A-PPP: Array-Aided Precise Point Positioning With Global Navigation Satellite Systems

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***Abstract*—In this paper, the global navigation satellite system (GNSS) precise point positioning (PPP) concept is generalized to array-aided PPP (A-PPP). A-PPP is a measurement concept that uses GNSS data, from multiple antennas in an array of known geometry, to realize improved GNSS parameter estimation (position, attitude, time and atmospheric delays). The concept is formulated such that it applies to each current and future multifrequency GNSS, stand-alone or in combination. A–PPP is made possible through solving a novel orthonormality-constrained multivariate (mixed) integer least-squares problem. It is shown that the integer matrix constraint is necessary to obtain a precise instantaneous attitude- and position solution, whereas the inclusion of the orthonormality constraint in the integer ambiguity objective function is essential to achieve high instantaneous probabilities of correct integer estimation. Different A-PPP applications are discussed, with their performances illustrated by means of empirical GPS results.**

***Index Terms*—Array-aided precise point positioning, global navigation satellite systems, multivariate integer least squares, orthonormality constrained integer least squares.**

I. Introduction

PRECISE point positioning (PPP), first described in [1], is a global navigation satellite system (GNSS) positioning method that processes pseudorange and carrier phase measurements from a standalone GNSS receiver to compute positions with a high, decimeter or centimeter, accuracy everywhere on the globe. By using satellite orbits and clocks, as well as other corrections (e.g., for Earth rotation, tides and ocean loading, phase wind-up, etc.), the GNSS receiver position along with other parameters, like atmospheric delays, can be estimated [2]–[4]. In recent years, services have been developed which allow high accuracy ephemeris data to be made available in real-time to users [5]–[7]. Such availability has created, and will continue to create, a wide range of PPP applications [8], [9]. Also, various forms of PPP are possible, like, e.g., single-frequency PPP using global ionospheric maps (GIMs) [3], [4], dual-frequency PPP using ionosphere-free combinations [2], or integer ambiguity resolution [10]–[13] enabled real-time kinematic (RTK) PPP [14]–[17]. Next to positioning, PPP is also used in remote sensing, [18], [19], as an ionospheric or tropospheric sensor, [20], [21], or for time-transfer [22]–[24].

In this paper, we extend the PPP concept to array-aided PPP (A-PPP). A-PPP is a GNSS measurement concept that uses GNSS data from multiple antennas in known formation to realize real-time precise attitude and improved positioning of a (stationary or moving) platform. It is assumed that the local antenna geometry is known in the body (platform) frame and that each of the antennas in the array collects GNSS pseudorange and carrier phase data. The A-PPP principle can then briefly be described as follows. The known array geometry in the platform frame enables successful integer carrier-phase ambiguity resolution, thereby realizing a two-order of magnitude improvement in the between-antenna GNSS pseudoranges. These very precise pseudoranges are then used to determine the platform's earth-fixed attitude, thus effectively making the platform a 3D direction finder. At the same time, the precision of the absolute pseudoranges and carrier phases are improved by exploiting the correlation that exists between these data and the very precise between-antenna pseudoranges. This improvement enables the improved platform parameter estimation. Also integrity improves, since with the known array geometry, redundancy increases, thus allowing improved error detection and multipath mitigation [25].

This contribution is organized as follows. In Section II, the GNSS models for precise point positioning and array-based attitude determination are presented. Their respective estimation problems are usually treated and solved independently. In Section III it is shown why and how this can be improved. A multivariate constrained formulation of the combined position-attitude model is introduced, which is structured as

$$\begin{aligned} \mathrm{E}(\boldsymbol{\mathcal{Y}}) &= \boldsymbol{M}\boldsymbol{\mathcal{B}} + \boldsymbol{N}\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{C}} \\ \mathrm{Cov}(\mathrm{vec}(\boldsymbol{\mathcal{Y}})) &= \boldsymbol{\Sigma} \otimes \boldsymbol{Q} \end{aligned} \tag{1}$$

with $\boldsymbol{\mathcal{Y}} = [\boldsymbol{y}_1, \boldsymbol{Y}]$ the random matrix of GNSS array observables and $\boldsymbol{\mathcal{B}} = [\boldsymbol{b}_1, \boldsymbol{RF}]$, $\boldsymbol{\mathcal{A}} = [\boldsymbol{a}_1, \boldsymbol{Z}]$, and $\boldsymbol{\mathcal{C}} = [\boldsymbol{d}_1, \boldsymbol{0}]$ the matrices containing the deterministic parameters that need to be estimated under the attitude orthonormality and ambiguity integer constraints

$$\boldsymbol{R} \in \mathbb{O}^{3\times q} \quad \text{and} \quad \boldsymbol{Z} \in \mathbb{Z}^{fs\times r} \tag{2}$$

$\boldsymbol{\mathcal{B}}$ is the matrix of antenna positions, $\boldsymbol{\mathcal{A}}$ the matrix of carrier phase ambiguities and $\boldsymbol{\mathcal{C}}$ the matrix of atmospheric delays and satellite-related terms. By means of a decorrelating transformation it is shown which improvements can be realized and how the PPP concept can be extended to array-aided PPP.

Manuscript received December 14, 2010; revised July 23, 2011 and October 26, 2011; accepted February 22, 2012. Date of publication March 05, 2012; date of current version May 11, 2012. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Sofia C. Olhede. This work was supported by an Australian Research Council (ARC) Federation Fellowship (project number FF0883188).

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Digital Object Identifier 10.1109/TSP.2012.2189854

An essential component of A-PPP processing is solving the constrained array estimation problem. This novel multivariate, orthonormality-constrained, mixed integer least-squares (ILS) problem is solved in Section IV. In contrast to the existing constrained ILS problems, as box-constrained ILS [26] and ellipsoid-constrained ILS [27], our problem is a mixed real/integer least-squares problem, of the multivariate type, with orthonormality constraints on the real-valued parameters. As is shown, the two type of constraints play a distinct role. The integer matrix constraint is necessary to obtain the most precise instantaneous attitude and position solution, whereas the inclusion of the orthonormality constraint in the ambiguity objective function is essential to achieve a high probability of correction integer estimation [28].

In the following, a frequent use is made of the Kronecker product $\otimes$ and the vec-operator. For their properties, see, e.g., [29], [30]. The expectation and covariance matrix of a random vector $\boldsymbol{x}$ are denoted as $\mathrm{E}(\boldsymbol{x})$ and $\mathrm{Cov}(\boldsymbol{x})$, respectively. For the covariance matrix of a random matrix $\boldsymbol{X}$, we often write $\boldsymbol{Q_{XX}}$ instead of $\mathrm{Cov}(\mathrm{vec}(\boldsymbol{X}))$. For the weighted squared norm, the notation $\|.\|_{\boldsymbol{Q}}^2 = (.)^T\boldsymbol{Q}^{-1}(.)$ is used. Although the terminology of weighted least-squares estimation is used throughout, the given least-squares (LS) estimators are also maximum likelihood estimators in the Gaussian case and best linear unbiased estimators (BLUEs) in the linear model case, since the inverse covariance matrix of the GNSS observables is used as weight matrix.

II. Positioning and Attitude

In this section we present the PPP observation equations for positioning and the array observation equations for attitude determination. Although these models are currently restricted to the usage of single- or dual-frequency GPS data, we formulate them for the general multifrequency case, thus enabling next generation GNSS application as well.

A. Precise Point Positioning

The undifferenced carrier-phase and pseudorange (code) observables of a GNSS receiver r tracking satellite s on frequency $f_j = c/\lambda_j$ (c is speed of light; λ_j is jth wavelength) are denoted as $\phi_{r,j}^s$ and $p_{r,j}^s$, respectively. When two satellites, s and t, are tracked, one can form the between-satellite, single-differenced (SD) phase, and code observables, of which the linear(ized) observation equations are given as [31]–[35]

$$\begin{aligned}\mathrm{E}(\phi_{r,j}^{st}) &= (\boldsymbol{g}_r^{st})^T\boldsymbol{b}_r - \mu_j i_r^{st} + \lambda_j a_{r,j}^{st} + c_{\phi,r}^{st}\\ \mathrm{E}(p_{r,j}^{st}) &= (\boldsymbol{g}_r^{st})^T\boldsymbol{b}_r + \mu_j i_r^{st} + c_{p,r}^{st}\end{aligned}\tag{3}$$

with the PPP correction terms, $c_{\phi,r}^{st} = \mathtt{t}_r^{st} - \delta\mathtt{s}_{,j}^{st} - \mathtt{o}_r^{st}$ and $c_{p,r}^{st} = \mathtt{t}_r^{st} - d\mathtt{s}_{,j}^{st} - \mathtt{o}_r^{st}$, assumed known. The unknown deterministic parameters in (3) are the receiver position coordinates in vector $\boldsymbol{b}_r$, the ionospheric delay i_r^{st} on frequency f_1 $\left(\mu_j = \lambda_j^2/\lambda_1^2\right)$ and the carrier-phase ambiguity $a_{r,j}^{st}$. The row-vector $\left(\boldsymbol{g}_r^{st}\right)^T$ contains the difference of the unit-direction vectors to satellites s and t. The between-satellite differencing has eliminated the receiver phase and the receiver code clock offsets. Likewise, the initial receiver phases are absent in the SD ambiguity, as it only contains the satellite initial phases and integer ambiguity, $a_{r,j}^{st} = -\varphi_{,j}^{st}(t_0) + z_{r,j}^{st}$. The ambiguity is constant in time as long as the receiver keeps lock.

The PPP corrections $c_{\phi,r}^{st}$ and $c_{p,r}^{st}$ consist of the tropospheric delay $\mathtt{t}_r^{st}$, the satellite phase and code clock delays, $\delta\mathtt{s}_{,j}^{st}$ and $d\mathtt{s}_{,j}^{st}$, and the receiver relevant orbital information $\mathtt{o}_r^{st}$ of the two satellites. The satellite ephemerides (orbit and clocks) is publicly available information that can be obtained from global tracking networks [5], [6].

For the tropospheric delay $\mathtt{t}_r$, one usually uses an *a priori* model, such as, e.g., the model of [36]. In case such modelling is not considered accurate enough, one may compensate by including the residual tropospheric zenith delay $\mathtt{t}_r^z$ as unknown parameter in (3). Then $\mathtt{t}_r = \mathtt{t}_r' + l_r\mathtt{t}_r^z$, with $\mathtt{t}_r'$ provided by the *a priori* model, l_r the satellite elevation dependent mapping function [37] and $\mathtt{t}_r^z$ the unknown to be estimated tropospheric zenith delay.

To write (3) in vector-matrix form, it is assumed that receiver r tracks $s+1$ satellites on f frequencies. Defining the $fs\times 1$ SD phase and code observation vectors as $\boldsymbol{\phi}_r = [\boldsymbol{\phi}_{r,1}^T,\ldots,\boldsymbol{\phi}_{r,f}^T]^T$ and $\boldsymbol{p}_r = [\boldsymbol{p}_{r,1}^T,\ldots,\boldsymbol{p}_{r,f}^T]^T$, where $\boldsymbol{\phi}_{r,j} = [\phi_{r,j}^{12},\ldots,\phi_{r,j}^{1(s+1)}]^T$, $\boldsymbol{p}_{r,j} = [p_{r,j}^{12},\ldots,p_{r,j}^{1(s+1)}]^T$, $j = 1,\ldots,f$, with a likewise definition for the atmospheric delays, the ambiguities and the corrections, the system of $2fs$ SD observation equations of receiver r follows as:

$$\begin{aligned}\mathrm{E}(\boldsymbol{\phi}_r) &= (\boldsymbol{e}_f\otimes\boldsymbol{G}_r)\boldsymbol{b}_r - (\boldsymbol{\mu}\otimes\boldsymbol{I}_s)\boldsymbol{i}_r\\ &\quad + (\boldsymbol{\Lambda}\otimes\boldsymbol{I}_s)\boldsymbol{a}_r + \boldsymbol{C}_{\phi;r}\\ \mathrm{E}(\boldsymbol{p}_r) &= (\boldsymbol{e}_f\otimes\boldsymbol{G}_r)\boldsymbol{b}_r + (\boldsymbol{\mu}\otimes\boldsymbol{I}_s)\boldsymbol{i}_r + \boldsymbol{C}_{p;r}\end{aligned}\tag{4}$$

with $\boldsymbol{e}_f = (1,\ldots,1)^T$, $\boldsymbol{G}_r = [(\boldsymbol{g}_r^{12})^T,\ldots,(\boldsymbol{g}_r^{1(s+1)})^T]^T$, $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_f)^T$, $\boldsymbol{\Lambda} = \mathrm{diag}[\lambda_1,\ldots,\lambda_f]$, $\boldsymbol{C}_{\phi;r} = \boldsymbol{e}_f\otimes(\boldsymbol{t}_r - \boldsymbol{o}_r) - \boldsymbol{\delta s}$ and $\boldsymbol{C}_{p;r} = \boldsymbol{e}_f\otimes(\boldsymbol{t}_r - \boldsymbol{o}_r) - \boldsymbol{ds}$. Note that in (4), the first satellite is used as reference (pivot) in defining the SD. This choice is not essential as any satellite can be chosen as pivot.

The system of SD observation (4) forms the basis for multifrequency PPP. In case of single-frequency PPP, $\boldsymbol{i}_r$ of (4) becomes part of $\boldsymbol{C}_{\phi;r}$ and $\boldsymbol{C}_{p;r}$, as the ionospheric delays are then provided externally by GIMs [3], [4], [8]. As demonstrated in [38] and [39], the single- and dual-frequency PPP convergence times depend significantly on the precision of the code and ionosphere-free observables. The variance reduction achieved by A-PPP (cf. Section III-C) will therefore reduce their convergence times.

B. The GNSS Array Model

Now consider a platform-fixed array of $r+1$ antennas/receivers, all tracking the same $s+1$ GNSS satellites on the same f frequencies. With two or more antennas, one can formulate the so-called double-differences (DD), which are between-antenna differences of between-satellite differences. For two antennas, q and r, tracking the same $s+1$ satellites on the same f frequencies, the DDs are defined as $\boldsymbol{\phi}_{qr} = \boldsymbol{\phi}_r - \boldsymbol{\phi}_q$ and $\boldsymbol{p}_{qr} = \boldsymbol{p}_r - \boldsymbol{p}_q$. In the DDs, both the receiver clock offsets and the satellite clock offsets get eliminated. Moreover, since double differencing eliminates all initial phases, the DD ambiguity vector $\boldsymbol{a}_{qr} = \boldsymbol{a}_r - \boldsymbol{a}_q$ is an integer vector. This is an important property. Inclusion of integer constraints into the model, strengthens the parameter estimation process and allows one to determine the noninteger parameters with a significantly improved accuracy [11], [40]. To emphasize the integerness of the DD ambiguity vector $\boldsymbol{a}_{qr}$, we write $\boldsymbol{z}_{qr} = \boldsymbol{a}_{qr}$.

The array size is assumed such that also the between-antenna differential contributions of orbital pertubations, troposphere, and ionosphere are small enough to be neglected. Hence, the two correction terms, $\boldsymbol{C}_{\phi;r}$ and $\boldsymbol{C}_{p;r}$, that are present in the between-satellite SD model (4), can be considered absent in the DD array model [5], [31]–[33]. Also, since the unit-direction vectors of two nearby antennas to the same satellite are the same for all practical purposes, we have $\boldsymbol{G} = \boldsymbol{G}_q = \boldsymbol{G}_r$. For two nearby antennas, q and r, the vectorial DD observation equations follow therefore from (4) as

$$\begin{aligned} \mathrm{E}(\boldsymbol{\phi}_{qr}) &= (\boldsymbol{e}_f \otimes \boldsymbol{G})\boldsymbol{b}_{qr} + (\boldsymbol{\Lambda} \otimes \boldsymbol{I}_s)\boldsymbol{z}_{qr} \\ \mathrm{E}(\boldsymbol{p}_{qr}) &= (\boldsymbol{e}_f \otimes \boldsymbol{G})\boldsymbol{b}_{qr} \end{aligned} \tag{5}$$

in which $\boldsymbol{b}_{qr} = \boldsymbol{b}_r - \boldsymbol{b}_q$ is the baseline vector between the two antennas.

In case of more than two antennas, the single-baseline model (5) can be generalized to a multibaseline array-model. Since the size of the array is assumed small, the model can be formulated in *multivariate* form, thus having the same design matrix as that of the single-baseline model (5). For the multivariate formulation, we take antenna 1 as the reference antenna (i.e., the master) and we define the $fs \times r$ phase and code observation matrices as $\boldsymbol{\Phi} = [\boldsymbol{\phi}_{12}, \ldots, \boldsymbol{\phi}_{1(r+1)}]$ and $\boldsymbol{\mathcal{P}} = [\boldsymbol{p}_{12}, \ldots, \boldsymbol{p}_{1(r+1)}]$, the $3 \times r$ baseline matrix as $\boldsymbol{B} = [\boldsymbol{b}_{12}, \ldots, \boldsymbol{b}_{1(r+1)}]$, and the $fs \times r$ DD integer ambiguity matrix as $\boldsymbol{Z} = [\boldsymbol{z}_{12}, \ldots, \boldsymbol{z}_{1(r+1)}]$. The multivariate equivalent to the DD single-baseline model (5) follows then as

$$\mathrm{E}\begin{bmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\mathcal{P}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_f \otimes \boldsymbol{G} & \boldsymbol{\Lambda} \otimes \boldsymbol{I}_s \\ \boldsymbol{e}_f \otimes \boldsymbol{G} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{Z} \end{bmatrix}. \tag{6}$$

The unknowns in this model are the matrices $\boldsymbol{B}$ and $\boldsymbol{Z}$. The matrix $\boldsymbol{B} \in \mathbb{R}^{3\times r}$ consists of the r unknown baseline vectors and the matrix $\boldsymbol{Z} \in \mathbb{Z}^{fs\times r}$ consists of the fsr unknown DD integer ambiguities.

The array geometry is described by the baseline matrix $\boldsymbol{B}$. Once $\boldsymbol{B}$ has been determined, the attitude of the platform can be determined if use is made of the known array geometry in the platform-fixed frame. Let $q = 1, 2, 3$ be the dimension of the antenna array (linear, planar or three dimensional) and let the coordinates in the platform-fixed frame of the known array geometry be given by the column vectors of the $q \times r$ matrix $\boldsymbol{F}$. Then $\boldsymbol{B}$ and $\boldsymbol{F}$ are related as

$$\boldsymbol{B} = \boldsymbol{R}\boldsymbol{F} \tag{7}$$

in which the $q \leq r$ column vectors of $\boldsymbol{R}$ are orthonormal, i.e., $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}_q$ or $\boldsymbol{R} \in \mathbb{O}^{3\times q}$. From this matrix equation, one can solve the attitude matrix $\boldsymbol{R}$ in a least-squares sense, once an estimate of $\boldsymbol{B}$ is available from solving (6) [41], [42].

III. Array-Aided Positioning

In this section, array-aided PPP is introduced as generalization of the PPP concept. Its various positioning applications are described together with the improvements that can be realized.

A. A Combined Position-Attitude Model

Usually the point positioning model (4) is processed independently from the attitude determination model (6). Moreover, in current GNSS attitude determination methods, also the integer estimation problem is treated separately from the attitude estimation process. Existing approaches either first resolve the integer ambiguities and then use the precise baseline estimates for attitude determination [43]–[45] or they use the baseline length constraints only for validation purposes [46], [47]. In this section, however, we combine the two models, (4) and (6), and show the improvement that a combined processing brings.

If we define $\boldsymbol{y}_1 = [\boldsymbol{\phi}_1^T, \boldsymbol{p}_1^T]^T$, $\boldsymbol{\gamma}_1 = [\boldsymbol{C}_{\phi;1}^T, \boldsymbol{C}_{p;1}^T]^T$, $\boldsymbol{Y} = [\boldsymbol{\Phi}^T, \boldsymbol{\mathcal{P}}^T]^T$, $\boldsymbol{H} = [\boldsymbol{\Lambda}^T, \boldsymbol{0}^T]^T$, $\boldsymbol{h} = [-\boldsymbol{\mu}^T, +\boldsymbol{\mu}^T]^T$, the models (4) and (6) can be written in the compact form

$$\begin{aligned} \mathrm{E}(\boldsymbol{y}_1) &= \boldsymbol{M}b_1 + \boldsymbol{N}\boldsymbol{a}_1 + \boldsymbol{d}_1 \\ \mathrm{E}(\boldsymbol{Y}) &= \boldsymbol{M}\boldsymbol{B} + \boldsymbol{N}Z \end{aligned} \tag{8}$$

where $\boldsymbol{M} = (\boldsymbol{e}_{2f} \otimes \boldsymbol{G})$, $\boldsymbol{N} = (\boldsymbol{H} \otimes \boldsymbol{I}_s)$, and $\boldsymbol{d}_1 = (\boldsymbol{h} \otimes \boldsymbol{I}_s)\boldsymbol{i}_1 + \boldsymbol{\gamma}_1$. Note that the two sets of observation equations have no parameters in common. This is the reason why one has treated the two equation sets of (8) separately. The first set is then used to determine the position of the array, i.e., to determine $\boldsymbol{b}_1$ from $\boldsymbol{y}_1$, while the second set is used to determine its attitude, i.e., to determine the rotation matrix $\boldsymbol{R}$ from $\boldsymbol{Y}$ via (7). However, despite the lack of common parameters in (8), the data of the two sets are correlated and therefore not independent. Thus in order to be able to solve the system (8) rigorously, one needs to take this correlation into account. This is possible if we know the complete covariance matrix of $[\boldsymbol{y}_1, \boldsymbol{Y}]$.

To determine the covariance matrix of $[\boldsymbol{y}_1, \boldsymbol{Y}]$, we first have to define the covariance matrix of the SD phase and code observables.

Definition 1 (SD Covariance Matrix): Let $\boldsymbol{\Upsilon} = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{r+1}]$, with $\boldsymbol{y}_i = [((\boldsymbol{I}_f \otimes \boldsymbol{D}_s^T)\boldsymbol{\varphi}_i)^T, ((\boldsymbol{I}_f \otimes \boldsymbol{D}_s^T)\boldsymbol{\rho}_i)^T]^T$ and $\boldsymbol{D}_s^T = [-\boldsymbol{e}_s, \boldsymbol{I}_s]$, where the undifferenced phase and code data vectors of antenna i are given as $\boldsymbol{\varphi}_i = [\boldsymbol{\varphi}_{i,1}^T, \ldots, \boldsymbol{\varphi}_{i,f}^T]^T$, $\boldsymbol{\varphi}_{i,j} = [\phi_{i,j}^1, \ldots, \phi_{i,j}^{s+1}]^T$ and $\boldsymbol{\rho}_i = [\boldsymbol{\rho}_{i,1}^T, \ldots, \boldsymbol{\rho}_{i,f}^T]^T$, $\boldsymbol{\rho}_{i,j} = [p_{i,j}^1, \ldots, p_{i,j}^{s+1}]^T$, $j = 1, \ldots, f$. Then the covariance matrix of $\mathrm{vec}(\boldsymbol{\Upsilon})$ is given as

$$\mathrm{Cov}(\mathrm{vec}(\boldsymbol{\Upsilon})) = \boldsymbol{Q}_r \otimes \boldsymbol{Q} \quad \text{with} \quad \boldsymbol{Q} = \boldsymbol{Q}_f \otimes \boldsymbol{Q}_s \tag{9}$$

where $\boldsymbol{Q}_s = \mathrm{blockdiag}[\boldsymbol{D}_s^T\boldsymbol{Q}_\phi\boldsymbol{D}_s, \boldsymbol{D}_s^T\boldsymbol{Q}_p\boldsymbol{D}_s]$ and $\boldsymbol{Q}_r, \boldsymbol{Q}_f, \boldsymbol{Q}_\phi$ and $\boldsymbol{Q}_p$ are positive definite cofactor matrices.

The structure of the covariance matrix $\mathrm{Cov}(\mathrm{vec}(\boldsymbol{\Upsilon}))$ has been defined such that it accomodates differences in the phase precision, differences in the code precision, frequency dependent tracking precision, satellite elevation dependency and differences in quality of the antenna/receivers in the array. The precision contribution of antenna/receivers and frequency can be specified through $\boldsymbol{Q}_r$ and $\boldsymbol{Q}_f$, while the cofactor matrices $\boldsymbol{Q}_\phi$ and $\boldsymbol{Q}_p$ identify the relative precision contribution of phase and code, including the satellite elevation dependency. The covariance between the phase observables and the code observables is assumed zero.

The required covariance matrix of $[\boldsymbol{y}_1, \boldsymbol{Y}]$ follows, with (9), from applying the variance-covariance propagation law (i.e., propagation of second order (central) moments) to $[\boldsymbol{y}_1, \boldsymbol{Y}] = \boldsymbol{\Upsilon}[\boldsymbol{C}_1, \boldsymbol{D}_r]$, with $\boldsymbol{D}_r$ the differencing matrix and $\boldsymbol{C}_1 = [1, 0, \dots 0]^T$. The complete structure of the combined positioning-attitude model can therefore be summarized as follows.

Definition 2 (Combined Position-Attitude Model): The multivariate observation equations and covariance matrix of the combined position-attitude model are given as

$$\begin{aligned} \mathrm{E}([\boldsymbol{y}_1, \boldsymbol{Y}]) &= \boldsymbol{M}[\boldsymbol{b}_1, \boldsymbol{B}] + \boldsymbol{N}[\boldsymbol{a}_1, \boldsymbol{Z}] + [\boldsymbol{d}_1, \boldsymbol{0}] \\ \mathrm{Cov}(\mathrm{vec}([\boldsymbol{y}_1, \boldsymbol{Y}])) &= \boldsymbol{\Sigma} \otimes \boldsymbol{Q} \end{aligned} \tag{10}$$

with cofactor matrices

$$\boldsymbol{\Sigma} = [\boldsymbol{C}_1, \boldsymbol{D}_r]^T \boldsymbol{Q}_r [\boldsymbol{C}_1, \boldsymbol{D}_r] \quad \text{and} \quad \boldsymbol{Q} = \boldsymbol{Q}_f \otimes \boldsymbol{Q}_s \tag{11}$$

and with the constraints $\boldsymbol{B} = \boldsymbol{R}F$, $\boldsymbol{R} \in \mathbb{O}^{3\times q}$, $\boldsymbol{Z} \in \mathbb{Z}^{fs\times r}$.

The nonzero correlation between $\boldsymbol{y}_1$ and $\boldsymbol{Y}$ is due to the nonzero term $\boldsymbol{C}_1^T \boldsymbol{Q}_r \boldsymbol{D}_r \neq 0$ in (11).

B. A Decorrelating Transformation

Although the equations of $\boldsymbol{y}_1$ and $\boldsymbol{Y}$ [cf. (10)] have no parameters in common, their nonzero correlation implies that treating the positioning problem independently from the attitude determination problem is suboptimal. To properly take the nonzero correlation into account, the two sets of observation equations need to be considered in an integral manner.

We now show how the nonzero correlation can be taken into account, while still being able to work with a system of observation equations that has the same structure as the original one (10). The idea is the following. We first decorrelate the two sets of data with an appropriate decorrelating transformation [cf. (12)]. Then we use the decorrelating transformation to reparametrize the parameters such that the positioning-parameters and the array-parameters are decoupled again. Thus a transformed system of decorrelated equations is obtained with the same structure as the orginal system and that therefore can be solved as such.

Theorem 1 (Decorrelated Positioning-Attitude Model): Let the invertible transformation $\boldsymbol{\mathcal{D}} : \mathbb{R}^{2f(r+1)s} \longrightarrow \mathbb{R}^{2f(r+1)s}$ be given as

$$\boldsymbol{\mathcal{D}} = \begin{bmatrix} 1 & -\boldsymbol{C}_1^T \boldsymbol{Q}_r \boldsymbol{D}_r (\boldsymbol{D}_r^T \boldsymbol{Q}_r \boldsymbol{D}_r)^{-1} \\ \boldsymbol{0} & \boldsymbol{I}_r \end{bmatrix} \otimes \boldsymbol{I}_{2fs} \tag{12}$$

and define $\mathrm{vec}([\bar{\boldsymbol{y}}, \boldsymbol{Y}]) = \boldsymbol{\mathcal{D}}\mathrm{vec}([\boldsymbol{y}_1, \boldsymbol{Y}])$. Then

$$\begin{aligned} \mathrm{E}([\bar{\boldsymbol{y}}, \boldsymbol{Y}]) &= \boldsymbol{M}[\bar{\boldsymbol{b}}, \boldsymbol{B}] + \boldsymbol{N}[\bar{\boldsymbol{a}}, \boldsymbol{Z}] + [\boldsymbol{d}_1, \boldsymbol{0}] \\ \mathrm{Cov}(\mathrm{vec}([\bar{\boldsymbol{y}}, \boldsymbol{Y}])) &= \boldsymbol{S} \otimes \boldsymbol{Q} \end{aligned} \tag{13}$$

with blockdiagonal cofactor matrix

$$\boldsymbol{S} = \mathrm{blockdiag}\left[(\boldsymbol{e}_r^T \boldsymbol{Q}_r^{-1} \boldsymbol{e}_r)^{-1}, (\boldsymbol{D}_r^T \boldsymbol{Q}_r \boldsymbol{D}_r)\right] \tag{14}$$

and with constraints $\boldsymbol{B} = \boldsymbol{R}\boldsymbol{F}$, $\boldsymbol{R} \in \mathbb{O}^{3\times q}$, $\boldsymbol{Z} \in \mathbb{Z}^{fs\times r}$, where $\boldsymbol{e}_r = (1, \dots, 1)^T$, $\mathrm{vec}([\bar{\boldsymbol{b}}, \boldsymbol{B}]) = \boldsymbol{\mathcal{D}}\mathrm{vec}([\boldsymbol{b}_1, \boldsymbol{B}])$, $\mathrm{vec}([\bar{\boldsymbol{a}}, \boldsymbol{Z}]) = \boldsymbol{\mathcal{D}}\mathrm{vec}([\boldsymbol{a}_1, \boldsymbol{Z}])$.

Proof: The proof is given in the Appendix.

Compare (13) and (14) to (10) and (11), respectively. The transformed set of (13) has the same structure as the original set (10), but since $\boldsymbol{S}$ is blockdiagonal, while $\boldsymbol{\Sigma}$ is not, it follows that the observation equations of the decorrelated $\bar{\boldsymbol{y}}$ and $\boldsymbol{Y}$ can be solved separately. Moreover, the same software packages can be used to solve for the parameters of (13) as has been used hiterto to solve for the parameters of (10). Importantly, however, with (13) the results will then be based on having taken the full covariance matrix into account.

C. The A-PPP Model and Its Applications

The decorrelating transformation (12) changed the positioning equations, but not those for attitude. Hence, it is the positioning that takes advantage of the array data when the full correlation between $\boldsymbol{y}_1$ and $\boldsymbol{Y}$ is taken into account. The model for $\bar{\boldsymbol{y}}$ will be referred to as the array-aided precise point positioning model.

Definition 3 (Array-Aided PPP Model): The observation equations and covariance matrix of the A-PPP model are given as

$$\begin{aligned} \mathrm{E}(\bar{\boldsymbol{y}}) &= \boldsymbol{M}\bar{\boldsymbol{b}} + \boldsymbol{N}\bar{\boldsymbol{a}} + \boldsymbol{d}_1 \\ \mathrm{Cov}(\bar{\boldsymbol{y}}) &= \left(\boldsymbol{e}_r^T \boldsymbol{Q}_r^{-1} \boldsymbol{e}_r\right)^{-1} \otimes \boldsymbol{Q} \end{aligned} \tag{15}$$

with the array-aided data vector $\bar{\boldsymbol{y}} = \boldsymbol{y}_1 - \boldsymbol{Y}\boldsymbol{D}_r^+ \boldsymbol{C}_1$ and $\boldsymbol{D}_r^+ = (\boldsymbol{D}_r^T \boldsymbol{Q}_r \boldsymbol{D}_r)^{-1} \boldsymbol{D}_r^T \boldsymbol{Q}_r$.

The precision of $\bar{\boldsymbol{y}}$ is always better than that of $\boldsymbol{y}_1$. This can be shown as follows. Since $1 = (\boldsymbol{C}_1^T \boldsymbol{e}_r)^2 = (\boldsymbol{C}_1^T \boldsymbol{Q}_r^{\frac{1}{2}} . \boldsymbol{Q}_r^{\frac{-1}{2}} \boldsymbol{e}_r)^2 = (\boldsymbol{C}_1^T \boldsymbol{Q}_r \boldsymbol{C}_1)(\boldsymbol{e}_r^T \boldsymbol{Q}_r^{-1} \boldsymbol{e}_r) \cos^2(\alpha)$ (cosine rule) and $\alpha \neq 0$, since $\boldsymbol{C}_1 \neq \boldsymbol{e}_r$ for $r > 1$, the strict inequality $(\boldsymbol{e}_r^T \boldsymbol{Q}_r^{-1} \boldsymbol{e}_r)^{-1} < (\boldsymbol{C}_1^T \boldsymbol{Q}_r \boldsymbol{C}_1)$ holds, and therefore, from (10) and (13), the matrix inequality

$$\mathrm{Cov}(\bar{\boldsymbol{y}}) < \mathrm{Cov}(\boldsymbol{y}_1) \tag{16}$$

follows. Hence, any linear function of $\bar{\boldsymbol{y}}$ will always have a smaller variance than the same function of $\boldsymbol{y}_1$. As an example, consider an array with n receivers that all are of the same quality. Then $\boldsymbol{Q}_r$ is a unit matrix and $\mathrm{Cov}(\bar{\boldsymbol{y}}_1) = \frac{1}{n}\mathrm{Cov}(\boldsymbol{y}_1)$. This '1 over n' rule improvement propagates then also into A-PPP's parameter estimation, thus resulting in improved results.

The A-PPP model can be applied in different ways. Although A-PPP, like PPP, can be used for other applications than positioning, e.g., remote sensing or time-transfer, attention will be restricted here to positioning. Three different positioning modes are considered: platform positioning, on-platform positioning and between-platform positioning. Each require different information from the array.

1) Platform Positioning (Without Ambiguity Resolution): This is the simplest A-PPP variant, as it can be solved in exactly the same way as any of the current PPP-variants. $\boldsymbol{Y}$ is the only array information that is required to construct $\bar{\boldsymbol{y}}$. Since the baseline matrix $\boldsymbol{B}$ and the ambiguity matrix $\boldsymbol{Z}$ do not need to be known, the solution of (15) can do without solving the attitude observation equations of model (13).

To interpret the platform positioning vector, recall from (13) that $\bar{\boldsymbol{b}} = \boldsymbol{b}_1 - \boldsymbol{B}\boldsymbol{D}_r^+ \boldsymbol{C}_1$. Since $\boldsymbol{B}\boldsymbol{D}_r^+ = [\boldsymbol{b}_1, \dots, \boldsymbol{b}_r]\boldsymbol{P}_{D_r}$ and $\boldsymbol{P}_{D_r} = \boldsymbol{I}_r - \boldsymbol{Q}_r^{-1} \boldsymbol{e}_r (\boldsymbol{e}_r^T \boldsymbol{Q}_r^{-1} \boldsymbol{e}_r)^{-1} \boldsymbol{e}_r^T$, it follows that $\bar{\boldsymbol{b}} = [\boldsymbol{b}_1, \dots, \boldsymbol{b}_r] \boldsymbol{Q}_r^{-1} \boldsymbol{e}_r (\boldsymbol{e}_r^T \boldsymbol{Q}_r^{-1} \boldsymbol{e}_r)^{-1}$. Hence, $\bar{\boldsymbol{b}}$ is the weighted least-squares combination of the $r + 1$ antenna positions. For

TABLE I
SINGLE-FREQUENCY, KINEMATIC (MOVING PLATFORM) N-E-U ONE-SIGMA POSITIONING PRECISION OF PPP AND FOUR-ANTENNA A-PPP

$[mm]$	PPP	A-PPP
σ_N	97	46
σ_E	89	48
σ_U	350	288

a diagonal weight matrix $\boldsymbol{Q}_r^{-1} = \text{diag}[w_1, \ldots, w_{r+1}]$, for instance, the position vector $\bar{\boldsymbol{b}}$ is equal to the weighted average

$$\bar{\boldsymbol{b}} = \frac{\sum_{i=1}^{r+1} w_i \boldsymbol{b}_i}{\sum_{i=1}^{r+1} w_i}. \tag{17}$$

Thus A-PPP, based on (15), determines the position of the "center of gravity" of the antenna configuration rather than that of a single antenna position. If needed, these two positions can be made to coincide by using a suitable symmetry in the array geometry. That is, $\bar{\boldsymbol{b}} = \boldsymbol{b}_1$ if $\sum_{i=1}^{r+1} w_i \boldsymbol{b}_{1i} = \boldsymbol{0}$.

Table I illustrates the single-frequency platform positioning performance of PPP and A-PPP. The experiment took place in Perth, Australia, on 30 July 2010 (05:24:00–07:03:59 UTC). The platform consisted of four Sokkia GSR2700 ISX antenna/receivers, three of which were placed in a triangle, 2 meters apart, with the fourth one exactly in the middle of the triangle. The 1-Hz single-frequency L1 GPS phase and code data were collected with a zero degree cut-off elevation angle. To allow for low velocity (pedestrian) platform movement the data was processed in kinematic mode, using as *a priori* PPP corrections, final IGS orbits, final IGS 30-second clock corrections, and final GIM maps. Table I shows the empirically determined North-East-Up (N-E-U) standard deviations (in millimeters) for kinematic PPP and A-PPP. The improvements are clearly visible, although horizontal positioning benefits more than vertical positioning. The height improvement is less, because the PPP and A-PPP common *a-priori* corrections uncertainty impacts the vertical component most.

2) Platform Positioning (With Ambiguity Resolution): PPP with integer ambiguity resolution is possible by means of externally provided corrections that transform the PPP ambiguities to integers [14]–[17]. The advantage of this PPP-RTK method over standard PPP is the considerable strengthening the integer constraints bring to the model. The question is now whether one can still take advantage of this in the A-PPP setup. Afterall, with A-PPP, the ambiguity vector $\bar{\boldsymbol{a}}$ of (15) remains noninteger even after the original SD ambiguities have been corrected to integers. The weighted average of integers is namely generally noninteger.

To resolve the problem of the nonintegerness of $\bar{\boldsymbol{a}}$, use is made of the relation

$$\bar{\boldsymbol{a}} = \boldsymbol{a}_1 - \boldsymbol{Z}\boldsymbol{D}_r^{+}\boldsymbol{C}_1 \tag{18}$$

which shows that one can undo the effect of averaging and express $\bar{\boldsymbol{a}}$ in $\boldsymbol{a}_1$, provided the integer matrix $\boldsymbol{Z}$ is known. Hence, the A-PPP RTK observation equations become

$$\text{E}(\bar{\bar{\boldsymbol{y}}}) = \boldsymbol{M}\bar{\boldsymbol{b}} + \boldsymbol{N}\boldsymbol{a}_1 + \boldsymbol{d}_1 \tag{19}$$

with the $\boldsymbol{Z}$-corrected observation vector given as

$$\bar{\bar{\boldsymbol{y}}} = \bar{\boldsymbol{y}} + \boldsymbol{N}\boldsymbol{Z}\boldsymbol{D}_r^{+}\boldsymbol{C}_1. \tag{20}$$

Since (19) has the same structure as the original PPP RTK equations, it can be solved in the same way. As to the required array information, now both $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are needed. $\boldsymbol{Y}$ is needed to obtain $\bar{\boldsymbol{y}}$ from $\boldsymbol{y}_1$, and $\boldsymbol{Z}$ is needed to obtain $\bar{\bar{\boldsymbol{y}}}$ from $\bar{\boldsymbol{y}}$. The A-PPP system (19) can therefore only be solved, after $\boldsymbol{Z}$ has been solved from the attitude equations of model (13).

Critical in the application of (19) is how fast and how well the integer matrix $\boldsymbol{Z}$ can be estimated. Preferably this should be on a single-epoch (instantaneous) basis, with a sufficiently high probability of correct integer estimation, i.e., $\text{Prob}[\check{\boldsymbol{Z}} = \boldsymbol{Z}] \approx 1$, where $\check{\boldsymbol{Z}}$ is the integer estimator of $\boldsymbol{Z}$[11], [48]. Only if this probability, also referred to as the ambiguity success rate, is sufficiently close to one, can one neglect the uncertainty in the integer estimator $\check{\boldsymbol{Z}}$ of $\boldsymbol{Z}$ and does $\text{Cov}(\bar{\bar{\boldsymbol{y}}}) \approx \text{Cov}(\bar{\boldsymbol{y}}) < \text{Cov}(\boldsymbol{y}_1)$ hold, meaning that one can take advantage of the improved precision of $\bar{\boldsymbol{y}}$ over $\boldsymbol{y}_1$. Section IV shows that such array integer ambiguity resolution is indeed possible with our method of integer $\boldsymbol{Z}$-estimation.

To illustrate the potential of A-PPP RTK, the GPS experiment of Table I was repeated but now with ionospheric- and satellite clock corrections provided by a regional dual-frequency CORS network [17]. The results showed cm-level positioning accuracy ($\sigma_N = 10, \sigma_E = 11, \sigma_U = 21[mm]$) and a one-minute time-to-fix, twice faster for A-PPP RTK than PPP RTK.

3) On-Platform Positioning: Next to determining the position of the platform, it is often also of importance to be able to determine the position of an arbitrary point *on* the platform. In many applications, for instance, the platform will be equipped with additional (remote sensing) sensors. The sensor positions are then needed so as to be able to colocate the remote sensing data with an earth-fixed frame.

Let $\boldsymbol{b}_S$ and $\boldsymbol{f}_S$ be the position vector of the sensor in the earth-fixed frame and in the platform-fixed frame, respectively. Then

$$\boldsymbol{b}_S = \bar{\boldsymbol{b}} + \boldsymbol{R}(\boldsymbol{f}_S - \bar{\boldsymbol{f}}) \tag{21}$$

with $\bar{\boldsymbol{f}} = -\boldsymbol{F}\boldsymbol{D}_r^{+}\boldsymbol{C}_1$ the counterpart of $\bar{\boldsymbol{b}}$ in the platform-fixed frame. Hence, since $\boldsymbol{f}_S$ and $\bar{\boldsymbol{f}}$ are assumed known, $\bar{\boldsymbol{b}}$ and $\boldsymbol{R}$ are needed to determine $\boldsymbol{b}_S$, the sensor position in the earth-fixed frame. Thus next to positioning, now also an attitude solution is needed.

As with $\bar{\boldsymbol{b}}$, the rotation matrix $\boldsymbol{R}$ can be determined with or without integer ambiguity resolution. But, as is shown in Section IV, the quality of $\boldsymbol{R}$ is rather poor for small sized arrays, when solved without the integer ambiguity constraints. Therefore the integer ambiguity resolved rotation matrix has preference and, as is shown in the next section, it can be determined with a high success rate with our method of integer $\boldsymbol{Z}$-estimation.

4) Between-Platform Positioning: The A-PPP concept can also be applied to the important field of relative navigation and formation flying. Examples of applications that can benefit from multiplatform A-PPP include land (robotics and cars [49], [50]), air (uninhabited air vehicles [41], [51]), and space (spacecraft formations and attitude [42], [52]) systems.

TABLE II
SINGLE-FREQUENCY, SINGLE-EPOCH, BETWEEN-PLATFORM AMBIGUITY SUCCESS RATES [%] FOR TWO SINGLE-ANTENNA AND TWO QUADRUPLE-ANTENNA PLATFORMS, SEPARATED BY SHORT (IONOSPHERE-FIXED) AND LONG (IONOSPHERE-FLOAT) DISTANCES (SF: SINGLE-FREQUENCY, DF: DUAL-FREQUENCY, XYZ-FIXED: STATION COORDINATES KNOWN, WL: WIDELANE)

[%]	Ionosphere-fixed		Ionosphere-float	
	SF	DF	XYZ-fixed	WL
1-1	23.8	100	69.1	5.9
4-4	96.4	100	97.0	97.1

Consider two A-PPP equipped platforms, P and Q, each having a system of observation equations like (19). By taking the between-platform difference, one gets

$$\mathrm{E}(\bar{\boldsymbol{y}}_{\mathtt{PQ}}) = \boldsymbol{M}\bar{\boldsymbol{b}}_{\mathtt{PQ}} + \boldsymbol{N}\boldsymbol{z}_{\mathtt{PQ}} + \boldsymbol{d}_{1\mathtt{PQ}} \tag{22}$$

with $\bar{\boldsymbol{y}}_{\mathtt{PQ}} = \bar{\boldsymbol{y}}_{\mathtt{Q}} - \bar{\boldsymbol{y}}_{\mathtt{P}}$ and a likewise definition for $\bar{\boldsymbol{b}}_{\mathtt{PQ}}$, $\boldsymbol{z}_{\mathtt{PQ}}$ and $\boldsymbol{d}_{1\mathtt{PQ}}$.

To solve (22), the $\boldsymbol{Y}$s and $\boldsymbol{Z}$s of both platforms are needed, but not their attitude. The rotation matrices of the two platforms, $\boldsymbol{R}_{\mathtt{P}}$ and $\boldsymbol{R}_{\mathtt{Q}}$, would be needed though, if next to the relative position, also the between-platform relative attitude, $\boldsymbol{R}_{\mathtt{PQ}} = \boldsymbol{R}_{\mathtt{Q}}\boldsymbol{R}_{\mathtt{P}}^T$, is required.

Note, importantly, that the DD ambiguity vector $\boldsymbol{z}_{\mathtt{PQ}}$ in (22) is integer. The following example illustrates how its success-rate can be improved by A-PPP. The success rates are given in Table II for the case the ionospheric delays are assumed absent in the model (ionosphere fixed), as for the case they are estimated as unknown parameters (ionosphere float). They are based on 1 Hz GPS phase and code tracking, with zero degree cut-off elevation angle, from two identical Sokkia-receiver equipped platforms, hundred meter apart, having the same configuration as in the experiment of Table I.

Table II shows the single-epoch succes rate improvement when going from a one-antenna equipped pair (1-1) to a quadruple-antennas equipped platform pair (4-4). The second column of Table II shows a significant improvement of the single-frequency (SF), ionosphere-fixed success rate, thus enabling faster single-frequency precise baseline positioning. Such improvement is not seen for the dual-frequency (DF) case. However, when compared with the SF results, we do see that the SF, multiantennas platform has a close to standard dual-frequency receiver performance.

The results of the fourth column indicate that A-PPP equipped CORS stations, having known coordinates [1], [17], can also benefit significantly. Finally, the last column of Table II indicates the A-PPP improvement of widelane (WL) ambiguity resolution. When positioning under ionosphere-float, full ambiguity resolution is often replaced by partial ambiguity resolution using the widelane [53].

We remark that the success rates of Table II are unconditional, since they are not conditioned on assuming the integer ambiguity matrices of both platforms, $\boldsymbol{Z}_{\mathtt{P}}$ and $\boldsymbol{Z}_{\mathtt{Q}}$, known [cf. (20)]. Hence, Table II's success rates give the probabilities of correctly estimating the between-platform integer ambiguities, irrespective of whether the integer array ambiguities of both platforms were estimated correctly or not. The method used for integer estimating the array ambiguities is described in the next section. There the improved performance of the method is also compared with the standard method of integer ambiguity resolution.

IV. CONSTRAINED ARRAY ESTIMATION

In the previous section it was shown that different A-PPP versions require different array information. For platform positioning without ambiguity resolution, it suffices to know $\boldsymbol{Y}$, (cf. 15). On-platform positioning, however, requires both $\boldsymbol{Y}$ and $\boldsymbol{R}$, [cf. (21)], while any version that includes integer ambiguity resolution needs $\boldsymbol{Z}$ as well. In order to make these A-PPP applications possible, it is shown in this section how to best estimate $\boldsymbol{Z}$ and $\boldsymbol{R}$.

A. The Array and its Constraints

To determine $\boldsymbol{R}$ and $\boldsymbol{Z}$, the array-part of model (13) needs to be solved. Would one only need $\boldsymbol{R}$, the simplest approach would be to solve model (13) with (7) in a least-squares sense while disregarding the integerness of $\boldsymbol{Z}$. In case of GNSS, however, this approach suffers from the drawback that a disregard of the integerness of $\boldsymbol{Z}$, implies that the baseline solution, and therefore the solution of $\boldsymbol{R}$ as well, is driven by the relatively poor code data.

Alternatively therefore, one could solve the array-part of model (13) for $\boldsymbol{B}$ in a least-squares sense, but now with the integerness of $\boldsymbol{Z}$ enforced, and then use this baseline solution to solve for $\boldsymbol{R}$. This second approach is an improvement over the first. Still, however, it can be further improved upon, since the determination of the integer matrix $\boldsymbol{Z}$ will then not have benefitted from the orthonormality of $\boldsymbol{R}$. As will be shown, this improvement turns out to be very significant indeed.

The above discussion makes clear that both constraints, the orthonormality constraint of $\boldsymbol{R}$ in $\boldsymbol{B} = \boldsymbol{RF}$ and the integer constraint on $\boldsymbol{Z}$, need to be enforced from the beginning. The aim of this section is therefore to show how the following, orthonormality-constrained, multivariate (mixed) integer model can be solved in a weighted least-squares sense.

Definition 4 (Constrained Array Model): The $2fs \times r$ matrix observation equation and covariance matrix of the constrained array model are given as

$$\mathrm{E}(\boldsymbol{Y}) = \boldsymbol{MRF} + \boldsymbol{NZ}, \mathrm{Cov}(\mathrm{vec}(\boldsymbol{Y})) = \boldsymbol{P} \otimes \boldsymbol{Q} \tag{23}$$

with the two sets of constraints

$$\boldsymbol{R} \in \mathbb{O}^{3\times q} \text{ and } \boldsymbol{Z} \in \mathbb{Z}^{fs\times r} \tag{24}$$

and where $\boldsymbol{P} = \boldsymbol{D}_r^T\boldsymbol{Q}_r\boldsymbol{D}_r$.

Thus the unknown parameters in this array model are the matrices $\boldsymbol{R}$ and $\boldsymbol{Z}$, constrained by (24).

B. The Role of Integer Ambiguity Resolution

To get a better understanding of the role played by integer ambiguity resolution in determining $\boldsymbol{R}$, let us for the moment disregard the first constraint, $\boldsymbol{R} \in \mathbb{O}^{3\times q}$, and consider, instead of the second constraint, the two extremes cases: $\boldsymbol{Z}$ is known or $\boldsymbol{Z}$ is completely unknown.

Lemma 1 ($\boldsymbol{Z}$ Known, $\boldsymbol{R}$ Unknown): Let the $\boldsymbol{Z}$-constrained LS-estimator of $\boldsymbol{R}$ in model (23) be defined as

$$\hat{\boldsymbol{R}}(\boldsymbol{Z}) = \arg\min_{\boldsymbol{R}\in\mathbb{R}^{3\times q}} \|\text{vec}(\boldsymbol{Y} - \boldsymbol{MRF} - \boldsymbol{NZ})\|^2_{\boldsymbol{Q_{YY}}} \quad (25)$$

with $\boldsymbol{Q}_{YY} = \text{Cov}(\text{vec}(\boldsymbol{Y}))$. Then $\hat{\boldsymbol{R}}(\boldsymbol{Z})$ and its covariance matrix are given as

$$\begin{aligned}\hat{\boldsymbol{R}}(\boldsymbol{Z}) &= \boldsymbol{M}^+(\boldsymbol{Y} - \boldsymbol{NZ})\boldsymbol{F}^+ \\ \boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})} &= (\boldsymbol{FP}^{-1}\boldsymbol{F}^T)^{-1} \otimes (\boldsymbol{M}^T\boldsymbol{Q}^{-1}\boldsymbol{M})^{-1}\end{aligned} \quad (26)$$

with the LS-inverses $\boldsymbol{M}^+ = (\boldsymbol{M}^T\boldsymbol{Q}^{-1}\boldsymbol{M})^{-1}\boldsymbol{M}^T\boldsymbol{Q}^{-1}$ and $\boldsymbol{F}^+ = \boldsymbol{P}^{-1}\boldsymbol{F}^T(\boldsymbol{FP}^{-1}\boldsymbol{F}^T)^{-1}$.

Proof: The proof is given in the Appendix.

This LS estimator of $\boldsymbol{R}$ is denoted as $\hat{\boldsymbol{R}}(\boldsymbol{Z})$ to emphasize its dependence on the value taken for $\boldsymbol{Z}$.

Through the presence of the matrices $\boldsymbol{M}$ and $\boldsymbol{F}$, we clearly recognize the contributions of both the receiver-satellite geometry, via $\boldsymbol{M}$, and the antenna-array geometry, via $\boldsymbol{F}$. Without the antenna-array geometry (i.e., $\boldsymbol{F} = \boldsymbol{I}$ or $\boldsymbol{R} = \boldsymbol{B}$), the solution would read $\hat{\boldsymbol{B}}(\boldsymbol{Z}) = \boldsymbol{M}^+(\boldsymbol{Y} - \boldsymbol{NZ})$. But with the antenna-array geometry included, a further least-squares mapping takes place, from $\hat{\boldsymbol{B}}(\boldsymbol{Z})$ to $\hat{\boldsymbol{R}}(\boldsymbol{Z}) = \hat{\boldsymbol{B}}(\boldsymbol{Z})\boldsymbol{F}^+$.

Now the other extreme, that of a completely unconstrained $\boldsymbol{Z}$-matrix, is considered.

Lemma 2 ($\boldsymbol{Z}$ Unknown, $\boldsymbol{R}$ Unknown): Let the unconstrained LS-estimators of $\boldsymbol{R}$ and $\boldsymbol{Z}$ in model (23) be defined as

$$\{\hat{\boldsymbol{R}}, \hat{\boldsymbol{Z}}\} = \arg\min_{\boldsymbol{R}\in\mathbb{R}^{3\times q}, \boldsymbol{Z}\in\mathbb{R}^{fs\times r}} \|\text{vec}(\boldsymbol{Y} - \boldsymbol{MRF} - \boldsymbol{NZ})\|^2_{\boldsymbol{Q_{YY}}}. \quad (27)$$

Then $\hat{\boldsymbol{R}}$, $\hat{\boldsymbol{Z}}$ and their covariance matrices are given as

$$\begin{aligned}\hat{\boldsymbol{R}} &= \bar{\boldsymbol{M}}^+(\boldsymbol{Y})\boldsymbol{F}^+ \\ \hat{\boldsymbol{Z}} &= \boldsymbol{N}^+(\boldsymbol{Y} - \boldsymbol{M}\hat{\boldsymbol{R}}\boldsymbol{F})\end{aligned} \quad (28)$$

and

$$\begin{aligned}\boldsymbol{Q}_{\hat{R}\hat{R}} &= (\boldsymbol{FP}^{-1}\boldsymbol{F}^T)^{-1} \otimes (\bar{\boldsymbol{M}}^T\boldsymbol{Q}^{-1}\bar{\boldsymbol{M}})^{-1} \\ \boldsymbol{Q}_{\hat{Z}\hat{Z}} &= (\boldsymbol{P}^{-1} \otimes \bar{\boldsymbol{N}}^T\boldsymbol{Q}^{-1}\bar{\boldsymbol{N}} \\ &\quad + \bar{\boldsymbol{P}}^{-1} \otimes \boldsymbol{N}^T\boldsymbol{Q}^{-1}\boldsymbol{P_M}\boldsymbol{N})^{-1}\end{aligned} \quad (29)$$

with $\bar{\boldsymbol{M}}^+ = (\bar{\boldsymbol{M}}^T\boldsymbol{Q}^{-1}\bar{\boldsymbol{M}})^{-1}\bar{\boldsymbol{M}}^T\boldsymbol{Q}^{-1}$, $\boldsymbol{N}^+ = (\boldsymbol{N}^T\boldsymbol{Q}^{-1}\boldsymbol{N})^{-1}\boldsymbol{N}^T\boldsymbol{Q}^{-1}$, $\bar{\boldsymbol{M}} = \boldsymbol{P}^\perp_{\boldsymbol{N}}\boldsymbol{M}$, $\boldsymbol{P}^\perp_{\boldsymbol{N}} = \boldsymbol{I} - \boldsymbol{NN}^+$, $\bar{\boldsymbol{N}} = \boldsymbol{P}^\perp_{\boldsymbol{M}}\boldsymbol{N}$, $\boldsymbol{P}^\perp_{\boldsymbol{M}} = \boldsymbol{I} - \boldsymbol{MM}^+$ and $\bar{\boldsymbol{P}}^{-1} = (\boldsymbol{I} - \boldsymbol{F}^+\boldsymbol{F})\boldsymbol{P}^{-1}$.

Proof: The proof is given in the Appendix.

Since $\boldsymbol{Z}$ is assumed unknown in (27), the precision of $\hat{\boldsymbol{R}}$ is, of course, poorer than that of $\hat{\boldsymbol{R}}(\boldsymbol{Z})$. Importantly, in case of GNSS, this difference is very significant. In case of GNSS, the precision of $\hat{\boldsymbol{R}}(\boldsymbol{Z})$ is driven by the very precise carrier-phase measurements, while the precision of $\hat{\boldsymbol{R}}$ is driven by the relatively low precision code measurements. Denoting the phase variance as σ^2_ϕ and the code variance as σ^2_p, the covariance matrices of the two attitude estimators can shown to be related as

$$\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})} \approx \frac{\sigma^2_\phi}{\sigma^2_p} \quad \boldsymbol{Q}_{\hat{\boldsymbol{R}}\hat{\boldsymbol{R}}} \quad (30)$$

where, in case of current GPS, $\frac{\sigma^2_\phi}{\sigma^2_p} \approx 10^{-4}$ [40]. This shows that a very large precision improvement in the determination of the attitude matrix $\boldsymbol{R}$ can be realized if one would be able to integer estimate $\boldsymbol{Z}$ with negligible uncertainty, i.e., with a success rate $\text{Prob}[\check{\boldsymbol{Z}} = \boldsymbol{Z}] \approx 1$. Achieving the latter, is the goal of integer ambiguity resolution [28].

C. Ambiguity Resolution Without Orthonormality Constraint

Although matrix $\boldsymbol{Z}$ is not known, we know that its entries are all integers. Hence, if one could estimate these entries with a probability of correct integer estimation that is sufficiently close to one, one could treat the integer estimated $\boldsymbol{Z}$ for all practical purposes as known and therefore indeed compute a very precise attitude matrix.

With the integer constraints included, the LS problem turns into a (mixed) integer least-squares (ILS) problem [13]. To determine its solution, we first write its objective function as a sum-of-squares.

Lemma 3 (Multivariate Orthogonal Decomposition): Let $\hat{\boldsymbol{R}}(\boldsymbol{Z})$, $\hat{\boldsymbol{R}}$, $\hat{\boldsymbol{Z}}$ and their covariance matrices be given as in (26), (28), and (29) and let $\hat{\boldsymbol{E}} = \boldsymbol{Y} - \boldsymbol{M}\hat{\boldsymbol{R}}\boldsymbol{F} - \boldsymbol{N}\hat{\boldsymbol{Z}}$. Then

$$\begin{aligned}\|\text{vec}(\boldsymbol{Y} - \boldsymbol{MRF} - \boldsymbol{NZ})\|^2_{\boldsymbol{Q_{YY}}} &= \|\text{vec}(\hat{\boldsymbol{E}})\|^2_{\boldsymbol{Q_{YY}}} \\ &+ \|\text{vec}(\hat{\boldsymbol{Z}} - \boldsymbol{Z})\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{Z}}\hat{\boldsymbol{Z}}}} + \|\text{vec}(\hat{\boldsymbol{R}}(\boldsymbol{Z}) - \boldsymbol{R})\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}}.\end{aligned} \quad (31)$$

Proof: The proof is given in the Appendix.

With this orthogonal decomposition the following result can be proven.

Lemma 4 ($\boldsymbol{Z}$ Integer, $\boldsymbol{R}$ Unknown): Let the $\boldsymbol{R}$-unconstrained (mixed) integer LS-estimators of $\boldsymbol{R}$ and $\boldsymbol{Z}$ in model (23) be defined as

$$\{\hat{\boldsymbol{R}}_{\text{U}}, \check{\boldsymbol{Z}}_{\text{U}}\} = \arg\min_{\boldsymbol{R}\in\mathbb{R}^{3\times q}, \boldsymbol{Z}\in\mathbb{Z}^{fs\times r}} \|\text{vec}(\boldsymbol{Y} - \boldsymbol{MRF} - \boldsymbol{NZ})\|^2_{\boldsymbol{Q_{YY}}}. \quad (32)$$

Then $\hat{\boldsymbol{R}}_{\text{U}}$ and $\check{\boldsymbol{Z}}_{\text{U}}$ are given as

$$\begin{aligned}\hat{\boldsymbol{R}}_{\text{U}} &= \hat{\boldsymbol{R}}(\check{\boldsymbol{Z}}_{\text{U}}) \\ \check{\boldsymbol{Z}}_{\text{U}} &= \arg\min_{\boldsymbol{Z}\in\mathbb{Z}^{fs\times r}} \|\text{vec}(\hat{\boldsymbol{Z}} - \boldsymbol{Z})\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{Z}}\hat{\boldsymbol{Z}}}}.\end{aligned} \quad (33)$$

Proof: Since the first term on the right-hand side (RHS) of (31) is a constant, while the third term on the RHS can be made zero for any $\boldsymbol{Z}$, i.e., by setting $\boldsymbol{R} = \hat{\boldsymbol{R}}(\boldsymbol{Z})$, the integer ambiguity matrix solution, $\check{\boldsymbol{Z}}_{\text{U}}$, follows from integer minimizing the second term on the RHS of (31). Substitution of this integer solution into $\hat{\boldsymbol{R}}(\boldsymbol{Z})$ gives the attitude matrix solution of (33). □

The estimators of (33) are given the suffix $(.)_{\text{U}}$ to emphasize that this solution is still $\boldsymbol{R}$-orthonormality unconstrained.

If the probability mass function of $\check{\boldsymbol{Z}}_{\text{U}}$ is sufficiently peaked at the true but unknown value $\boldsymbol{Z}$, i.e., $\text{Prob}[\check{\boldsymbol{Z}}_{\text{U}} = \boldsymbol{Z}] \approx 1$, then the uncertainty in $\check{\boldsymbol{Z}}_{\text{U}}$ can be neglected for all practical purposes and the covariance matrix of $\hat{\boldsymbol{R}}_{\text{U}}$ can be approximated by $\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}$, being the covariance matrix of the very precise estimator $\hat{\boldsymbol{R}}(\boldsymbol{Z})$, cf. (30). Hence, if a precise *orthonormal* attitude matrix is asked for, one can use as estimator $\check{\boldsymbol{R}}_{\text{U}} = \check{\boldsymbol{R}}(\check{\boldsymbol{Z}}_{\text{U}})$, with $\check{\boldsymbol{R}}(\boldsymbol{Z})$ defined as the orthonormality-constrained least-squares solution

$$\check{\boldsymbol{R}}(\boldsymbol{Z}) = \arg\min_{\boldsymbol{R}\in\mathbb{O}^{3\times q}} \|\text{vec}(\hat{\boldsymbol{R}}(\boldsymbol{Z}) - \boldsymbol{R})\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}}. \quad (34)$$

This problem reduces to Wahba's problem [54]–[56], also known as the "orthogonal Procrustes problem" [57], [58], in case the covariance matrix of $\hat{\boldsymbol{R}}(\boldsymbol{Z})$ would have the special structure $\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})} = (\boldsymbol{F}\boldsymbol{P}^{-1}\boldsymbol{F}^T)^{-1} \otimes \boldsymbol{I}$, with $\boldsymbol{P}$ diagonal. The many solution methods of Wahba's problem have been reviewed in [59]. One of the simplest is based on the singular value decomposition (SVD) of the baseline matrix. It allows a direct computation of the attitude matrix.

For our GNSS array, $\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}$ is fully populated, and therefore no such direct solution, as with Wahba's problem, can be used to solve (34). This nonlinear least-squares problem is therefore iteratively solved by means of one of the gradient descent methods, like the Gauss-Newton method, having a local linear rate of convergence, or the Newton method, having a local quadratic rate of convergence. These gradient descent methods work well when initialized by a good starting value. In our case, the solution of Wahba's problem provides a good starting value. For a numerical-statistical analysis of nonlinear least-squares procedures as used in positioning, we refer to, e.g., [60]–[62].

D. Ambiguity Resolution With Orthonormality Constraint

As is shown in [45], the required high probability of correct integer estimation of $\check{\boldsymbol{Z}}_{\mathrm{U}}$ is feasible in the GNSS multifrequency case ($f > 1$), but generally problematic in the single-frequency case. This shows that the single-frequency array model needs a further strengthening.

To increase the strength of the array model, we now include the orthonormality constraint $\boldsymbol{R} \in \mathbb{O}^{3\times q}$ from the start. With this constraint rigorously incorporated into the integer estimation process, a higher probability of correct integer estimation can be achieved. The inclusion of the constraint $\boldsymbol{R} \in \mathbb{O}^{3\times q}$ is thus not so much for the purpose of forcing the solution of $\boldsymbol{R}$ to be orthonormal per se, but rather to aid the integer ambiguity resolution process.

With both the integer constraint and the orthonormality constraint included, the minimization problem becomes a constrained (mixed) ILS problem.

Theorem 2 (Z Integer, R Orthonormal): Let the orthonormality-constrained (mixed) integer LS-estimators of $\boldsymbol{R}$ and $\boldsymbol{Z}$ in model (23) be defined as

$$\{\check{\boldsymbol{R}}, \check{\boldsymbol{Z}}\} = \arg \min_{\boldsymbol{R}\in\mathbb{O}^{3\times q}, \boldsymbol{Z}\in\mathbb{Z}^{fs\times r}} \|\mathrm{vec}(\boldsymbol{Y} - \boldsymbol{M}\boldsymbol{R}\boldsymbol{F} - \boldsymbol{N}\boldsymbol{Z})\|^2_{\boldsymbol{Q}_{\boldsymbol{Y}\boldsymbol{Y}}}. \tag{35}$$

Then $\check{\boldsymbol{R}}$ and $\check{\boldsymbol{Z}}$ are given as

$$\check{\boldsymbol{R}} = \check{\boldsymbol{R}}(\check{\boldsymbol{Z}}) \text{ and } \check{\boldsymbol{Z}} = \arg \min_{\boldsymbol{Z}\in\mathbb{Z}^{fs\times r}} J(\boldsymbol{Z}) \tag{36}$$

with the ambiguity objective function given as

$$J(\boldsymbol{Z}) = \|\mathrm{vec}(\hat{\boldsymbol{Z}} - \boldsymbol{Z})\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{Z}}\hat{\boldsymbol{Z}}}} + \|\mathrm{vec}(\hat{\boldsymbol{R}}(\boldsymbol{Z}) - \check{\boldsymbol{R}}(\boldsymbol{Z}))\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}}. \tag{37}$$

Proof: The proof is given in the Appendix.

Note, importantly, that the ambiguity objective function (37) differs from that of $\check{\boldsymbol{Z}}_{\mathrm{U}}$ [cf. (33)] by the presence of the $\boldsymbol{R}$-dependent second term. The presence of both constraints is therefore felt when evaluating this objective function. In integer minimizing $J(\boldsymbol{Z})$, not only the weighted distance between $\boldsymbol{Z}$ and $\hat{\boldsymbol{Z}}$ counts [as is the case in (33)], but also the weighted distance between $\hat{\boldsymbol{R}}(\boldsymbol{Z})$ and its closest orthonormal matrix $\check{\boldsymbol{R}}(\boldsymbol{Z})$. The weights are determined by the inverses of $\boldsymbol{Q}_{\hat{\boldsymbol{Z}}\hat{\boldsymbol{Z}}}$ and $\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}$, respectively. And as remarked earlier, in case of GNSS, the covariance matrix $\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}$ is driven by the very precise carrier-phase data [cf. (30)]. Thus the second term in the ambiguity objective function (37) receives a relatively large weight and contributes significantly to the improved success rate performance of $\check{\boldsymbol{Z}}$ over $\check{\boldsymbol{Z}}_{\mathrm{U}}$. This method is therefore also our method of choice for realizing array-aided PPP.

TABLE III
SINGLE-FREQUENCY, SINGLE-EPOCH, UNCONSTRAINED- (U) AND CONSTRAINED-(C) AMBIGUITY SUCCESS RATES [%] FOR TWO DUAL-ANTENNA PLATFORMS AND ONE TRIPLE-ANTENNA PLATFORM. THE SUCCESS RATES ARE GIVEN FOR A VARYING NUMBER OF TRACKED SATELLITES (5–9)

#	Dual Ant. 1		Dual Ant. 2		Triple Ant.	
Sats	U [%]	C [%]	U [%]	C [%]	U [%]	C [%]
5	8.9	79.3	8.9	74.6	5	99.6
6	42.0	98.2	40.1	97.1	38.9	100
7	75.5	99.7	76.1	99.8	78.5	100
8	97.4	100	96.9	100	97.7	100
9	99.8	100	99.7	100	99.8	100

The following experiment illustrates the very high success rates that can be achieved when working with the ambiguity objective function (37), instead of with the standard quadratic ambiguity objective function of (33). Located at a stationary point in Limburg, the Netherlands, a fixed array of three antennas (a Trimble Zephyr Geodetic L1/L2, the Master, and two Trimble Geodetic W Groundplane, the auxiliaries), connected to three Trimble receivers (a Trimble R7 and two Trimble SSi), was used to collect (10:44–13:29 UTC) and process 1 Hz data, with a zero cut-off elevation angle. The two baselines formed by the three antennas have lengths 2.214 m and 1.742 m, with a 66.4–degree relative orientation.

The single-frequency, single-epoch success rates are given in Table III as function of the number of tracked satellites and the number of antennas used. For each configuration, the unconstrained (U) and constrained (C) success rates are given, based on using (33) and (37), respectively. The number of tracked satellites was artificially reduced to show the robustness against constellation availability. Also, different baselines have been included in the model: for the two single-baseline (dual-antennas) cases only the baseline length is used as *a priori* constraint, whereas for the two-baseline (triple-antennas) case the complete geometry is used to construct the $\boldsymbol{F}$ matrix of (7).

As the results show (compare the U- and C-columns), the success rates improve dramatically when the constraints are exploited using (37). For the worst scenario, with only five satellites in view, the inclusion of the dual-antennas length constraint is sufficient to increase the success rate from about 9% to about 75%–79%. The constrained success rate increases even further to 99.6% when the full sets of constraints for the three antennas is exploited. Also note that the constrained success rates are far more robust against variability in number of tracked satellites than the unconstrained success rates are. The results of Table III are typical for the performance of the ambiguity objective function (37). The high success rates show that real-time A-PPP platform ambiguity resolution is possible and that reinitialization, in case of a complete loss of lock, only requires a few epochs at most.

The estimation theory and method presented is not restricted to a minimum or maximum antenna separation. But the implicit assumption is of course that the antennas do not interfere with one another. As to the suitability of antenna spacing: smaller spacing makes the integer estimation process simpler (if less than 0.5 wavelength the simplest integer rounding techniques get closer to optimal performance in terms of maximizing the success rate). Larger spacing, however, improves platform's attitude resolution performance [cf. (26) and (29)].

E. The Integer Ambiguity Search

An integer search is needed to solve for the integer minimizer of $J(\boldsymbol{Z})$ [cf. (37)]. Such search is conceptually simple in principle. The search can be confined to any nonempty discrete set of the type

$$\Omega(\chi^2) = \{\boldsymbol{Z} \in \mathbb{Z}^{fs\times r} \mid J(\boldsymbol{Z}) \leq \chi^2\} \tag{38}$$

where χ^2 is a user-defined positive constant; it controls the size of the search space. If $\Omega(\chi^2)$ is nonempty, then the integer minimizer is, by definition, contained in it. It is then found by first collecting all integer matrices inside $\Omega(\chi^2)$, followed by selecting the one that returns the smallest function value $J(\boldsymbol{Z})$.

Although conceptually simple, the actual search in our A-PPP case turns out to be somewhat more complex. To appreciate this complexity, several issues need to be addressed. First, consider the shape of the search space $\Omega(\chi^2)$. The search space would be ellipsoidal, in case the second, $\boldsymbol{R}$-dependent, term in $J(\boldsymbol{Z})$ would be absent [cf. (37)]. The presence of this attitude-dependent term, however, turns $\Omega(\chi^2)$ into a nonellipsoidal, nonconvex search space. This effect is emphasized the more so, since the covariance matrix $\boldsymbol{Q}_{\hat{\boldsymbol{Z}}\hat{\boldsymbol{Z}}}$ in the first term of $J(\boldsymbol{Z})$ is driven by the relatively poor code precision, while the covariance matrix $\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}$ in the second term of $J(\boldsymbol{Z})$ is driven by the very precise carrier phase precision.

Second, consider the user-defined positive constant χ^2 in (38). It determines the size of the search space. Any choice $\chi^2 = J(\boldsymbol{Z}_0)$, with $\boldsymbol{Z}_0$ an integer matrix, guarantees that the search space is nonempty. At the same time, however, one would like χ^2 to be small enough, so as to have a search space with not too many integer candidates. Therefore some care needs to be excersized in choosing $\boldsymbol{Z}_0$. Any $\boldsymbol{Z}_0$ which is too far from the actual integer minimizer can be expected to result in a too large χ^2, especially due to the amplifying effect of the second term in $J(\boldsymbol{Z})$. We therefore choose $\boldsymbol{Z}_0 = \lceil \hat{\boldsymbol{Z}}^* \rfloor$, where $\hat{\boldsymbol{Z}}^*$ is the *real-valued* minimizer of $J(\boldsymbol{Z})$ and $\lceil . \rfloor$ denotes rounding to nearest integer. It is our experience that this choice works very well. Note that $\hat{\boldsymbol{Z}}^*$ is the orthonormality-constrained least-squares solution of the ambiguity matrix $\boldsymbol{Z}$.

Third, consider the actual evaluation of $J(\boldsymbol{Z})$ (cf. 37). Any such evaluation also requires the evaluation of $\hat{\boldsymbol{R}}(\boldsymbol{Z})$ and therefore, for any candidate $\boldsymbol{Z}$, the solution of a nonlinear constrained least-squares problem like (34). Since such minimization for every candidate in $\Omega(\chi^2)$ is a computational burden on the search, the search efficiency can be improved if one would be able to work with an easier-to-evaluate function of which the level set would still be a good approximation to $\Omega(\chi^2)$. We therefore work with an easy-to-evaluate, sharp upper bounding function $J'(\boldsymbol{Z})$, having as level set $\Omega'(\chi^2) = \{\boldsymbol{Z} \in \mathbb{Z}^{fs\times r} \mid J'(\boldsymbol{Z}) \leq \chi^2\}$. Then

$$J(\boldsymbol{Z}) \leq J'(\boldsymbol{Z}) \text{ and } \Omega'(\chi^2) \subset \Omega(\chi^2). \tag{39}$$

Note that for any orthonormal matrix $\boldsymbol{R}_0 \in \mathbb{O}^{3\times q}$, the function

$$J'(\boldsymbol{Z}) = \|\text{vec}(\hat{\boldsymbol{Z}} - \boldsymbol{Z})\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{Z}}\hat{\boldsymbol{Z}}}} + \|\text{vec}(\hat{\boldsymbol{R}}(\boldsymbol{Z}) - \boldsymbol{R}_0)\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}} \tag{40}$$

is an upper bound for $J(\boldsymbol{Z})$. Thus $J'(\boldsymbol{Z})$ can expected to be a good approximation to $J(\boldsymbol{Z})$, if $\boldsymbol{R}_0$ is a good approximation to $\hat{\boldsymbol{R}}(\boldsymbol{Z})$. This suggests that we take the closed form solution of Wahba's problem as our choice for $\boldsymbol{R}_0$. And indeed, it is our experience that this choice results in a sharp upper bound. The drawback of this choice is, however, that it requires the solution of an SVD for every candidate $\boldsymbol{Z}$. Our method of choice is therefore to use, instead of the SVD, the (weighted) Gramm-Schmidt orthogonalization of $\hat{\boldsymbol{R}}(\boldsymbol{Z})$. This is faster to execute than the SVD and still results in a sharp enough upper bound.

The actual search is similar as used in [10] and proceeds as follows. We start the search for an integer candidate in the initial search space $\Omega'(\chi_0^2) \subset \Omega(\chi_0^2)$, where $\chi_0^2 = J'(\boldsymbol{Z}_0)$. Let this candidate be $\boldsymbol{Z}_1$. Then $J'(\boldsymbol{Z}_1) = \chi_1^2 < \chi_0^2$, which gives the shrunken search space $\Omega'(\chi_1^2)$, in which again an integer candidate is searched, say $\boldsymbol{Z}_2$. This iterative process of *'search and shrink'* is repeated until the integer minimizer of $J'(\boldsymbol{Z})$, say $\check{\boldsymbol{Z}}'$, is found. Since this minimizer need not be the minimizer of $J(\boldsymbol{Z})$ (although in practice it usually is), the search space $\Omega(\bar{\chi}^2) \supset \Omega'(\bar{\chi}^2)$, with $\bar{\chi}^2 = J'(\check{\boldsymbol{Z}}')$, is searched. The sought-for minimizer $\check{\boldsymbol{Z}}$ is then selected from the candidates in $\Omega(\bar{\chi}^2)$. In practice, with our choice of bounding function, $\Omega(\bar{\chi}^2)$ contains only a few candidates and usually even only one. As a result the integer minimizer of $J(\boldsymbol{Z})$ can be found efficiently.

V. Summary and Conclusion

In this paper, the GNSS A-PPP concept was introduced as a generalization of PPP. A-PPP is a GNSS measurement concept that uses GNSS data from multiple antennas in known formation to realize improved GNSS parameter estimation (position, attitude, time, and atmospheric delays).

For its stochastic model a general structure was introduced so as to accomodate differences in phase precision, differences in code precision, frequency dependent tracking precision, satellite elevation dependency and also differences in quality of the antenna/receivers in the array. By means of a decorrelating transformation, applied to a combined positioning and attitude model, it was shown which improvements array-aiding brings to the different forms of positioning. The improvements can be exploited in different ways, e.g., to improve accuracy, to reduce convergence time, to achieve higher success rates or to improve between-platform positioning. The A-PPP improvements were illustrated by means of empirical results obtained from GPS experiments.

To enable fast and accurate A-PPP, a novel orthonormality-constrained multivariate (mixed) integer least-squares problem was introduced and solved. It was shown that its integer matrix constraint is necessary to obtain the most precise instantaneous attitude- and position solution, whereas the inclusion of the orthonormality constraint in the ambiguity objective function is essential to achieve high instantaneous probabilities of correct integer estimation. We also discussed the nonlinear A-PPP ambiguity objective function and presented a search for its integer matrix minimizer.

The A-PPP principle is generally applicable. It applies to single-, dual-, and multifrequency GNSS receivers, as well as to any current and future GNSS (e.g., Europe's Galileo and China's Compass), standalone or in combination. The integer ambiguity resolved array-aiding concept is not restricted to GNSS, as it may apply to, e.g., acoustic phase-based positioning [63] and other interferometric techniques as well.

APPENDIX

Proof of Theorem 1: Application of the one-to-one D-transformation, $\text{vec}([\bar{\boldsymbol{y}}, \boldsymbol{Y}]) = \boldsymbol{\mathcal{D}}\text{vec}([\boldsymbol{y}_1, \boldsymbol{Y}])$, to the multivariate observation equations of (10), directly gives those of (13).

To derive the covariance matrix $\text{Cov}(\text{vec}([\bar{\boldsymbol{y}}, \boldsymbol{Y}]))$, we first substitute $\text{vec}([\boldsymbol{y}_1, \boldsymbol{Y}]) = \boldsymbol{\Upsilon}[\boldsymbol{C}_1, \boldsymbol{D}_r]$, with $\boldsymbol{\Upsilon} = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{r+1}]$, into $\text{vec}([\bar{\boldsymbol{y}}, \boldsymbol{Y}]) = \boldsymbol{\mathcal{D}}\text{vec}([\boldsymbol{y}_1, \boldsymbol{Y}])$. This gives

$$\text{vec}([\bar{\boldsymbol{y}}, \boldsymbol{Y}]) = \left([\boldsymbol{P}_{\boldsymbol{D}_r}^{\perp}\boldsymbol{C}_1, \boldsymbol{D}_r]^T \otimes \boldsymbol{I}_{2fs}\right)\text{vec}(\boldsymbol{\Upsilon}) \quad \text{(A1)}$$

in which $\boldsymbol{P}_{D_r}^{\perp}$ denotes the orthogonal projector $\boldsymbol{P}_{D_r}^{\perp} = \boldsymbol{I} - \boldsymbol{D}_r(\boldsymbol{D}_r^T\boldsymbol{Q}_r\boldsymbol{D}_r)^{-1}\boldsymbol{D}_r^T\boldsymbol{Q}_r$. With $\text{Cov}(\text{vec}(\boldsymbol{\Upsilon}) = \boldsymbol{Q}_r \otimes \boldsymbol{Q}$ [cf. (9)], an application of the variance-covariance propagation law to (A1) gives

$$\text{Cov}(\text{vec}([\bar{\boldsymbol{y}}, \boldsymbol{Y}])) = \boldsymbol{S} \otimes \boldsymbol{Q} \quad \text{(A2)}$$

with

$$\boldsymbol{S} = [\boldsymbol{P}_{D_r}^{\perp}\boldsymbol{C}_1, \boldsymbol{D}_r]^T\boldsymbol{Q}_r[\boldsymbol{P}_{D_r}^{\perp}\boldsymbol{C}_1, \boldsymbol{D}_r]. \quad \text{(A3)}$$

Since $\boldsymbol{D}_r^T\boldsymbol{Q}_r\boldsymbol{P}_{D_r}^{\perp} = 0$ and the projector $\boldsymbol{P}_{D_r}^{\perp}$ can alternatively be expressed as $\boldsymbol{P}_{D_r}^{\perp} = \boldsymbol{Q}_r^{-1}\boldsymbol{e}_r(\boldsymbol{e}_r^T\boldsymbol{Q}_r^{-1}\boldsymbol{e}_r)^{-1}\boldsymbol{e}_r^T$, because $\boldsymbol{D}_r^T\boldsymbol{e}_r = \boldsymbol{0}$, we finally obtain

$$\begin{aligned}\boldsymbol{S} &= \text{blockdiag}\left[\boldsymbol{C}_1^T\boldsymbol{Q}_r\boldsymbol{P}_{D_r}^{\perp}\boldsymbol{C}_1, (\boldsymbol{D}_r^T\boldsymbol{Q}_r\boldsymbol{D}_r)\right]\\ &= \text{blockdiag}\left[(\boldsymbol{e}_r^T\boldsymbol{Q}_r^{-1}\boldsymbol{e}_r)^{-1}, (\boldsymbol{D}_r^T\boldsymbol{Q}_r\boldsymbol{D}_r)\right]. \quad \text{(A4)}\end{aligned}$$

This concludes the proof of the theorem. □

Proof of Lemma 1: The system of multivariate normal equations of the LS-problem (25) is given as

$$(\boldsymbol{F}\boldsymbol{P}^{-1}\boldsymbol{F}^T \otimes \boldsymbol{M}^T\boldsymbol{Q}^{-1}\boldsymbol{M})\hat{\boldsymbol{R}}(\boldsymbol{Z}) = (\boldsymbol{F}\boldsymbol{P}^{-1} \otimes \boldsymbol{M}^T\boldsymbol{Q}^{-1})\text{vec}(\boldsymbol{Y}). \quad \text{(A5)}$$

With the use of the Kronecker product property $(\boldsymbol{A} \otimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1}$ ($\boldsymbol{A}$ and $\boldsymbol{B}$ invertible matrices), inversion of (A5) gives $\hat{\boldsymbol{R}}(\boldsymbol{Z})$ of (26). The covariance matrix $\boldsymbol{Q}_{\hat{R}(Z)\hat{R}(Z)}$ follows from an application of the variance-covariance propagation law to the expression for $\hat{\boldsymbol{R}}(\boldsymbol{Z})$. □

Proof of Lemma 2: The multivariate normal equations of the LS-problem (35) are given as

$$\begin{bmatrix}\boldsymbol{F}\boldsymbol{P}^{-1}\boldsymbol{F}^T \otimes \boldsymbol{M}^T\boldsymbol{Q}^{-1}\boldsymbol{M} & \boldsymbol{F}\boldsymbol{P}^{-1} \otimes \boldsymbol{M}^T\boldsymbol{Q}^{-1}\boldsymbol{N}\\ \boldsymbol{P}^{-1}\boldsymbol{F}^T \otimes \boldsymbol{N}^T\boldsymbol{Q}^{-1}\boldsymbol{M} & \boldsymbol{P}^{-1} \otimes \boldsymbol{N}^T\boldsymbol{Q}^{-1}\boldsymbol{N}\end{bmatrix} \times \begin{bmatrix}\text{vec}(\hat{\boldsymbol{R}})\\ \text{vec}(\hat{\boldsymbol{Z}})\end{bmatrix} = \begin{bmatrix}(\boldsymbol{F}\boldsymbol{P}^{-1} \otimes \boldsymbol{M}^T\boldsymbol{Q}^{-1})\text{vec}(\boldsymbol{Y})\\ (\boldsymbol{P}^{-1} \otimes \boldsymbol{N}^T\boldsymbol{Q}^{-1})\text{vec}(\boldsymbol{Y})\end{bmatrix}. \quad \text{(A6)}$$

After reduction for $\hat{\boldsymbol{Z}}$, the reduced normal equations are obtained as

$$[\boldsymbol{F}\boldsymbol{P}^{-1}\boldsymbol{F}^T \otimes \bar{\boldsymbol{M}}^T\boldsymbol{Q}^{-1}\bar{\boldsymbol{M}}]\text{vec}(\hat{\boldsymbol{R}}) = [\boldsymbol{F}\boldsymbol{P}^{-1} \otimes \bar{\boldsymbol{M}}^T\boldsymbol{Q}^{-1}]\text{vec}(\boldsymbol{Y}) \quad \text{(A7)}$$

with $\bar{\boldsymbol{M}} = \boldsymbol{P}_N^{\perp}\boldsymbol{M}, \boldsymbol{P}_N = \boldsymbol{N}(\boldsymbol{N}^T\boldsymbol{Q}^{-1}\boldsymbol{N})^{-1}\boldsymbol{N}^T\boldsymbol{Q}^{-1}$ and $\boldsymbol{P}_N^{\perp} = \boldsymbol{I} - \boldsymbol{P}_N$. Inversion of (A7) gives $\hat{\boldsymbol{R}}$ of (28).

With $\hat{\boldsymbol{R}}$ given, the normal equation for $\hat{\boldsymbol{Z}}$ follows from (A6) as

$$\begin{aligned}&(\boldsymbol{P}^{-1} \otimes \boldsymbol{N}^T\boldsymbol{Q}^{-1}\boldsymbol{N})\text{vec}(\hat{\boldsymbol{Z}})\\ &\quad = (\boldsymbol{P}^{-1} \otimes \boldsymbol{N}^T\boldsymbol{Q}^{-1})\text{vec}(\boldsymbol{Y} - \boldsymbol{M}\hat{\boldsymbol{R}}\boldsymbol{F}). \quad \text{(A8)}\end{aligned}$$

Inversion gives $\hat{\boldsymbol{Z}}$ of (28). The covariance matrices $\boldsymbol{Q}_{\hat{R}\hat{R}}$ and $\boldsymbol{Q}_{\hat{Z}\hat{Z}}$ of (29) follow from an application of the variance-covariance propagation law to the expressions of $\hat{\boldsymbol{R}}$ and $\hat{\boldsymbol{Z}}$ in (28). □

Proof of Lemma 3: Denote the objective function $\|\text{vec}(\boldsymbol{Y} - \boldsymbol{M}\boldsymbol{R}\boldsymbol{F} - \boldsymbol{N}\boldsymbol{Z})\|_{\boldsymbol{Q}_{\boldsymbol{YY}}}^2$, which is a quadratic form in $\boldsymbol{x} = \text{vec}(\boldsymbol{R}, \boldsymbol{Z})$, as $\mathcal{E}(\boldsymbol{x})$. Since its gradient vanishes at its minimizer $\hat{\boldsymbol{x}} = \text{vec}(\hat{\boldsymbol{R}}, \hat{\boldsymbol{Z}})$, the quadratic form can be written as the sum of its zero-order and second-order term

$$\mathcal{E}(\boldsymbol{x}) = \mathcal{E}(\hat{\boldsymbol{x}}) + (\hat{\boldsymbol{x}} - \boldsymbol{x})^T\boldsymbol{N}(\hat{\boldsymbol{x}} - \boldsymbol{x}) \quad \text{(A9)}$$

with $\boldsymbol{N}$ being the normal matrix of (A6) (it is $\frac{1}{2}$ times the Hessian matrix of $\mathcal{E}(\boldsymbol{x})$). Define the blocktriangular transformation

$$\boldsymbol{T} = \begin{bmatrix}\boldsymbol{I} & \boldsymbol{F}^{+T} \otimes \boldsymbol{M}^{+}\boldsymbol{N}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}. \quad \text{(A10)}$$

Then

$$\boldsymbol{T}(\hat{\boldsymbol{x}} - \boldsymbol{x}) = \text{vec}(\hat{\boldsymbol{R}}(\boldsymbol{Z}) - \boldsymbol{R}, \hat{\boldsymbol{Z}} - \boldsymbol{Z}) \quad \text{(A11)}$$

and

$$\boldsymbol{T}^{-T}\boldsymbol{N}\boldsymbol{T}^{-1} = \text{blockdiagonal}\left[\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}^{-1}, \boldsymbol{Q}_{\hat{\boldsymbol{Z}}\hat{\boldsymbol{Z}}}^{-1}\right]. \quad \text{(A12)}$$

Hence

$$\begin{aligned}&(\hat{\boldsymbol{x}} - \boldsymbol{x})^T\boldsymbol{N}(\hat{\boldsymbol{x}} - \boldsymbol{x})\\ &= \|\text{vec}(\hat{\boldsymbol{R}}(\boldsymbol{Z}) - \boldsymbol{R})\|_{\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}}^2 + \|\text{vec}(\hat{\boldsymbol{Z}} - \boldsymbol{Z})\|_{\boldsymbol{Q}_{\hat{\boldsymbol{Z}}\hat{\boldsymbol{Z}}}}^2. \quad \text{(A13)}\end{aligned}$$

This combined with (A9) concludes the proof. □

Proof of Theorem 2: Using the orthogonal decomposition (31), we can write

$$\begin{aligned}&\min_{\boldsymbol{R}\in\mathbb{O}^{3\times q}, \boldsymbol{Z}\in\mathbb{Z}^{fs\times r}} \|\text{vec}(\boldsymbol{Y} - \boldsymbol{M}\boldsymbol{R}\boldsymbol{F} - \boldsymbol{N}\boldsymbol{Z})\|_{\boldsymbol{Q}_{\boldsymbol{YY}}}^2\\ &= \|\text{vec}(\hat{\boldsymbol{E}})\|_{\boldsymbol{Q}_{\boldsymbol{YY}}}^2\\ &\quad + \min_{\boldsymbol{Z}\in\mathbb{Z}^{fs\times r}}\Big(\|\text{vec}(\hat{\boldsymbol{Z}} - \boldsymbol{Z})\|_{\boldsymbol{Q}_{\hat{\boldsymbol{Z}}\hat{\boldsymbol{Z}}}}^2\\ &\quad + \min_{\boldsymbol{R}\in\mathbb{O}^{3\times q}} \|\text{vec}(\hat{\boldsymbol{R}}(\boldsymbol{Z}) - \boldsymbol{R})\|_{\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}}^2\Big). \quad \text{(A14)}\end{aligned}$$

Hence, if we define $\check{\boldsymbol{R}}(\boldsymbol{Z}) = \arg\min_{\boldsymbol{R}\in\mathbb{O}^{3\times q}} \|\text{vec}(\hat{\boldsymbol{R}}(\boldsymbol{Z}) - \boldsymbol{R})\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}}$, the integer ambiguity matrix minimizer of (A14) is as follows:

$$\check{\boldsymbol{Z}} = \arg\min_{\boldsymbol{Z}\in\mathbb{Z}^{fs\times r}} \left(\|\text{vec}(\hat{\boldsymbol{Z}} - \boldsymbol{Z})\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{Z}}\hat{\boldsymbol{Z}}}} + \|\text{vec}(\hat{\boldsymbol{R}}(\boldsymbol{Z}) - \check{\boldsymbol{R}}(\boldsymbol{Z}))\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{R}}(\boldsymbol{Z})\hat{\boldsymbol{R}}(\boldsymbol{Z})}} \right) \quad \text{(A15)}$$

and the corresponding orthonormal attitude matrix minimizer as $\check{\boldsymbol{R}} = \check{\boldsymbol{R}}(\check{\boldsymbol{Z}})$. □

Acknowledgment

The author would like to acknowledge P. Buist and G. Giorgi of the Delft University of Technology, Delft, the Netherlands. They have been instrumental in helping implement and test the multivariate estimation algorithms. L. Huisman and D. Odijk of the Curtin University of Technology, Perth, Australia, executed and analyzed the A-PPP experiments.

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