

Efficient Approximations for the Arctangent Function

This article provides several efficient approximations for the arctangent function using Lagrange interpolation and minimax optimization techniques. These approximations are particularly useful when processing power, memory, and power consumption are important issues. In addition to comparing the errors and the computational workload of these approximations, we also extend them to all four quadrants.

ARCTANGENT APPROXIMATIONS

The evaluation of the arctangent function is commonly encountered in real-time signal processing applications. Numerous algorithms are available to implement the arctangent function when computational cost is unimportant. The most direct solution is based on the Taylor series expansion on $[-1, 1]$. However, this series converges slowly for arguments close to one and hence is inefficient.

Iterative algorithms, such as the coordinated rotation digital computer (CORDIC) algorithm (requiring only shifts and add operations), have been successfully used to compute trigonometric functions [1], [2]. However, their sequential nature makes them less attractive when speed is a major concern; attempts to increase speed have been at the expense of additional hardware [3]. Lookup-table-based approaches to the computation of inverse trigonometric functions are very fast but require considerable memory [4], [5].

Polynomial and rational function approximations that have been proposed in the literature [3] are more suitable for numerical coprocessors. Approximations using polynomials of large degrees are computationally expensive. Rational approximations are in principle more accurate than polynomial approximations for the same number of coefficients. However, the

required division operations are relatively complex to implement; iterative techniques, such as those based on Newton's method, are often used.

The aforementioned approaches are best suited for applications where processing power and/or memory are readily available. However, for many applications, simpler and more efficient ways of evaluating $\arctan(x)$ are desirable. This article proposes simple approximations for evaluating the arctangent function that may be easily implemented in hardware with limited memory and processing power.

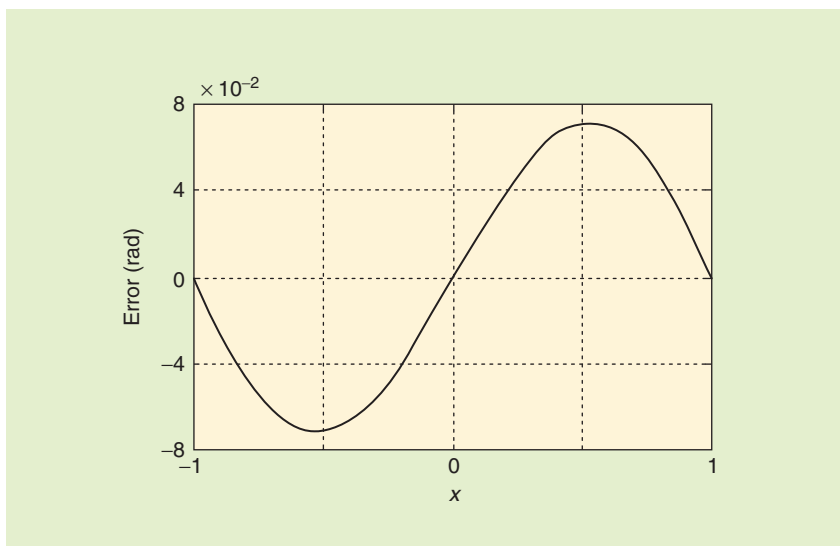
PROPOSED METHODOLOGY AND APPROXIMATIONS

Approximations to the arctangent function can be obtained using second- and third-order polynomials and simple rational functions. Lagrange interpolation-based and minimax criterion-based approaches are used to obtain the polynomial coefficients. The following is our initial derivation of an arctangent approximation using Lagrange interpolation.

Consider the three points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$. Let $\Phi(x) = \arctan(1+x)/(1-x)$, $-1 \leq x \leq 1$. According to the Lagrange interpolation formula, we have

$$\begin{aligned} \phi(x) &\approx \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\phi(x_0) \\ &+ \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\phi(x_1) \\ &+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\phi(x_2) \\ &= \frac{\pi}{4}(x+1), \quad -1 \leq x \leq 1. \quad (1) \end{aligned}$$

It can be verified that $\Phi(x) = \pi/4 + \arctan(x)$; hence, $\arctan(x)$ can be approximated by the first-order formula



[FIG1] Approximation errors using (2).

$$\arctan(x) \approx \frac{\pi}{4}x, \quad -1 \leq x \leq 1. \quad (2)$$

This linear approximation has been used in [6] for FM demodulation due to its minimal complexity. It requires only a scaling operation by a fixed constant and can be computed in one cycle in most processors. Figure 1 shows the deviation of this linear (first-order) approximation from the arctangent function on the interval $[-1, 1]$. This error function, given in (3), is antisymmetric with respect to $x = 0$ and attains a maximum of about 0.07 rad (4°) at $x_{\max} = \pm(4/\pi - 1)^{1/2}$

$$\Psi(x) = \arctan(x) - \frac{\pi}{4}x, \quad -1 \leq x \leq 1. \quad (3)$$

Here's a useful trick to improve the accuracy of the first-order approximation (2). It can be seen from Figure 1 that the error curve is approximately quadratic on the interval $[0, 1]$ and hence can be approximated by a second-order polynomial. Following this strategy and applying Lagrange's interpolation formula to $\Psi(x)$ using $x_0 = 0$, $x_1 = x_{\max}$, and $x_2 = 1$, we obtain the following formula:

$$\Psi(x) \approx 0.285x(1 - |x|), \quad -1 \leq x \leq 1, \quad (4)$$

where the odd symmetry of $\Psi(x)$ has been applied. A second-order, and more accurate, approximation for $\arctan(x)$ with a maximum absolute error of 0.0053 rad (0.3°) is thus obtained as

$$\arctan(x) \approx \frac{\pi}{4}x + 0.285x(1 - |x|), \quad -1 \leq x \leq 1. \quad (5)$$

It is also of interest to approximate the error function, $\Psi(x)$, given by (3), by a second-order polynomial of the form $(\alpha x^2 - \alpha x)$ that passes through the endpoints of the interval $[0, 1]$. The optimum parameter, $\alpha > 0$, may be obtained using the following minimax criterion

$$J = \min_{\alpha} \left\{ \max_{0 \leq x \leq 1} \{ |\Psi(x) - \alpha x(x - 1)| \} \right\}. \quad (6)$$

The above criterion (6) does the following: for every α , the maximum absolute error between $\Psi(x)$ and the second-order polynomial $(\alpha x^2 - \alpha x)$ is determined. This error, as a function of α , is then minimized. This will yield a unique α that gives the least error J ; hence the name minimax criterion, and the approximation is called minimax approximation.

Using an extensive computer search, the optimum $\alpha \approx 0.273$ and the following approximation for the $\arctan(x)$ with a maximum absolute error of 0.0038 rad (0.22°) are obtained:

$$\arctan(x) \approx \frac{\pi}{4}x + 0.273x(1 - |x|), \quad -1 \leq x \leq 1. \quad (7)$$

A third-order polynomial is also a good candidate to fit the error curve shown in Figure 1. When the third-order polynomial is constrained to be of the form $\alpha x^3 + \beta x$, the best minimax approximation for the arctangent function is given by

$$\arctan(x) \approx \frac{\pi}{4}x + x(0.186982 - 0.191942x^2), \quad -1 \leq x \leq 1. \quad (8)$$

The maximum absolute error for this approximation is 0.005 rad (0.29°) and is worse than that given by (7).

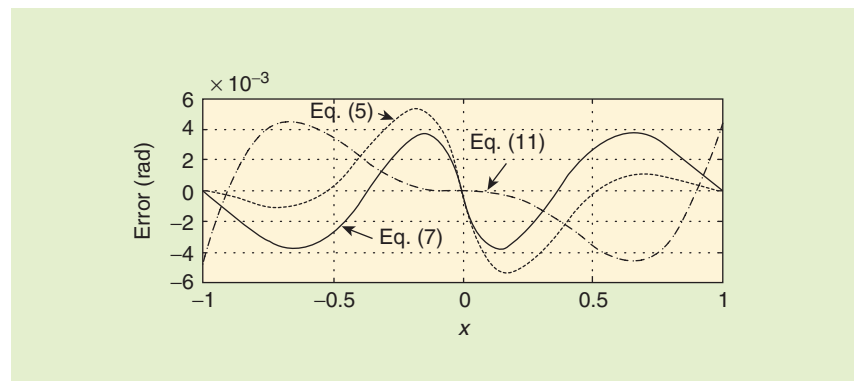
[TABLE 1] MAXIMUM ERROR AND COMPUTATIONAL WORKLOAD.

EQUATION	ERROR (RAD.)	ADDS	MULTIPLIES	DIVIDES
(2)	0.07	0	1	0
(5)	0.0053	1	2	0
(7)	0.0038	1	2	0
(8)	0.005	1	3	0
(9)	0.0015	2	3	0
(10)	0.0047	1	2	1
(11)	0.0049	2	1	1

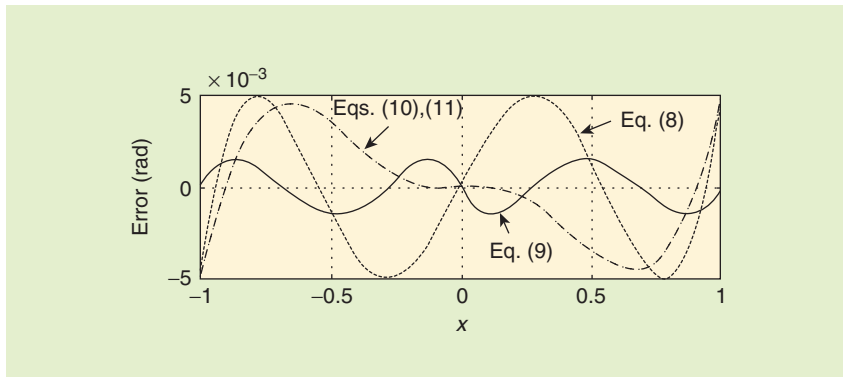
A better third-order polynomial for approximating the error is of the form $x(x - 1)(\alpha x - \beta)$ for x in the interval $[0, 1]$. Using the minimax criteria, the following polynomial with a maximum absolute error of 0.0015 rad (0.086°) is identified as the optimal approximation to $\arctan(x)$

$$\arctan(x) \approx \frac{\pi}{4}x - x(|x| - 1) \times (0.2447 + 0.0663|x|), \quad -1 \leq x \leq 1. \quad (9)$$

Another candidate for approximating the arctangent function is from the class of rational functions of the form $\tau(x) = x/(1 + \beta x^2)$ in the interval $[-1, 1]$. For $0 \leq \beta \leq 1$, the first derivative of $\tau(x)$ is positive and the second derivative of $\tau(x)$ is negative. This implies that $\tau(x)$ has a shape very similar to that of the arctangent function in the same interval. Using minimax criteria, the following approximation with the maximum absolute error of about 0.0047 rad (0.27°) is obtained:



[FIG2] Approximation errors using second-order polynomials (5), (7) and a rational (11).



[FIG3] Approximation errors using cubic polynomials (8), (9) and rationals (10), (11).

$$\arctan(x) \approx \frac{x}{1 + 0.28086x^2}, \quad -1 \leq x \leq 1. \quad (10)$$

Recently, a similar idea was presented in [5] with a maximum absolute error of 0.0049 rad (0.28°). The approximation in [5] is

$$\arctan(x) \approx \frac{x}{1 + 0.28125x^2}, \quad -1 \leq x \leq 1. \quad (11)$$

The scaling constant, 0.28125, was chosen to permit multiplication to be performed (as discussed later) with two arithmetic binary right shifts and a single addition. This yields a negligible increase in maximum error.

DISCUSSION

Here we compare the various arctangent approximations presented in this article. Table 1 contains the maximum error and the number of adds, multiplies, and divides for various arctan approximations. The third-order approximation given by (9) has the least error among the proposed approximations but has the highest computational cost. The second-order approximation given by (7) has the next lowest error and has fewer operations than (9). Hence, (7) provides a favorable compromise between accuracy and computational cost. The linear approximation given by (2) has only one multiplication operation but has the least accuracy. With a single-cycle multiply and accumulate (MAC) processor, the evaluation of the arctangent using (7) would take only two cycles if the sign

information of the argument is already available. However, one needs to check the sign of the argument, which may take an extra cycle.

Comparison of the approximations to $\arctan(x)$ using the proposed two second-order approximations given by (5) and (7) are shown in Figure 2. These approximations have maximum errors that are an order of magnitude better than that of the linear approximation (2). Furthermore, the second-order approximation given by (5) provides better accuracy for the subintervals $1 > x > 0.5$ and $-1 < x < -0.5$, where the maximum error is only about 0.001 rad (0.057°). No such observation can be made for (7).

Figure 3 shows that (9) provides the best accuracy amongst the third-order approximations considered in this article, while the rational approximations (10) and (11) have essentially identical errors. The error approaches zero at $x = \pm 1$ for (4), (7), and (9) (see Figures 2 and 3).

The rational approximations given by (10) and (11) are computationally more expensive than the second-order ones given by (5) and (7), though this cost increase is partly offset by the elimination of the need for sign comparisons. Approximations using (10) or (11) have a division operation that may slow down the processing. The approximation using (11) and the approximation given by (5) intersect at approximately $x_c = 0.3933$ in the interval $[0, 1]$. For values of $|x| > x_c$, the proposed approximation given by (5) yields better accuracy as shown in Figure 2. For values of $|x| < x_c$, the approximation in (11) gives better accuracy.

An alternative approach to improving the accuracy, though at the expense of computational cost, would be to combine (5) and (11). This combined equation is given by

$$\arctan(x) \approx \gamma A + (1 - \gamma)B, \quad (12)$$

where A is given by (11) and B is given by (5). The weighting variable γ is unity when $0 \leq |x| \leq x_c$ and zero otherwise. Using this approach, the maximum error is less than 0.0025 rad (0.14°), as can be seen from Figure 2.

[TABLE 2] ARCTAN APPROXIMATIONS VERSUS OCTANT LOCATIONS.

OCTANT	APPROXIMATION (7)	APPROXIMATION (10)
I, VIII	$\frac{Q}{I} [1.0584 - \text{sign}(Q) \cdot 0.273 \cdot (Q/I)]$	$\frac{Q/I}{1 + 0.28086 \cdot \frac{Q^2}{I^2}}$
II, III	$\frac{\pi}{2} - \frac{I}{Q} [1.0584 - \text{sign}(I) \cdot 0.273 \cdot (I/Q)]$	$\frac{\pi}{2} - \frac{I/Q}{1 + 0.28086 \cdot \frac{I^2}{Q^2}}$
IV, V	$\text{sign}(Q) \cdot \pi + \frac{Q}{I} [1.0584 + \text{sign}(Q) \cdot 0.273 \cdot (Q/I)]$	$\text{sign}(Q) \cdot \pi \cdot \frac{Q/I}{1 + 0.28086 \cdot \frac{Q^2}{I^2}}$
VI, VII	$-\frac{\pi}{2} - \frac{I}{Q} [1.0584 + \text{sign}(I) \cdot 0.273 \cdot (I/Q)]$	$-\frac{\pi}{2} - \frac{I/Q}{1 + 0.28086 \cdot \frac{I^2}{Q^2}}$

FOUR QUADRANT APPROXIMATIONS

The arctan algorithms presented thus far are applicable for angles in the range of $-\pi/4$ to $\pi/4$ rad. Here we extend the angular range to $-\pi$ to π radians. Let $z = I + jQ$ be any complex number and let $x = Q/I$. The four quadrant arctangent function $\text{atan2}(I, Q)$ can be evaluated over the four quadrants by substituting Q/I for x in the approximations presented in this article. Due to space limitations, we only consider the extensions for approximations given by (7) and compare them to the four quadrant arctangent expressions given for (10).

To obtain the four quadrant arctangent approximations, the range over which the approximation operates is extended. The complex plane is divided into eight octants where octant I, for example, covers the angle range of 0 to $\pi/4$ rad. The four quadrant calculations are thus reduced to the first octant calculations, and using the rotational symmetries of arctangent function the approximations for the other octants can be easily obtained. Table 2 provides the four quadrant approximations based on (7) and (10). It should be noted that Table 2 uses the following definition for determining the sign value of an argument:

$$\text{sign}(z) = \begin{cases} 1, & z \geq 0 \\ -1, & z < 0. \end{cases} \quad (13)$$

Many desirable features of the four quadrant rational approximation based on (11) are given in [5]. The multiplication by 0.28125 in the denominator of (11) can be implemented as a sum of two arithmetic right shifts, $x^2/4$ and $x^2/32$. (This fact accounts for the two adds and single multiply requirements for (11) in Table 1.) The constants appearing in the four quadrant approximations using either (5) or (10) cannot be implemented with shifts. However, the complexity of the four quadrant arctangent approximations using any of (5), (10), or (11) are the same. The four quadrant arctangent approximation using (7) has better accuracy and deserves consideration.

CONCLUSION

Simple approximations to the arctangent function and four quadrant arctangent functions have been introduced. The second-order polynomial in (7) provides a favorable compromise between accuracy and computational cost. Furthermore, it is well suited for implementation in hardware.


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