t,\alpha_i,0)+\sum_{i=0}^{N}V_{\tilde{x}}(t,0,\alpha_i)F_i^T(t-\alpha_i)+G(t)\Psi_w(t)G^T(t)-V_{\tilde{x}}(t,0,0)H^T(t)\Psi_v^{-1}(t)H(t)V_{\tilde{x}}(t,0,0). \quad (13)$$

Liang's (12) is incorrect because the relations describing a system, such as (1) and (2), as well as the estimate and estimation error equations, are all valid for $t\geqslant t_0$ (see (4), (7) and (8)).[1] Therefore, introducing a delay, say α, in any of these equations, which Liang has done, simply implies that these equations are now valid for $t-\alpha\geqslant t_0$ instead of $t\geqslant t_0$. Further, using this procedure, one obtains the relation for the estimation error $\tilde{x}(t-\alpha,0)=\{x(t-\alpha)-\hat{x}(t-\alpha,0)\}$ for $t-\alpha\geqslant t_0$. However, the equation for estimation error needed, besides the one for $\tilde{x}(t,0)=\{x(t)-\hat{x}(t,0)\}$, to derive the expression for the evolution of covariance $V_{\tilde{x}}(t,0,\alpha)$, is $\tilde{x}(t,\alpha)=\{x(t-\alpha)-\hat{x}(t,\alpha)\}$ for $t\geqslant t_0$. This error probably stems from a similar error made in [5] where the relations for computing the smoothed estimate and covariance are inaccurate as reported.

It is evident that (12) and (13) form only a part of the optimal filter given by (3)–(8). Some of the reasons for the inaccuracies are stated in what follows.

Remark 1: Using the following definitions of Liang, which are a bit confusing,

$$\tilde{x}(t-\alpha_i)=x(t-\alpha_i)-\hat{x}(t-\alpha_i) \quad (14)$$

$$\hat{f}_i[\tilde{x}(t-\alpha_i)+\hat{x}(t-\alpha_i),t-\alpha_i]=E\{f_i[\tilde{x}(t-\alpha_i)+\hat{x}(t-\alpha_i),t-\alpha_i]/Z(t)\} \quad (15)$$

when the system is linear, then (15), with the aid of (14), yields

$$\hat{f}_i[\tilde{x}(t-\alpha_i)+\hat{x}(t-\alpha_i),t-\alpha_i]=E\{F_i(t-\alpha_i)x(t-\alpha_i)/Z(t)\}=F_i(t-\alpha_i)\hat{x}(t-\alpha_i/t)=F_i(t-\alpha_i)\hat{x}(t,\alpha_i) \quad (16)$$

rather than

$$\hat{f}_i[\tilde{x}(t-\alpha_i)+\hat{x}(t-\alpha_i),t-\alpha_i]=F_i(t-\alpha_i)\hat{x}(t-\alpha_i/t-\alpha_i)=F_i(t-\alpha_i)\hat{x}(t-\alpha_i,0) \quad (17)$$

for $i=1,2,\cdots,N$. Evidently, the results such as given by (17) have been employed in arriving at the estimation schemes.[1]

Remark 2: Keeping in view the above remark, it is obvious from Liang's algorithms (see also [5, p. 111]) that no provision is made for computing the smoothed estimate $\hat{x}(t,\sigma)$ and covariance $V_{\tilde{x}}(t,\sigma,\alpha)$, $0<\sigma$, $\alpha\leqslant\alpha_N$. This is due to the fact that the cost function which should have been minimized is [1], [4]

$$J(t,\sigma)=E\{(x(t-\sigma)-\hat{x}(t,\sigma))^T(x(t-\sigma)-\hat{x}(t,\sigma))\};\qquad 0\leqslant\sigma\leqslant\alpha_N \quad (18)$$

instead of

$$J(t)=E\{(x(t)-\hat{x}(t,0))^T(x(t)-\hat{x}(t,0))\}. \quad (19)$$

Here, without any loss of generality, the weighting matrix is taken to be an identity matrix.

Remark 3: In contrast to nondelay systems, which are governed by the conditional probability density function $p(x;t/Z(t))$, time delay systems are characterized by the conditional probability density functional $p(x(-s),0\leqslant s\leqslant\alpha_N;t/Z(t))$ [6]. Therefore, trying to manipulate the conditional probability density function for systems without time delays to yield results for delay systems contributes to most of the inaccuracies in the results presented by Liang. Moreover, if the system does not involve any delay, then the estimator for time delay systems must reduce to a combined filter and smoother [1], [6] but not just to the filter for nondelay systems.

Remark 4: Since the suboptimal filters for nonlinear continuous systems, as well as the optimal filter for linear systems with time delays have been previously reported in the literature [1]–[4], Liang would have realized that his results were in error had he referred to these publications, or, at least, to [1].

Remark 5: Liang erred in stating that the results for correlated state and observation noise processes are not available [6]–[8]. With regard to the algorithm for correlated noise processes, it has the same drawbacks as the one for the uncorrelated case, and hence the preceding remarks are also applicable.

Remark 6: The estimators of Liang, excluding (12)[1], are of a similar type as the suboptimal algorithm presented in [9], because the implicit constraint[1] is that the delayed states $x(t-\alpha_i)$, $i=1,2,\cdots,N$, act as forcing functions with

$$x(t-\alpha_i)\simeq\hat{x}(t-\alpha_i,0). \quad (20)$$

The estimators therefore are not exact and/or new but grossly suboptimal even for the linear case. This is also the reason for the estimators to agree with the ones for systems involving no delay.

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Correction to "An Invalid Norm Appearing in Control and Estimation"

T. H. KERR

Abstract—Two counterexamples are presented in the above correspondence [1] to demonstrate that $\|A\|_3$, the minimum of the standard column-sum norm ($\|A\|_1$) and the row-sum norm ($\|A\|_2$), is not a valid norm. While these observations are correct, there is an error in [1] in the argument used to justify the use of $\|A\|_3$ as a convergence test. This error is corrected here.

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The author is with The Analytic Sciences Corporation (TASC), Reading, MA 01867.

For a valid justification that the use of $\|A\|_3$ in a convergence test is acceptable (where $\|A\|_3 \triangleq \min\{\|A\|_1, \|A\|_2\}$), the three contiguous sentences following (14) in [1] that discuss the Hilbert norm (or induced Euclidean norm) $\|A\|_4$ should properly read as follows.

From [2] or [3, p. 183], the Hilbert norm $\|A\|_4$ can be upper bounded by both

$$\|A\|_4 \leqslant n^{1/2} \cdot \|A\|_1 \quad \text{for all } A \tag{15}$$

$$\|A\|_4 \leqslant n^{1/2} \cdot \|A\|_2 \quad \text{for all } A \tag{16}$$

(where n is the dimension of the square matrix A). From the above two inequalities, the following upper bound may be inferred.

$$\|A\|_4 \leqslant n^{1/2} \cdot \min\{\|A\|_1, \|A\|_2\} \quad \text{for all } A \tag{17}$$

(i.e., the minimum of two upper bounds is again an upper bound). Thus, it can be guaranteed via (17) that if either $\|A\|_1$ or $\|A\|_2$ is less than some specified criterion, say $\epsilon' = \epsilon / n^{1/2}$, then $\|A\|_4$ is also less then ϵ.

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Book Reviews

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Brian F. Doolin
Associate Editor—Book Reviews
Flight Dynamics and Contr. Branch, 211-1
NASA Ames Res. Cen.
Moffett Field, CA 94035

System Theory: A Unified State-Space Approach to Continuous and Discrete-Time Systems—L. Padulo and M. A. Arbib (Philadelphia: Saunders, 1974, 799 pp.). *Reviewed by P. S. Krishnaprasad.*

P. S. Krishnaprasad received the Ph.D. degree from Harvard University, Cambridge, MA. Currently he is an Assistant Professor in the Department of Systems Engineering, Computer Engineering, and Information Sciences at Case Western Reserve University, Cleveland, OH. His interests are in geometric and algebraic system theory and identification problems.

In the past decade, system theory has taken various paths of development. One may identify among these the completion of Kalman's module-theoretic approach, generalizations to various algebras and category-theoretic foundations, the structure theory of linear systems in its various forms due to Wonham and Morse, Brunovsky, Rosenbrock, Popov and Kalman, the pioneering work on nonlinear systems due to Brockett, Hermann, Sussmann and others, and more recently the differential and algebraic-geometric investigations motivated by the identification problem. As new facts are being added at an exciting pace, it becomes increasingly necessary to provide textbook treatments of the subject that unify the various results, both for pedagogical reasons and to reveal common threads of ideas that may otherwise be overlooked. The work of Padulo and Arbib is an introductory book on system theory written with this aim in mind. At the time of writing this review, this book had been available for four years. Before then, since 1970 or so, researchers and students in the field have had access to the standard treatises [1]–[3] and the monograph [4]. We make here some inevitable comparisons.

First, the authors' approach is unusual in that they have set out to include practically all the relevant mathematical background either as part of the text or as guided exercises. This partly explains the length of the book. Secondly, they make effective use of discrete-time systems to illustrate the basic ideas before transferring these to a continuous-time setting.

The first chapter introduces the concept of state and defines in crisp notation the notions of stationarity, causality, and system representation.

The second chapter deals with generators of state transition. Here the authors also introduce the elements of linear algebra to describe smooth systems with several degrees of freedom. The examples used are mostly from circuit theory and elementary mechanics.

The third chapter begins with the idea of input–output map and linearity. The authors then define linearization of maps and apply it to systems.

The fourth chapter begins with the fundamental ideas of control. Taking an automata-theorist's viewpoint, the authors give most general definitions of the concepts of controllability, reachability, and distinguishability, relate these to system interconnections and give tests in terms of system response functions. They then specialize these concepts to discrete-time linear systems. The chapter closes with a discussion of discrete-time transfer functions. All the relevant results on linear algebra and normed spaces used in proofs here are developed in some detail.

The fifth chapter is on continuous-time systems. Beginning with the contraction mapping fixed-point theorem, the authors establish the basic result on integration of ordinary differential equations. They then restrict discussion to linear systems and close the chapter with a discussion on equivalence.