# Reviews

# Integral Evaluation Using the Mellin Transform and Generalized Hypergeometric Functions: Tutorial and Applications to Antenna Problems

George Fikioris, *Senior Member, IEEE*

***Abstract*—This is a tutorial presentation of the Mellin-transform (MT) method for the exact calculation of one-dimensional definite integrals, and an illustration of the application of this method to antenna/electromagnetics problems. Once the basics have been mastered, one quickly realizes that the MT-method is extremely powerful, often yielding closed-form expressions very difficult to come up with other methods or to deduce from the usual tables of integrals. Yet, as opposed to other methods, the MT-method is very straightforward to apply; it usually requires laborious calculations, but little ingenuity. In fact, the MT-method is used by Mathematica to symbolically calculate definite integrals. The first part of this paper is a step-by-step tutorial, proceeding from first principles. It includes basic information on Mellin-Barnes integrals and generalized hypergeometric functions, and summarizes the key ideas of the MT-method. In the remaining parts, the MT-method is applied to three examples from the antenna area. The results here are believed to be new, at least in the antenna/electromagnetics literature. In our first example, we obtain a closed-form expression, as a generalized hypergeometric function, for the power radiated by a constant-current circular-loop antenna; this quantity has been extensively discussed recently. Our second example concerns the admittance of a 2-D slot antenna. In both these examples, the exact closed-form expressions are applied to improve upon existing formulas in standard antenna textbooks. In our third example, finally, we obtain a very simple expression for an integral arising in recent, unpublished studies of unbounded, biaxially anisotropic media.**

***Index Terms*—Antenna theory, integration (mathematics), Mellin transforms.**

## I. Introduction

A QUICK look through any advanced antenna textbook, or through any issue of these TRANSACTIONS, reveals that definite integrals play an important role in theoretical antenna and electromagnetics studies. In radiation problems, workers in the field integrate along wire antennas, over planar apertures, and over enclosing surfaces; they obtain exact solutions to differential equations using integral transforms, and solve integral equations approximately using projection methods, such as Galerkin's method, which introduce additional integrations. More often than not, the integrals encountered are complicated and one resorts to numerical-integration techniques. But investigating whether the integrals can be evaluated analytically is always worth some effort: Closed-form expressions are usually preferable for numerical calculations, especially when the expressions involve special functions computable by packaged routines. Furthermore, such expressions can be useful for further analytical work and can provide better physical insight. This paper reviews a powerful technique for evaluating one-dimensional definite integrals exactly, applies this technique to certain antenna/electromagnetics problems, and uses the resulting closed-form expressions to obtain further interesting, physically revealing results.

The power of the technique is evidenced by the fact that it is used by modern packages that perform symbolic integration. In particular, it forms an important part of Mathematica's[1] routine `Integrate[]` which, according to S. Wolfram [1], "can evaluate $(\ldots)$ most definite integrals listed in standard books of tables." Furthermore, the technique "has been used in an essential manner" [2] for the creation of the integral tables in the monumental, three-volume reference work [3]–[5]. We call the technique the "Mellin-transform method" (MT-method), because taking a Mellin transform (MT) is the method's initial and key step. But it is known in the literature [1] with other names, such as the Marichev-Adamchik method [2], [6]–[8].

A key feature of the MT-method is that it often provides results in terms of generalized hypergeometric functions $({}_pF_q)$ or, more generally, in terms of Meijer $G$-functions or Mellin-Barnes (MB) integrals. The ${}_pF_q$, which are defined by convergent series, can often be re-written in terms of simpler special functions; thus, the ${}_pF_q$ are frequently—but not always—just a convenient intermediate step. The same is true for the $G$-function, whose definition is more involved and intimately related to the MT. However, even expressions involving ${}_pF_q$ can nowadays be very useful for numerical calculations:

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The author is with the Department of Electrical and Computer Engineering, National Technical University of Athens, GR 157-73 Zografou, Athens, Greece (e-mail: gfiki@cc.ece.ntua.gr)

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[1]Mathematica is a registered product of Wolfram Research, Inc., Champaign, IL 61820-7237 USA.

For the numerical computation of ${}_pF_q$, today's packaged routines use sophisticated methods and do not rely exclusively on the aforementioned series definition. Such routines—which will, hopefully, further improve in the near future—can be used as black boxes by the user. Today, packaged routines exist even for the more general $G$-function.

Why take the trouble to learn the MT-method? Why not just use modern symbolic integration routines? Given an integral, one can (and, in the author's opinion, should!) first attempt evaluation with Mathematica or other packages. One should also try lookup in integral tables. Nevertheless, as our paper will explicate, learning the method is worthwhile for a number of reasons: i) The method is (once the basics have been mastered) easy to apply. ii) We can sometimes combine the MT-method with additional manipulations to yield further useful results. iii) Learning the MT-method serves as an excellent introduction to the ${}_pF_q$ and even more so, to the $G$-function. Thus, familiarity with the method can help us appreciate and understand our answers. iv) Intermediate expressions (especially the expression as a MB-integral) can greatly help further analytical work including, in particular, asymptotic analysis. Thus sometimes one is interested in more than just the final result. v) Finally, many workers (the author included!) simply like to check their results independently. This is especially true when the integrals contain several parameters upon which the form of the answer depends.

It is worth mentioning that, even in the primarily mathematical literature, the MT-method is often underutilized. For example, it is barely mentioned in D. Zwillinger's 1992 *Handbook of Integration* [9]. As another example, when discussing the technique (in the very much related context of asymptotic expansions), M. J. Ablowitz and A. S. Fokas ([10, p. 504]) state, "This method, although often quick and easy to apply, is not widely known."

The first part of this paper, Section II, is a tutorial on the MT-method, concluding with a first example of the method's application. Section II proceeds from first principles and includes a review of the gamma function, the Mellin transform, the ${}_pF_q$, and MB-integrals. Our treatment here is introductory: We pay little attention to generality or to the many mathematical subtleties of our subject. In particular, we do not provide validity conditions for formulas involving general functions. Section II is, necessarily, not very different from other introductory treatments of the subject [8], [11]–[13]. Some elementary complex analysis, including the idea of analytic continuation, is a prerequisite for understanding Section II which, otherwise, is self-contained. The next two sections present two original example-integrals, arising in antenna problems (loop, microstrip, and aperture antennas), to which the MT-method is applied. In each example, we give background information (theory and/or applications), use the MT-method to evaluate the relevant integrals exactly and discuss, apply, or interpret the exact results. Section V presents a similar discussion for a certain integral arising in unpublished studies on biaxially anisotropic media; our treatment here combines the MT-method with additional manipulations and is more advanced. Many of the derivations in this paper are somewhat sketchy; an interested reader will certainly benefit by working out omitted details.

## II. Tutorial on the MT-Method

Let $f(x)$ denote a complex-valued function of the real, positive variable $x$. The MT of $f(x)$ will be denoted by $\tilde{f}(z)$ and, alternatively, by the more complete notation $\mathrm{MT}\{f(x);z\}$. The definition of the MT involves an integral

$$\tilde{f}(z) = \mathrm{MT}\{f(x);z\} = \int_0^\infty x^{z-1} f(x)\,dx. \tag{1}$$

The new variable $z$, which is taken to be complex, must be restricted to those values for which the integral converges. In general, we have convergence at $x=0$ only if $\mathrm{Re}\{z\}$ is larger than a certain value, and at $x=\infty$ only if $\mathrm{Re}\{z\}$ is smaller than a certain value. Thus, if the MT of $f(x)$ [as defined in (1)] exists at all, it exists in a vertical strip in the complex $z$-plane. Furthermore, under mild conditions on $f(x)$, it can be shown ([6], p. 39) that $\tilde{f}(z)$ is an analytic function of $z$ in that strip. The strip will be referred to by the term "strip of initial definition" (SID). For the application considered here, analytic continuation of $\tilde{f}(z)$ to other complex values of $z$ is always necessary.

### A. Mellin Transform: Basic Properties

We now turn to some properties of the MT. First of all, the MT is related to other, more usual, transforms. For example, if $\mathrm{FT}\{f(x);z\} = \int_{-\infty}^{\infty} f(x)e^{ixz}\,dx$ is the Fourier transform of $f(x)$, one has $\tilde{f}(z) = \mathrm{FT}\{f(e^x);-iz\}$. And if $\mathrm{LT}\{f(x);z\}$ is the usual (one-sided) Laplace transform of $f(x)$ then $\tilde{f}(z) = \mathrm{LT}\{f(e^{-x});z\}+\mathrm{LT}\{f(e^x);-z\}$. Thus, many properties of the LT and the FT can be rephrased for the MT. For instance, the aforementioned analyticity of the MT in a vertical strip can be viewed as a consequence of the well-known analyticity of the LT in a right-half plane.

It can be shown (via the Fourier or Laplace inversion formula) that the inversion formula for the MT is ([6, p. 39]), [11]

$$f(x) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} x^{-z}\tilde{f}(z)\,dz \tag{2}$$

where the integration path is a vertical line in the complex $z$-plane, *lying within the SID,* as shown in Fig. 1. Formula (2) uniquely determines $f(x)$ from $\tilde{f}(z)$. Let $\delta_L < \mathrm{Re}\{z\} < \delta_R$ be the SID of $\tilde{f}(z)$. The reader is invited to show, directly from the definition (1), the following elementary properties of the MT:

$$\mathrm{MT}\{f(\alpha x);z\} = \alpha^{-z}\tilde{f}(z), \qquad \delta_L < \mathrm{Re}\{z\} < \delta_R, \quad \alpha > 0 \tag{3}$$

$$\mathrm{MT}\{x^\alpha f(x);z\} = \tilde{f}(z+\alpha), \qquad \delta_L - \mathrm{Re}\{\alpha\} < \mathrm{Re}\{z\} < \delta_R - \mathrm{Re}\{\alpha\} \tag{4}$$

$$\mathrm{MT}\{f(x^\alpha);z\} = \frac{1}{|\alpha|}\tilde{f}\left(\frac{z}{\alpha}\right) \quad \begin{cases} \alpha\delta_L < \mathrm{Re}\{z\} < \alpha\delta_R, & \text{if } \alpha > 0, \\ \alpha\delta_R < \mathrm{Re}\{z\} < \alpha\delta_L, & \text{if } \alpha < 0. \end{cases} \tag{5}$$

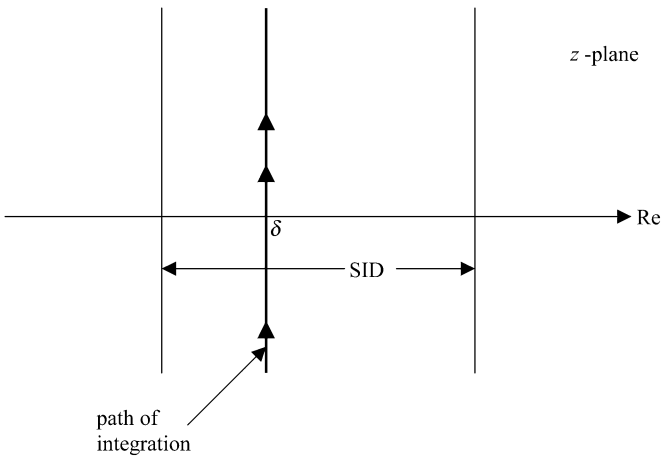


Fig. 1. The integration path in the inversion formula (2) is a vertical line in the complex $z$-plane, lying within the strip of initial definition SID.

### B. Mellin Transform: Parseval Formula and Related Properties

The Fourier or Laplace transform of the product of two functions is given by the convolution of the individual transforms (where convolution is defined differently for the two transforms.) The corresponding statement for the Mellin transform is

$$\int_0^\infty g(y)h(y)y^{z-1}\,dy = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \tilde{g}(w)\tilde{h}(z-w)\,dw \tag{6}$$

in which $\delta$ belongs to the SID of $\tilde{g}(w)$. Once again, the right-hand side (RHS) is a convolution of sorts. To show (6), begin from its left-hand side (LHS), use (2) to introduce $\tilde{g}(w)$, and interchange the order of integration. A slight generalization of (6) is

$$\int_0^\infty g(xy)h(y)y^{z-1}\,dy = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \tilde{g}(w)\tilde{h}(z-w)x^{-w}\,dw, \quad x>0 \tag{7}$$

which is a combination of (6) and (3). The special case $z=1$ of (7) is

$$\int_0^\infty g(xy)h(y)\,dy = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \tilde{g}(z)\tilde{h}(1-z)x^{-z}\,dz \tag{8}$$

in which $\delta$ belongs both to the SID of $\tilde{g}(z)$ and to the SID of $\tilde{h}(1-z)$—for (8) to hold, it is necessary that these two SIDs overlap. Note that the special case $x=1$ of (8) is what is usually called the Parseval formula for the MT.

Formula (8) forms the core of the MT-method. It is worth re-stating (8) somewhat differently, and outlining an alternative derivation. The operation on the LHS is the so-called Mellin convolution of the two functions $g(x)$ and $h(x)$; we (but not all authors!) denote it by $(g \oslash h)(x)$ so that, by definition

$$(g \oslash h)(x) = \int_0^\infty g(xy)h(y)\,dy, \quad x>0. \tag{9}$$

The fundamental difference from the more usual types of convolution is that the product $xy$, not the difference $x-y$, is the argument of one of the two integrand functions. By virtue of the inversion formula, we can re-write (8) and (9) as

$$\mathrm{MT}\{(g \oslash h)(x); z\} = \tilde{g}(z)\tilde{h}(1-z) \tag{10}$$

which can also be shown directly from (9), (3), and (1), with no recourse to an inversion formula. In the RHS, one has a product of MTs (one of them is actually reflected and translated), so that formula (10) is, in a certain sense, the reverse of (6), with the Mellin convolution in the original ($x$-) domain and a product in the transform ($z$-) domain.

### C. Gamma Function; Psi Function; Pochhammer's Symbol

The gamma function $\Gamma(z)$ is defined as the MT of $e^{-x}$, viz.

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx, \quad \mathrm{Re}\{z\}>0. \tag{11}$$

The restriction $\mathrm{Re}\{z\}>0$, necessary for convergence of the integral at $x=0$, means that the SID is, in this case, the entire right-half complex $z$-plane. It can be shown that

$$\Gamma(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{1}{z+n} + \int_1^\infty x^{z-1}e^{-x}\,dx \qquad z \neq 0,-1,-2,\ldots \tag{12}$$

(split (11) into $\int_0^1 + \int_1^\infty$; expand $e^{-x}$ into Taylor series in first integral; integrate term-by-term.) In the derivation of (12), it was assumed that $\mathrm{Re}\{z\}>0$. However, the RHS of (12) is analytic for all $z$ except $z=0,-1,\ldots$. Thus, (12) provides the analytic continuation of $\Gamma(z)$ to (complex) values of $z$ for which the defining integral (11) did not make sense.

The gamma function has many properties. We give the ones most useful for the MT-method. From (12), it is seen that, at $z=0,-1,-2,\ldots,-n,\ldots,\Gamma(z)$ has simple poles with corresponding residues $1,-1,1/2,\ldots,(-1)^n/n!,\ldots$. The recurrence formula

$$\Gamma(z+1) = z\Gamma(z) \tag{13}$$

is easily shown by integrating (11) by parts. With $\Gamma(1)=1$, it follows that

$$\Gamma(n+1) = n!, \quad n=0,1,2,\ldots. \tag{14}$$

It is possible to show [14] that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{15}$$

which is called the reflection formula. As a consequence of (15), $\Gamma(1/2) = \sqrt{\pi}$. With the recurrence formula, one can further determine the values $\Gamma(3/2), \Gamma(5/2), \ldots$. As an additional consequence of (15), $1/\Gamma(z)$ is analytic for all $z$. That is, $\Gamma(z)$ has no zeros. The familiar Stirling's formula [14] is an asymptotic approximation to $\Gamma(z)$ for large, complex arguments. Finally, the duplication formula [14] is

$$\Gamma(2z) = \frac{1}{2\sqrt{\pi}} 2^{2z} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right). \tag{16}$$

The derivative of $\Gamma(z)$ is usually computed through the psi function, defined by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{17}$$

It can be shown [14] that

$$\psi(n+1) = -\gamma + \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right), \quad n = 0, 1, 2, \ldots. \tag{18}$$

where the quantity within parentheses is to be interpreted as 0 when $n = 0$ and where $\gamma = -\psi(1) \simeq 0.57721567$ is Euler's constant. We finally define Pochhammer's symbol by

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad n = 0, 1, 2, \ldots. \tag{19}$$

This symbol, which will be used in the definition of the ${}_pF_q$, satisfies the recursion formula

$$(z)_{n+1} = (z+n)(z)_n, \quad n = 0, 1, 2, \ldots. \tag{20}$$

### *D. Simple Applications of Mellin Transforms and the Gamma Function*

To familiarize readers with the foregoing material, and to prepare for what follows, we now provide simple examples involving the MT and $\Gamma(z)$.

*Application 1:* Let us calculate $\mathrm{MT}\{e^{-x} - 1; z\}$, and directly verify the inversion formula.

With $f(x) = e^{-x} - 1$, the integrand of (1) behaves like $x^z$ as $x \to 0$ and like $x^{z-1}$ as $x \to \infty$. Thus, the integral converges if $-1 < \mathrm{Re}\{z\} < 0$. For such values of $z$

$$\begin{aligned} \mathrm{MT}\{e^{-x} - 1; z\} &= \int_0^\infty (e^{-x} - 1) x^{z-1}\, dx \\ &= \Gamma(z), \quad -1 < \mathrm{Re}\{z\} < 0 \end{aligned} \tag{21}$$

(integrate by parts; identify resulting integral with $\Gamma(z+1)$; use (13).) Thus, the Mellin transforms of $e^{-x}$ and $e^{-x} - 1$ are both $\Gamma(z)$. The two SID, however, do not overlap. Therefore, the two corresponding inversion formulas (2) differ because $\delta$ changes.

To directly verify that $(2\pi i)^{-1} \int_{\delta - i\infty}^{\delta + i\infty} x^{-z} \Gamma(z)\, dz = e^{-x} - 1$ (for $-1 < \delta < 0$; compare with the inversion formula for $\Gamma(z)$), collapse the contour until it wraps around the poles at $z = -1, -2, \ldots$ on the negative real axis—a procedure to be described, somewhat loosely, as "closing the contour at left." As will be discussed (briefly) in Section VI, this procedure is indeed legitimate so that, by the residue theorem

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} x^{-z} \Gamma(z)\, dz \\ &\quad = \sum_{n=1}^{\infty} \mathrm{Res}\{x^{-z}\Gamma(z); z = -n\} \\ &\quad = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x} - 1. \end{aligned} \tag{22}$$

In (22), and throughout this paper, $\mathrm{Res}\{f(z); z = z_0\}$ denotes the residue of $f(z)$ at $z_0$.

*Application 2:* One can easily verify the identity

$$\Gamma(z - n) = (-1)^n \frac{\Gamma(z)}{(1-z)_n}, \quad n = 0, 1, 2 \ldots \tag{23}$$

by induction: The cases $n$ and $n+1$ can be related by (13) and (20).

*Application 3:* For $\alpha \neq 0, x^{-z}\Gamma(\alpha z + \beta)$ has simple poles at $z = p_n$, where

$$p_n = -\frac{\beta + n}{\alpha}, \quad n = 0, 1, 2, \ldots. \tag{24}$$

These poles are equispaced and form a semi-infinite lattice. For the important special case of real $\alpha$, the lattice is parallel to the real axis. For any nonzero $\alpha$, the corresponding residues involve powers of $x$

$$\begin{aligned} &\mathrm{Res}\left\{x^{-z}\Gamma(\alpha z + \beta), z = p_n\right\} \\ &\quad = \frac{1}{\alpha} \frac{(-1)^n}{n!} x^{-p_n} \\ &\quad = \frac{1}{\alpha} \frac{(-1)^n}{\Gamma(n+1)} x^{-p_n}, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{25}$$

In the special case $\alpha > 0$ ($\alpha < 0$), the lattice continues indefinitely to the left (to the right), and the powers of $x$ ascend (descend).

*Application 4:* Let us find the poles and residues of $\Gamma(z)\Gamma(z-1)x^{-z}$.

At $z = 1$, there is a simple pole with residue $1/x$. At $z = 0, -1, -2, \ldots$, there are double poles. The residue at $z = 0$ is found by first writing

$$\begin{aligned} \Gamma(z)\Gamma(z-1)x^{-z} &= \Gamma(z) \frac{\Gamma(z)}{z-1} x^{-z} \\ &= \frac{1}{z^2} \frac{1}{z-1} [\Gamma(z+1)]^2 x^{-z}. \end{aligned} \tag{26}$$

The functions $\frac{1}{z-1}, [\Gamma(z+1)]^2$, and $x^{-z}$ are analytic at $z = 0$ and can be expanded in Taylor series about that point. In particular, $x^{-z} = 1 - z \ln x + (1/2) z^2 (\ln x)^2 + O(z^3)$. When the corresponding series are multiplied, the desired residue is the coefficient of $z$. It is

$$\mathrm{Res}\{\Gamma(z)\Gamma(z-1)x^{-z}; z = 0\} = \ln x + 2\gamma - 1. \tag{27}$$

TABLE I
SELECTED FUNCTIONS $f(x)$, TOGETHER WITH THEIR MTs $\tilde{f}(z)$, AND THE SID OF $\tilde{f}(z)$

| Entry | $f(x)$ | $\tilde{f}(z)$ | SID |
|---|---|---|---|
| 1 | $(x^2+1)^{-1}$ | $\frac{1}{2}\Gamma\left(\frac{z}{2}\right)\Gamma\left(1-\frac{z}{2}\right)$ | $0<\mathrm{Re}\{z\}<2$ |
| 2 | $\begin{cases}(1-x^2)^{-1/2}, 0<x<1\\ 0, x>1\end{cases}$ | $\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{z}{2}+\frac{1}{2}\right)}$ | $\mathrm{Re}\{z\}>0$ |
| 3 | $\begin{cases}0, 0<x<1\\ (x^2-1)^{-1/2}, x>1\end{cases}$ | $\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)}{\Gamma\left(1-\frac{z}{2}\right)}$ | $\mathrm{Re}\{z\}<1$ |
| 4 | $\left(\frac{\sin x}{x}\right)^2$ | $-\frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{z}{2}-1\right)}{\Gamma\left(\frac{3}{2}-\frac{z}{2}\right)}$ | $0<\mathrm{Re}\{z\}<2$ |
| 5 | $J_\nu(x)$ | $\frac{1}{2}\left(\frac{1}{2}\right)^{-z}\frac{\Gamma\left(\frac{\nu}{2}+\frac{z}{2}\right)}{\Gamma\left(1+\frac{\nu}{2}-\frac{z}{2}\right)}$ | $-\mathrm{Re}\{\nu\}<\mathrm{Re}\{z\}<\frac{3}{2}$ |
| 6 | $J_\mu(x)J_\nu(x)$ | $\frac{1}{2}\left(\frac{1}{2}\right)^{-z}\Gamma(1-z)\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}+\frac{z}{2}\right)$ $\times\left[\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}-\frac{z}{2}\right)\Gamma\left(1+\frac{\nu}{2}+\frac{\mu}{2}-\frac{z}{2}\right)\right]^{-1}$ $\times\left[\Gamma\left(1-\frac{\nu}{2}+\frac{\mu}{2}-\frac{z}{2}\right)\right]^{-1}$ | $-\mathrm{Re}\{\nu+\mu\}<\mathrm{Re}\{z\}<1$ |

The residue at any other double pole can be found in a similar manner. For $n=0,1,\ldots$, the reader is invited to show that the final answer is

$$\mathrm{Res}\{\Gamma(z)\Gamma(z-1)x^{-z}; z=-n\} = \frac{x^n}{n!(n+1)!}\left[\ln x - 2\psi(n+1) - \frac{1}{n+1}\right]. \quad (28)$$

Besides powers of $x$, the residues at these double poles also involve the logarithm of $x$. Residue calculations like this are important for the MT-method, so further information is provided in our discussion-Section VI.

## E. Table Lookup of Mellin Transforms; Mellin-Barnes Integrals

When calculating integrals with the MT-method, one needs to find the MTs of functions involved in the integrand. This is usually done using symbolic programs such as Mathematica or Matlab,[2] or published tables of MTs. We present our own short Table I, which shows several functions $f(x)$, their MTs $\tilde{f}(z)$, and the corresponding SIDs. The specific functions have been selected for the simple reason that they will be useful when evaluating our example-integrals.

A primary goal of the MT-method is to find a Mellin-Barnes (MB) integral representation of the integral to be evaluated, so we proceed to discuss MB-integrals. What one immediately observes is that each $\tilde{f}(z)$ of Table I has been written as a product, in which the factors have the form $\Gamma(a+Az)$, $[\Gamma(a+Az)]^{-1}$, or $\alpha^{-z}$, where all $A$'s are real. Let us call this a "standard product." The integrands of the corresponding inversion integrals will also be "standard products" multiplied by $x^{-z}$. For instance, as a consequence of Entry 4 of Table I and (2) and (3), one has

$$\left(\frac{\sin 3x}{3x}\right)^2 = -\frac{\sqrt{\pi}}{4}\frac{1}{2\pi i} \times \int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma\left(\frac{z}{2}-1\right)}{\Gamma\left(\frac{3}{2}-\frac{z}{2}\right)}(3x)^{-z}\,dz, \quad 0<\delta<2. \quad (29)$$

Convergent integrals like the one in the RHS of (29)—with integrands of the aforementioned type, integrated along proper contours in the $z$-plane—are called Mellin-Barnes integrals (MB-integrals) ([6, p. 11], [15, §1.19]). They are very important for us, because *"most" functions $f(x)$ can be written as MB-integrals, or as linear combinations of MB-integrals. In other words, their MTs $\tilde{f}(z)$ can be written as linear combinations of standard products*. (This statement is formulated precisely in [6].)

A standard product can often be written in other "non-standard" forms [simple illustrations of this fact are provided by the reflection and recurrence formulas (15) and (13)]. Since MB-integrals are important for us, *it is preferable to use "standard products" when possible*. This brings us back to discussing published tables of MTs. Many well-known tables—such as the standard table of integrals [16]—do not always express their MTs as standard products, so it is preferable to use tables that so do. By far the most extensive such table is ([5, 8.4]) in the three-volume work by Prudnikov, Brychkov, and Marichev. (We also mention the table in [6], as well as the much shorter tables in [12].)

[2]Matlab is a registered trademark of The MathWorks, Natick, MA.

Table I will be used shortly, when we deal with our example-integrals. For now, the reader may wish to familiarize him/herself with the aforementioned published tables by using them to verify Table I. Note that: i) The table ([5, 8.4]) should be used in conjunction with the MT-properties of Section II-A. For example, Entry 4 of Table I comes from [5, 8.4.5.11] and (5) and (4). ii) Formula [5, 8.4.19.15], which can be used to derive Entry 6, can be simplified with aid of (16), resulting in one less gamma function.

### *F. Generalized Hypergeometric Functions*

The MT-method often gives results in terms of the generalized hypergeometric function ${}_pF_q$. The most striking difference of this "function" with the more usual "special functions of mathematical physics" (the Bessel function $J_\nu$, say) is that ${}_pF_q$ is much more general; it involves many parameters and, depending on their values, often reduces to more usual functions. "Very many" functions (including $J_\nu$) have ${}_pF_q$ representations.

The extensive table ([5, Ch. 7]) can be searched in a systematic manner to see whether a given ${}_pF_q$ can be reduced to a more usual special function. It is obviously very useful for our purposes and will be referred to as the "reduction table" for the ${}_pF_q$. A similar table can be found online [17]. We now proceed to discuss ${}_pF_q$ in more detail.

The generalized hypergeometric series of order $(p,q)$ is defined as a power series in $z$ and is denoted by ${}_pF_q(\alpha_1,\alpha_2,\ldots,\alpha_p;\beta_1,\beta_2,\ldots,\beta_q;z)$. The expressions for the power-series coefficients involve the $p$ numbers $\alpha_l$ and the $q$ numbers $\beta_l (p,q = 0,1,\ldots)$, called upper and lower parameters, respectively. The precise definition is

$$ {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!} \quad (30) $$

where all $\beta_l \neq 0,-1,-2,\ldots$.[3] In (30), and throughout this paper, empty products or sums are to be interpreted, in the usual manner, as unity or zero, respectively. Note that the order is reduced when an upper and a lower parameter are equal.

For the defining series to make sense, it must converge, at least for some $z$. The convergence/divergence can be examined by application of the ratio test for power series and Stirling's formula. Let us first mention that the series *diverges* for all nonzero $z$ (zero radius of convergence) when $p \geq q+2$ ([5, 7.2.3]). On the other hand ([5, 7.2.3]):

*Case 1:* When $p \leq q$, the series converges for all complex $z$ and defines the so-called generalized hypergeometric function.

*Case 2:* When $p = q+1$, the series converges inside the unit $z$-circle and diverges outside, so that the radius of convergence here equals 1. In this case, the generalized hypergeometric function—still denoted by ${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)$—is defined i) by the series (30) when $|z| < 1$; ii) by the analytic continuation of (30) when $|z| \geq 1$.

There now exist packaged routines for the numerical calculation of ${}_pF_q$, which should greatly enhance the use of ${}_pF_q$ in engineering applications. *For numerical computation, today's packaged routines do not rely exclusively on the series definition (30)*[4] but, rather, on the vast number of properties of the ${}_pF_q$. Extensive lists of properties are in [5, Ch. 7] and online in [17]. *When numerical results are of primary concern, it is today often sufficient to express the quantity of interest in terms of ${}_pF_q$, and to use the aforementioned routines as black boxes.* More generally, when one encounters series in theoretical work, *it is always beneficial to attempt to identify the series with a ${}_pF_q$* because of the many tabulated properties of ${}_pF_q$ and because of the ease in numerical evaluation of the ${}_pF_q$.

By definition, the ${}_pF_q$ are *ascending* series in $z$, so the first few terms are often simple approximations for small $|z|$. In physical problems, such approximations can be especially revealing. But a ${}_pF_q$ can help provide simple approximations even for large $|z|$, as large-$|z|$ asymptotic expansions for ${}_pF_q$ have been derived and tabulated [17], [18].

A yet more general function than ${}_pF_q$ is the Meijer-$G$ function. Roughly speaking, the $G$-function is a special type of MB-integral in which all coefficients $A$ of the factors $\Gamma(a + Az), [\Gamma(a + Az)]^{-1}$ are 1 or $-1$. Today, numerical computation of $G$ can be done by packaged routines (in Mathematica 5.0, for example). A definition of $G$ that is usually adequate is provided in [19, §2.1]. The $G$-function representation of ${}_pF_q$ is [5, 8.4.51.1].

### *G. The MT-Method for Integral Evaluation: Basic Ideas*

We finally come to the MT-method itself. It applies to integrals $f(x)$ which are Mellin convolutions. That is, the integral $f(x)$ to be evaluated can be written as

$$ f(x) = \int_0^\infty g(xy)h(y)\,dy = (g \oslash h)(x), \quad x > 0 \quad (31) $$

where $x$ is a positive parameter. Many integrals have this form (all Laplace transforms, for example), or can be written in terms of integrals having this form (all Fourier transforms, for example). We first give a general (but sketchy) description of the MT-method.

*First Step:* Apply formula (8) to obtain a complex-integral representation of $f(x)$, viz.

$$ f(x) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \tilde{g}(z)\tilde{h}(1-z)x^{-z}\,dz \quad (32) $$

which will, hopefully, be a MB-integral. By the discussion in Section II.B, this step is the same as taking the MT of (31) (with respect to $x$), using formula (10) and, finally, the inversion formula (2), in which $\tilde{f}(z) = \tilde{g}(z)\tilde{h}(1-z)$. In (32), $\delta$ belongs both to the SID of $\tilde{g}(z)$ *and* to the SID of $\tilde{h}(1-z)$, which must overlap.

*Second Step:* Very often, straightforward manipulation of (32) yields a $G$-function representation of $f(x)$. In this paper, we prefer to emphasize the following procedure: In (32), determine the singularities to the *left* of the contour, which will hopefully be poles (not necessarily simple), located at $z = z_0, z_1, \ldots$. Then, close the contour at left (we will briefly

[3] This restriction is required because $(-m)_n$ vanishes for sufficiently large $n$; see (23).

[4] To quickly persuade oneself of this, note that both Mathematica and Matlab can handle ${}_pF_q$ when $|z| > 1$ in Case 2.

discuss when this is legitimate in Section VI), and apply the residue theorem to obtain a series representation for $f(x)$

$$f(x) = \sum_{n=0}^{\infty} \mathrm{Res}\left\{\tilde{f}(z)x^{-z}; z = z_n\right\}. \tag{33}$$

Typically, (33) is an *ascending* series expansion. When this is the case, the first few terms of the series are sometimes simple, physically revealing approximations for small $x$. Also, it often happens (but not always, as we will see) that one can identify the series in (33) with a ${}_pF_q$. We now present a simple example illustrating the MT-method.

### H. The MT-Method for Integral Evaluation: A First Example

Consider the integral

$$f(x) = \int_0^1 \left(\frac{\sin xy}{xy}\right)^2 \frac{1}{\sqrt{1-y^2}}\,dy. \tag{34}$$

(In Section IV, we will discuss how such an integral arises in antenna problems and specify the meaning of $x$.) Let us, in this first example, apply the MT-method without omitting details. The integral $f(x)$ can be written as in (31), where

$$g(x) = \left(\frac{\sin x}{x}\right)^2$$
$$h(x) = \begin{cases}(1-x^2)^{-1/2}, & \text{if } 0 < x < 1,\\ 0, & \text{if } x > 1.\end{cases} \tag{35}$$

The MT $\tilde{g}(z)$ can be found directly as Entry 4 of Table I. From Entry 2, we deduce that

$$\tilde{h}(1-z) = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)}{\Gamma\left(1-\frac{z}{2}\right)}, \quad \mathrm{Re}\{z\} < 1. \tag{36}$$

The SID of $\tilde{g}(z)$ and $\tilde{h}(1-z)$ do overlap; the strip of overlap is $0 < \mathrm{Re}\{z\} < 1$ so that (32) gives the following MB-integral representation of $f(x)$:

$$f(x) = -\frac{\pi}{8}\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma\left(\frac{z}{2}-1\right)\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)}{\Gamma\left(1-\frac{z}{2}\right)\Gamma\left(\frac{3}{2}-\frac{z}{2}\right)}x^{-z}\,dz \qquad 0 < \delta < 1. \tag{37}$$

Each gamma function in (37) contributes to the integrand a semi-infinite lattice of poles (if the function is in the numerator), or zeros (if in the denominator). The locations of these poles and zeros can be determined using Application 3 of Section II-D. Evidently, a pole contribution at a specified location cancels a zero contribution at the same location—for example, there is no pole or zero at $z = 3$. It follows that, to the *left* of the inversion path, there are *simple* poles at $0, -2, -4, \ldots$. For reasons to be outlined in Section VI, it is legitimate to close the contour at left. Therefore, (33) is, in our case

$$f(x) = -\frac{\pi}{8}\sum_{n=0}^{\infty}\mathrm{Res}\left\{\frac{\Gamma\left(\frac{z}{2}-1\right)\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)}{\Gamma\left(1-\frac{z}{2}\right)\Gamma\left(\frac{3}{2}-\frac{z}{2}\right)}x^{-z}; z=-2n\right\}. \tag{38}$$

To evaluate the residues, set $z = -2n$ except in $\Gamma(z/2-1)$, to which the poles at $z = -2n = p_{n+1}$ are due. Then, use Application 3 of Section II-D once again

$$f(x) = -\frac{\pi}{8}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}+\frac{2n}{2}\right)}{\Gamma\left(1+\frac{2n}{2}\right)\Gamma\left(\frac{3}{2}+\frac{2n}{2}\right)} \times \mathrm{Res}\left\{\Gamma\left(\frac{z}{2}-1\right)x^{-z}; z = p_{n+1}\right\}$$
$$= \frac{\pi}{4}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma(1+n)\Gamma\left(\frac{3}{2}+n\right)}\frac{(-1)^n}{\Gamma(n+2)}x^{2n}. \tag{39}$$

Now set $\Gamma(1+n) = n!$ and express the three remaining gamma functions in terms of Pochhammer's symbol using (19). By Section II-C, $\Gamma(1) = 1$ and $\Gamma(1/2) = 2\Gamma(3/2) = \sqrt{\pi}$. Finally, compare with the definition (30) of ${}_pF_q$ (Case 1 in Section II-F) to obtain

$$f(x) = \frac{\pi}{2}\,{}_1F_2\left(\frac{1}{2}; \frac{3}{2}, 2; -x^2\right). \tag{40}$$

Equation (40) (or the equivalent series form (39), which is less preferable for numerical calculation by modern routines) is our final result. (Entry 7.14.2.46 of the "reduction table" in [5, Ch. 7] actually provides a "simplified" answer. But that answer involves rather unusual special functions—a Laguerre function and a modified Struve function—so it will not be repeated here.) For completeness, let us also mention that (39) can be obtained by more elementary methods. In (34), expand $\sin^2 xy/(xy)^2$ into its Taylor series (this can be done by writing $\sin^2 xy = (1-\cos 2xy)/2$ and using the well-known series for the cosine). Then, integrate term-by-term, using Entry 2 of Table I, which is a standard tabulated integral.

We now proceed to further antenna problems to which the MT-method can be applied. Most of the final *exact* results that follow (as well as those that precede) have been verified numerically. That is, the final result agrees with numerical evaluation of the original integral. Such checks are always a good idea when possible.

## III. Power Radiated By Certain Circular Antennas

### A. Constant-Current Circular-Loop Antennas

Circular, thin-wire loop antennas are one of the most basic types of radiators and are discussed in standard textbooks, e.g., [20, Ch. 5]. Simple in construction, they are used for frequencies from about 3 MHz up to microwave. Electrically small loops are rather poor radiators (the radiation resistance $R_r$ is usually smaller than the loss resistance) and are therefore used when efficiency is not of primary importance. Large loops have a larger $R_r$ (our result (48) will illuminate this) and are therefore used primarily as elements of directional arrays—such as helical antennas and Yagi-Uda arrays—with the loop circumference and inter-element spacing chosen to achieve the desired directional properties.

Many studies have focused on the case of *constant* loop current $I_0$. Such studies are practically useful for at least two reasons: For sufficiently small loops (and, also, for large inter-element spacings in the case where the loop is an array element) the current is truly constant. Second, constant current distributions can be achieved even for large loops ([20, p. 249]): one divides the loop into sections and feeds each section with a different feed line. Often, all lines are driven from a common source.

An accurate method for determining the field radiated by a constant-current, circular-loop, thin-wire antenna proceeds from the standard integral ([20, (5-14)]) for the vector potential **A**, which is $\phi$-directed. The distance from loop to observation point is approximated ([20, (5-43)]) subject to the usual condition $r \gg a$, where $a$ is the loop radius and $(r, \theta, \phi)$ are spherical coordinates with origin at the loop's center and $z$-axis perpendicular to the loop. This leads to an integral which can be evaluated in terms of the Bessel function $J_1$. The resulting expression is then used in the familiar formulas ([20, §3.6]) relating **A** to the radiated fields. With $\zeta_0 = 120\pi$ ohms, the nonzero components are ([20, (5-54)])

$$E_\phi = -\zeta_0 H_\theta = \frac{I_0 ka\,\zeta_0 e^{-jkr}}{2r} J_1(ka \sin\theta). \qquad (41)$$

The radiated power ([20, (5-58)]), found by integrating over a large sphere, therefore equals $(\pi/2)(ka)^2\zeta_0|I_0|^2 f(ka, 1, 1, 1)$, with the more general integral $f(x, \mu, \nu, \tau)$ defined by

$$f(x,\mu,\nu,\tau) = \int_0^{\pi/2} J_\mu(x\sin\theta) J_\nu(x\sin\theta)\sin^\tau\theta\, d\theta, \qquad x > 0. \qquad (42)$$

The reason for the more general notation will be explained in the next section. Once the radiated power is found, the directivity and radiation resistance $R_r$ easily follow ([20, §5.3.2]).

### B. Circular-Patch Microstrip Antennas; Cavity Model

The cavity model is one of the most popular methods for the analysis of circular microstrip antennas ([20, Ch. 14]). One treats the region between patch and ground plane as a cavity bounded above and below by electric conductors and by a magnetic conductor along the patch's perimeter. Within this model, the radiated power can be shown to be proportional to the quantity ([20, (14-75) and (14-72)])

$$2f(ka,0,0,1) - f(ka,0,0,3) - 2f(ka,0,2,3) + 2f(ka,2,2,1) - f(ka,2,2,3) \qquad (43)$$

where $f(x, \mu, \nu, \tau)$ was defined in (42), and where $a$ is the "effective radius" ([20, (14-67)]) of the patch. Details of the derivation of (43) are provided in [20, §14.3], [21], and [22]. With the radiated power determined, one can immediately find the directivity ([20, (14-80)]).

The integral $f$ in (42) thus comes up in at least two antenna problems. $f$ has deserved a great deal of attention. Recently, $f$ has been the subject of much discussion in the *IEEE Antennas and Propagation Magazine* [23]–[30]. Some of these papers are referred to in the 3rd (2005) edition of the standard textbook [20]. For the loop antenna ($\mu = \nu = \tau = 1$), [20] proposes numerical evaluation of (42) and provides a computer program for doing so. [26] mentions an additional application in which $f$ arises, namely, the circular loop with a co-sinusoidal current. Finally, the exact evaluation and/or the asymptotics of $f$ (more precisely, of special or of more general cases of $f$) have been much discussed in other (more mathematical) contexts [31]–[37]. In the next section, we provide a closed-form expression for $f$, in terms of a ${}_pF_q$, by straightforward application of the MT-method.

### C. Integral Evaluation

Change the variable $\sin\theta = y$ in (42) to obtain the more suitable expression

$$f(x,\mu,\nu,\tau) = \int_0^1 \frac{J_\mu(xy)J_\nu(xy)}{\sqrt{1-y^2}} y^\tau\, dy, \quad x > 0 \qquad (44)$$

which is (31) with $g(x) = J_\mu(x)J_\nu(x)$, $h(x) = x^\tau(1-x^2)^{-1/2}$ for $0 < x < 1$, and $h(x) = 0$ for $x > 1$. To avoid unnecessary complications, let us assume that the complex quantities $\mu, \nu$ and $\tau$ have nonnegative real parts. Because of the Bessel functions, (44) might appear more difficult than our previous example (34). With the MTs $\tilde{g}(z)$ and $\tilde{h}(z)$ obtainable from Table I and (4), however, the MT-method can be applied just as before. We omit lengthy intermediate formulas, and directly give the result as a MB-integral

$$\begin{aligned} f(x,\mu,\nu,\tau) &= \frac{\sqrt{\pi}}{4}\frac{1}{2\pi i} \\ &\times \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(1-z)\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}+\frac{z}{2}\right)}{\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}-\frac{z}{2}\right)\Gamma\left(1+\frac{\nu}{2}+\frac{\mu}{2}-\frac{z}{2}\right)} \\ &\times \frac{\Gamma\left(\frac{1}{2}-\frac{z}{2}+\frac{\tau}{2}\right)}{\Gamma\left(1-\frac{\nu}{2}+\frac{\mu}{2}-\frac{z}{2}\right)\Gamma\left(1-\frac{z}{2}+\frac{\tau}{2}\right)}\left(\frac{x}{2}\right)^{-z} dz \\ &- \mathrm{Re}\{\nu+\mu\} < \delta < 1. \end{aligned} \qquad (45)$$

Once again, we have *simple* poles to the left of the contour, contributed here by $\Gamma(\nu/2 + \mu/2 + z/2)$. Closing the contour at left (see Section VI) and calculating residues leads to a convergent series. With (16) and (30), the series can be identified with a ${}_3F_4$. The result is

$$\begin{aligned} f(x,\mu,\nu,\tau) &= \frac{\sqrt{\pi}}{2}\left(\frac{x}{2}\right)^{\mu+\nu} \frac{\Gamma(\lambda)}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma\left(\lambda+\frac{1}{2}\right)} \\ &\times {}_3F_4\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}, 1+\frac{\nu}{2}+\frac{\mu}{2}, \lambda; \mu+1, \nu+1, \right. \\ &\left. \mu+\nu+1, \lambda+\frac{1}{2}; -x^2\right) \end{aligned} \qquad (46)$$

where $\lambda = (1 + \nu + \mu + \tau)/2$. For general $\mu, \nu$, and $\tau$ in (46), the "reduction table" in [5, Ch. 7] gives no simpler form, so that (46) is our final result.

The series form corresponding to (46) can be determined by more direct methods [26]. (In (44), expand $J_\mu(xy)J_\nu(xy)$ into its Taylor series ([16, 8.442.1]); integrate term-by-term using Entry 2 of Table I.) Nonetheless, none of [20]–[30] mention the ${}_3F_4$ whose use—as discussed in Section II-F and, further, in Section III-D below—presents several advantages.

For many of the cases in [20]–[30], it is possible to lower the order in (46). For instance, the ${}_3F_4$ reduces to a ${}_2F_3$ when $\nu = \mu$ and further reduces to a ${}_1F_2$ when, also, $\tau = 1$

$$f(x,\mu,\mu,1) = \frac{x^{2\mu}}{\Gamma(2\mu+2)}\,{}_1F_2\left(\mu+\frac{1}{2};2\mu+1,\mu+\frac{3}{2};-x^2\right). \tag{47}$$

The reduction table in [5, Ch. 7] gives certain simpler forms for special cases of (47)—especially when $\mu = 0$ or $\mu = 1$—but, once again, those forms involve rather unusual special functions and will not be repeated here.

### *D. Application to Electrically Large Loop Antennas*

We return to the loop antenna with $ka = C/\lambda$, where $C$ is the circumference, so that the relevant integral $f$ equals the expression in (47) with $\mu = 1$. The first few terms in the definition (30) for the ${}_1F_2$ easily provide a small-$C/\lambda$ approximation for the power, or for $R_r$. We do not dwell on this. Instead, we focus on the nontrivial case where $C/\lambda$ is *large*. We use two terms of the large-$|z|$ asymptotic expansion of the ${}_1F_2$, which can be found in [17]. The following result for the radiation resistance $R_r$ is thus easily obtained

$$R_r \sim 60\pi^2\frac{C}{\lambda}\left[1+\frac{1}{\sqrt{\pi}}\left(\frac{C}{\lambda}\right)^{-1/2}\cos\left(2\frac{C}{\lambda}+\frac{\pi}{4}\right)\right] \text{ ohms.} \tag{48}$$

The first (linear) term $60\pi^2(C/\lambda)$ is the approximation provided as [20, (5-60)] (derived in [20] directly from the integral). The second term grows and oscillates. Fig. 2 shows the exact $R_r$ (as calculated from (47)—numerical integration of (44) of course gives a coincident curve), together with the first (linear) term, and the full approximation (48). It is seen that the previously published approximation is greatly improved (compare also to [20, Fig. 5.10]). In fact, (48) sheds light on the interesting way in which $R_r$ grows, and provides very good quantitative accuracy: The error is less than 5% even for $C/\lambda$ as small as 4; the error decreases (non-monotonically) as $C/\lambda$ increases.

## IV. Aperture Admittance of a 2-D Slot Antenna

Aperture antennas, especially rectangular ones, are very common at microwave frequencies. Many analysis methods ([20, Ch. 12]) assume an infinite, planar, perfectly conducting ground plane with a known tangential aperture field $E_{\tan}$, and proceed to determine the complete fields from Maxwell's equations. One often assumes that $E_{\tan}$ is constant and parallel to the rectangle's small side ([20, §12.5, §12.9]). For simplicity, it is sometimes further assumed that the rectangle is infinitely long ([20, p. 718]), [38], as in a parallel-plate waveguide with 90-degree bends; the radiated fields in this simpler 2-D problem approximate those of long, narrow rectangular slots. Let our 2-D slot lie on the $xy$-plane, with width $b$ parallel to the $y$-axis, and with $E_{\tan} = E_y = E_0$. This assumed field corresponds to the dominant (TEM) field in an infinitely long, parallel-plate waveguide.

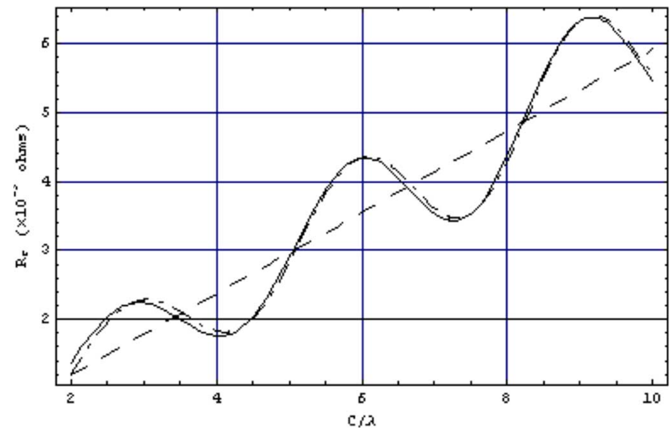


Fig. 2. Radiation resistance $R_r$ of circular loop as function of circumference $C/\lambda$: Exact $R_r$ (solid line), together with linear approximation (i.e., first term of (48); dashed curve), and full approximation (48) (dot-dashed curve).

The complete fields are most easily found by the spectral-domain method, which in this case amounts to taking a Fourier transform in $y$. If $\mathcal{E}_y(k_y)$ and $\mathcal{H}_x(k_y)$ are the transforms of the tangential, on-aperture, spatial-domain fields $E_y(y) = E_0$ and $H_x(y)$, the former turn out to be ([20, p. 718])

$$\mathcal{E}_y(k_y) = \frac{\zeta_0}{k}\sqrt{k^2-k_y^2}\,\mathcal{H}_x(k_y) = bE_0\frac{\sin(k_y b/2)}{(k_y b/2)}. \tag{49}$$

The aperture admittance $Y_a = G_a + jB_a$ is defined by adapting the equation $Y_a = 2P^*/|V|^2$ from ordinary circuit theory: Here, $V = bE_0$ is the aperture voltage, and $P$ is the radiated power per unit length, determined by integrating $E_yH_x = E_0H_x$ along $y$. By Parseval's theorem, $P$ can also be found from the spectral-domain fields as $P = \int_{-\infty}^{\infty}\mathcal{E}_y\mathcal{H}_x^*\,dk_y$. Substituting (49) and taking the *imaginary part* shows that the susceptance $B_a$ is $B_a = 2f(kb/2)/(\lambda\zeta_0)$, where $f$ is the integral [20, p. 720], [38]

$$f(x) = \int_1^{\infty}\left(\frac{\sin xy}{xy}\right)^2\frac{1}{\sqrt{y^2-1}}\,dy, \quad x>0. \tag{50}$$

We note that the *real part*—which, when multiplied by $2/(\lambda\zeta_0)$ equals the conductance $G_a$ ([20, p. 720]), [38]—is our very first example-integral (Section II-H, (34), with $x = kb/2$). Neither [20] nor [38] contain an evaluated form for (50) or (34).

The integral in (50) is more interesting than those of (34) and (42) in two respects: i) Because of the infinite integration interval and the slowly-decaying, oscillatory integrand, direct numerical evaluation of (50) is less straightforward (i.e., it is less accurate and requires more computer time, as further discussed in Section VI), and ii) it is much more difficult to come up with our final result [(52) below] using other methods. With the aid of (31), (32), and Table I, the expression as a MB-integral turns out to be

$$f(x) = -\frac{\pi}{4}\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma(z-1)\Gamma(z)}{\Gamma(\frac{3}{2}-z)\Gamma(\frac{1}{2}+z)}x^{-2z}\,dz \qquad 0<\delta<1 \tag{51}$$

in which a change of variable was made so that the coefficients of $z$ in the gamma functions are 1 or $-1$. A $G$-function representation of $f(x)$ follows directly from (51), see Section VI.

We prefer to find a more classical—and in a sense more revealing—expression as follows.

As discussed in Section VI, one can close the contour of (51) at left. Within the closed contour, there are *double* poles, located at $z = 0, -1, -2\ldots$. The residues at these poles can be found as in Application 4 of Section II-D or, more systematically, with the aid of a lemma provided in our discussion-Section VI. One thus obtains

$$f(x) = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{(n!)^2(n+1)(2n+1)} \left[-\ln x + \psi(n+1) + \frac{4n+3}{2(n+1)(2n+1)}\right]. \quad (52)$$

The ascending series of (52) involves *two* convergent power series, one of which is multiplied by $\ln x$. We stress that the logarithm appears because of the double poles in the integrand of the MB-integral. Note that the series multiplying $\ln x$ can be identified with a ${}_pF_q$ [coincidentally, it is the same ${}_pF_q$ that occurs in (40)], but not the other series.

For narrow slots (that is, small values of $x = kb/2$), the series in (52) converges very rapidly and is particularly useful for numerical computation (at least in this example; there is no beforehand guarantee that a series arising from the MT-method will converge rapidly.) To illustrate, when $kb = 10$, one gets an accuracy of 3% with 12 terms, 0.005% with 15 terms, and 0.0005% with 16 terms. When $kb$ is smaller, fewer terms are required: With two terms, the approximation for the aperture susceptance $B_a$ is

$$B_a \sim \frac{2}{\lambda\zeta_0}\left[-\ln\frac{kb}{2} - \gamma + \frac{3}{2} - \frac{(kb)^2}{24}\left(-\ln\frac{kb}{2} - \gamma + \frac{19}{12}\right)\right]. \quad (53)$$

Formula (53), which is simple enough for "back-of-the-envelope" calculations, is an improvement to the "quasi-static result" of [38], ([20, p. 720]). The quasi-static result essentially corresponds to keeping one term in (52). The improvement is significant: With $kb = 2$, there is a 2.6% error with two terms, compared to a 19% error with one term.

## V. An Integral Arising in the Theory of Biaxially Anisotropic Media

The unpublished studies [39], [40] (which are somewhat similar to the recent papers [41], [42]) deal with the Green's function in unbounded, biaxially anisotropic media, with the aim of understanding the behavior of the two types of waves possible in such media. The problem is interesting in that, in its usual form [43], the Sommerfeld radiation condition is not applicable; that form requires isotropic media. To replace the radiation condition, [39]–[42] (see also [44]) initially assume a small loss, choose the solution that is bounded at infinity and, finally, take the limit of that solution for zero loss.

In [39], the Green's function is known in cylindrical coordinates through its inverse Fourier transform. The integral over the radial Fourier variable is then performed. A key constituent of the resulting expression is the integral [39], [40]

$$f_{2m+1}(x) = \int_0^{\infty} \frac{J_{2m+1}(xy)}{y^2+1}\,dy \qquad x > 0, \quad m = 0,1,2,\ldots \quad (54)$$

which we will evaluate using the MT-method combined with additional manipulations. Our treatment here is more advanced, but the final answer (61) will be particularly simple.

For reasons to become apparent, we will first deal with the more general integral obtained by replacing the odd, positive integer $2m+1$ by a complex parameter $\nu$, viz.

$$f_\nu(x) = \int_0^{\infty} \frac{J_\nu(xy)}{y^2+1}\,dy, \quad x > 0, \quad \mathrm{Re}\{\nu\} > -1 \quad (55)$$

and take the limit $\nu \to 2m+1$ as a final step. With the aid of Table I, one finds

$$f_\nu(x) = \frac{1}{4}\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma\left(\frac{\nu}{2}+\frac{z}{2}\right)\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)\Gamma\left(\frac{1}{2}+\frac{z}{2}\right)}{\Gamma\left(1+\frac{\nu}{2}-\frac{z}{2}\right)} \times \left(\frac{x}{2}\right)^{-z} dz \quad (56)$$

in which $\max\{-1, -\mathrm{Re}\{\nu\}\} < \delta < 1$. A $G$-function expression for $f_\nu(x)$ is given in Section VI. For general $\nu$, and to the left of the path in (56), there are two lattices of *simple* poles. Closing the contour (see Section VI) and calculating residues, one obtains

$$f_\nu(x) = \sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n\left[\frac{x}{4}\frac{\Gamma\left(\frac{\nu}{2}-\frac{1}{2}-n\right)}{\Gamma\left(\frac{\nu}{2}+\frac{3}{2}+n\right)} + \frac{1}{2}\left(\frac{x}{2}\right)^{\nu}\frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}+n\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}-n\right)}{n!\,\Gamma(1+\nu+n)}\right]. \quad (57)$$

The gamma functions can be expressed in terms of Pochhammer's symbols using (19) and (23). The resulting series can be immediately identified with ${}_pF_q$'s, so that

$$f_\nu(x) = \frac{x}{4}\frac{\Gamma\left(\frac{\nu}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{\nu}{2}+\frac{3}{2}\right)}\,{}_1F_2\left(1; \frac{3}{2}+\frac{\nu}{2}, \frac{3}{2}-\frac{\nu}{2}; \frac{x^2}{4}\right) + \frac{1}{2}\left(\frac{x}{2}\right)^{\nu}\frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)}{\Gamma(1+\nu)}\,{}_0F_1\left(1+\nu; \frac{x^2}{4}\right). \quad (58)$$

This time, further simplification is possible using the aforementioned "reduction tables" of ${}_pF_q$. Specifically, from [5, 7.14.3.6] and [5, 7.13.1.1] one obtains

$$f_\nu(x) = i s_{0,\nu}(ix) + \frac{\pi}{2}\frac{1}{\cos(\pi\nu/2)} I_\nu(x) \quad (59)$$

in which $I_\nu$ is the usual modified Bessel function and $s_{0,\nu}$ is the Lommel function discussed in [16, 8.57] (see also [45]), or [5, II.12]. Like all manipulations in our previous examples, those used up to now [to obtain (59) from (55)] have been straightforward.

The answer (59) is simple enough, but both terms become infinite in the case $\nu = 1, 3, \ldots$, which is precisely the case we are interested in. Since the original integral (54) is finite, the two terms in (59) must combine to give a quantity that remains finite in the limit $\nu \to 2m+1$. To calculate this quantity, use [16, 8.570.2] to express $s_{0,\nu}$ in terms of $J_\nu, Y_\nu$, and the Lommel function $S_{0,\nu}$. Then, combine the $J_\nu$ and $Y_\nu$ with the $I_\nu$ of (59) employing usual Bessel-function identities. One can readily arrive at

$$f_\nu(x) = iS_{0,\nu}(ix) - ie^{-i\nu\pi/2}K_\nu(x) \qquad (60)$$

which is an alternative to (59) expression for general $\nu$. Clearly, the RHS of (60) is finite when $\nu \to 2m+1$. Furthermore, by [16, 8.573.2], or [5, II.12], in the limit, $S_{0,\nu} = S_{0,2m+1}$ reduces to a polynomial, a formula for which is provided in [16, 8.590.1], or [5, II.24]. Using that formula yields our final expression for $f_{2m+1}(x)$

$$f_{2m+1}(x) = \frac{1}{2}\sum_{n=0}^{m}(-1)^n\frac{(m+n)!}{(m-n)!}\left(\frac{2}{x}\right)^{2n+1} + (-1)^{m+1}K_{2m+1}(x). \qquad (61)$$

This simple answer consists of an odd polynomial of degree $2m+1$ in $1/x$, plus a modified Bessel function. (61) is excellent both for numerical evaluation and for further analytical work. This is especially true for *large* values of $x$, where $K_{2m+1}(x)$ is very small and the polynomial strongly dominates. (Large $x$ is of interest in [39]–[42]).

## VI. Discussion

- In [12, Ch. 5] and [13] one finds a very simple set of *sufficient* conditions enabling one to "close the contour at left." Those conditions are satisfied in all cases of this paper.
- Calculations of residues at double poles can be laborious. For this reason, we give a useful lemma, which can be readily shown from the properties of Section II-C:
  If $n = 0, 1, 2, \ldots$ and $g(z)$ is analytic and non-zero at $z = -n$, then $[\Gamma(z)]^2 g(z)x^{-z}$ has a double pole at $z = -n$, and the residue there is

$$\operatorname{Res}\left\{[\Gamma(z)]^2 g(z)x^{-z}; z=-n\right\} = \frac{x^n}{(n!)^2}\left[-g(-n)\ln x + 2\psi(n+1)g(-n) + g'(-n)\right]. \qquad (62)$$

  Many other expressions, arising when applying the MT-method and involving gamma functions, can be written in a form appropriate for the application of the above lemma. To use the lemma to verify (28), for example, substitute $\Gamma(z-1)$ by $\Gamma(z)/(z-1)$. An identical substitution allows one to show (52) from (51).
- For completeness, we give a $G$-function expression for the integral (50) of Section IV. It is

$$f(x) = \frac{1}{2x^2} - \frac{\pi}{4}\, G_{13}^{20}\left(x^2 \middle| \begin{matrix} & \frac{1}{2} & \\ 0 & -1 & -\frac{1}{2}\end{matrix}\right). \qquad (63)$$

  For the integral (55) of Section V, one has

$$f_\nu(x) = \frac{1}{2}\, G_{13}^{21}\left(\frac{x^2}{4} \middle| \begin{matrix} & \frac{1}{2} & \\ \frac{1}{2} & \frac{\nu}{2} & -\frac{\nu}{2}\end{matrix}\right). \qquad (64)$$

  We note that the $G$-function reduction table ([5, 8.4.52]) provides no simplification for (64).
- It has already been mentioned that the MT-method is important for Mathematica's symbolic routine (SR) `Integrate[]`. SRs can also help when one applies the MT-method on his/her own, as SRs can be used in many intermediate steps. Such steps include the "lookup" of MTs, "messy" manipulations such as the calculation of residues, and the simplification of complicated expressions. SRs are powerful tools, and it pays to be flexible when using them. When applied to the RHS of (64), for example, Mathematica 5.0's routine `FullSimplify[]` *does not* yield (58), even when $x > 0$ is assumed. When applied to the RHS of (64) *minus* the RHS of (58), however, `FullSimplify[]` *does* yield zero. Here, `FullSimplify[]` verifies the answer but cannot produce it from scratch.
- One may have the view that, like the routines for the ${}_pF_q$, modern numerical-integration routines can themselves often be used as black boxes. We discuss this by focusing on integrals like those in Sections IV and V, which have an infinite upper integration limit, and oscillations due to a factor $\cos xy$ or $J_\nu(xy)$ in the integrand. Let us consider three relevant routines (or types of routines), whose use seems to be widespread.
  - i) As far as accuracy is concerned, the best routine the author is aware of is Mathematica's `NIntegrate[]`, provided the `Method->Oscillatory` option is used. The user sets the upper integration limit to `Infinity`. To give an example, one gets highly accurate results by evaluating (54) in this manner; but the computer time is significantly less if one uses (61).
  - ii) Even for the types of integrals discussed here, it is not always possible to use the `Method->Oscillatory` option, e.g., when the $\nu$ in $J_\nu(xy)$ is negative or complex. In such cases, a Mathematica user can resort to `NIntegrate[]` without the `Method->Oscillatory` option; the integration limit can still be set to `Infinity`. The accuracy in such cases is significantly less than before. For example, when $x = 3$ and $\nu = -0.9$ in (55), one gets a result correct to within only 0.3%, accompanied by a warning message. By contrast, (58) quickly yields answers which, as far as the author can tell, are highly accurate.
  - iii) The numerical integration routines in Matlab 7.0 (`quad, quadl`) do not allow the user to specify an infinite integration limit. With such types of routines, one often specifies an integration limit large enough to yield a desired accuracy. For slowly decaying integrands, such "truncation methods" may not work well at all. As an example, consider the integral $\int_1^\infty y^{-\nu}\cos xy\,dy (\nu > 0, x > 0)$ which, as the

reader may wish to verify, can be evaluated in terms of a ${}_1F_2$. A truncation method amounts to numerically integrating $\int_1^M y^{-\nu} \cos xy\, dy$. For $\nu = \frac{1}{2}$, both the integral and its truncated version can be evaluated in terms of Fresnel integrals ([20, ch. 13, App. IV]) so, for this case, we can compute the best one can do by *any* truncation method. As it turns out, the required values of $M$ are very, very large. For $x = 2, M = 12\,000$ is necessary for 1% accuracy, and $M = 48\,000$ is necessary for 0.5% accuracy. For either value of $M$, an actual numerical-integration routine will certainly provide much less accuracy. The situation deteriorates even more if $\nu$ decreases.

- We finally give some further references to the topics of this paper. References [11], [46], and [47] are introductory treatments of the MT. They include brief descriptions of the MT-method, as well as short discussions of (and references to) other applications of the MT. Gamma and related functions are treated in most textbooks on complex variables and special functions; besides [14], we mention [10], [15], and [48]. Reference [15] discusses the gamma function, the ${}_pF_q$, and the $G$-function, and includes derivations; more extensive references for these topics are [18] and [19], while many relevant formulas can be found in [5], [16], and [17]. On MB-integrals, see [6], [15], [18], [48], and the comprehensive book [49]. [8] and [11]–[13] contain simple, informative discussions relevant to the MT-method, not too different from the general material in Sections II-A to II-G; more detailed expositions can be found in the pioneering (but readable) works [6], [2]. The origins of what we call the "MT-method" go far back: The idea of the Mellin inversion formula appeared in an 1876 memoir by Riemman, and the first accurate discussion was given by Mellin in 1896 and 1902. What we now call "MB-integrals" were first introduced by Pincherle in 1888 [50], developed theoretically by Mellin by 1910, and used by Barnes in 1908 to discuss the asymptotic expansion of certain special functions. Biographies of Mellin and Barnes can be found in [49].

## VII. Summary and Conclusion

What we call the "MT-method" is an extremely powerful technique for the exact evaluation of definite integrals. While, in many cases, completely straightforward, the method is not as widely known as it should. It can often be combined with other methods and it is applicable to a wide class of integrals. It is a significant constituent of certain modern symbolic integration packages, and has been employed in an essential manner to compile what may be the most comprehensive published table of integrals.

When applicable, the MT-method typically yields ascending series (which often involve logarithms, or powers of logarithms), or expressions involving the generalized hypergeometric function ${}_pF_q$, or Meijer's $G$-function. Because such expressions can be automatically handled by modern numerical routines, they are much more useful than in the past. Because the ${}_pF_q$ and $G$ possess a vast number of documented properties, such expressions can also be a good first step for further analytical work. Often, though, expressions involving ${}_pF_q$ and $G$ are merely an intermediate step as they can be simplified by lookup in extensive tables, or by symbolic routines.

To apply the method, one should have some familiarity with the ${}_pF_q$ and MB-integrals, and possess some experience with certain lookup tables. More importantly, one should have a good working knowledge of the basics of the Mellin transform, as well as of gamma and related functions. All these topics, which can be understood by one familiar with complex analysis, are discussed in the tutorial Section II. This section places little emphasis on mathematical details or fine points. The remaining sections illustrate the MT-method by treating example-integrals, all arising from antenna problems. All answers are suitable for numerical evaluation and most are believed to be new, at least in the antenna/electromagnetics literature. Two of these answers lead, additionally, to simple approximate formulas for the integral which significantly improve upon formulas of standard antenna textbooks. These sections thus explicitly illustrate highly desirable features of the MT-method, in the specific context of antenna theory.

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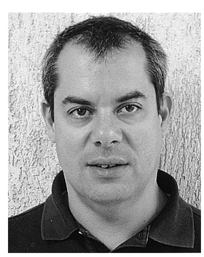

**George Fikioris** (S'90–M'94–SM'05) received the Diploma of Electrical Engineering from the National Technical University of Athens, Greece (NTUA) in 1986, and the S.M. and Ph.D. degrees in engineering sciences from Harvard University, Cambridge, MA, in 1987 and 1993, respectively.

From 1993 to 1998, he was an Electronics Engineer with the Air Force Research Laboratory, Hanscom AFB, MA. From 1999 to 2002, he was a Researcher with the Institute of Communication and Computer Systems at the NTUA. Since 2002, he has been a Lecturer at the Department of Electrical and Computer Engineering, NTUA where, in July 2006, he was elected Assistant Professor. He is the author or coauthor of over 25 papers in technical journals, and numerous papers in conferences. He coauthored *Cylindrical Antennas and Arrays* (Cambridge, U.K.: Cambridge Univ. Press, 2002). His research interests include electromagnetics, antennas, and applied mathematics.

Dr. Fikioris is a member of the American Mathematical Society and of the Technical Chamber of Greece.