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Rank Reduction Direction-of-Arrival Estimators With an Improved Robustness Against Subarray Orientation Errors

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Abstract—The problem of signal direction-of-arrival (DOA) estimation in partly calibrated arrays composed of multiple subarrays with unknown intersubarray parameters and imperfectly known subarray orientations is studied. The recently developed spectral and root versions of the rank reduction estimator (RARE) are applicable to scenarios where no intersubarray calibration is available, but unfortunately, these algorithms are very sensitive to subarray orientation errors. Therefore, the conventional RARE algorithms can be applied to partly calibrated arrays with subarray orientation errors only if these errors are negligibly small. In this correspondence, a new robust modification of the RARE algorithm with a reduced sensitivity to subarray misorientations is proposed, and expressions for the corresponding stochastic Cramér–Rao bound (CRB) are derived. The performance of our robust RARE algorithm is demonstrated to be close to this bound in the case of moderate subarray orientation errors.

Index Terms—Direction-of-arrival (DOA) estimation, partly calibrated arrays, rank reduction estimator (RARE).

I. INTRODUCTION

THE problem of direction-of-arrival (DOA) estimation in large sparse subarray-based sensor arrays has recently attracted a significant interest because using such arrays, it is possible to enlarge the array aperture without a corresponding increase in hardware/software costs [1]–[8]. In such arrays, the aperture of each subarray is typically much smaller than the aperture of the whole array and, therefore, each subarray can be assumed to be well calibrated. The calibration of the whole array, on the other hand, may be poor due to completely unknown or imprecisely known (e.g., perturbed) intersubarray displacements, imperfect time synchronization of different subarrays, unknown channel mismatches between subarrays, or a combination of the above-mentioned imperfections [6].

In the case when each particular subarray is calibrated but there is no calibration between subarrays (i.e., all intersubarray parameters are unknown), the recently developed rank reduction estimator (RARE) can be efficiently used to estimate the signal DOAs. In the most general case when all subarrays have arbitrary geometries, the spectral RARE technique of [6] can be directly applied. In the particular case of linear identically oriented subarrays whose interelement spacings are integer multiples of the known shortest baseline, the root-RARE algorithm of [7] can be used. The latter algorithm is search free and, therefore, its computational cost is substantially lower than that of spectral RARE. A serious shortcoming of both the spectral and root-RARE algorithms is that they may be quite sensitive to subarray orientation errors, which may easily occur in practice [9], [10].

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In this correspondence (see also [11]), we develop new spectral and root modifications of the RARE algorithm that are robust against subarray orientation errors. We also derive the corresponding stochastic Cramér–Rao bound (CRB) to which we compare the performance of the proposed estimators. Our simulation results demonstrate a substantially improved robustness of the proposed RARE techniques as compared with the conventional RARE algorithms in scenarios in which subarray misorientations occur. It is also shown that, in such scenarios, the performance of the proposed robust RARE algorithms is substantially closer to the CRB than that of the conventional RARE algorithms.

The rest of our correspondence is organized as follows. The conventional RARE algorithms are revisited in Section II. In Section III, new RARE algorithms are formulated that have an improved robustness against subarray misorientations. In Section IV, a stochastic CRB is derived that corresponds to the problem considered. Section V contains simulation results. Conclusions are given in Section VI.

II. CONVENTIONAL RARE ALGORITHMS

Assume that an array of M omnidirectional sensors consists of K arbitrary nonoverlapping subarrays, and the k th subarray has $M_k \geq 1$ sensors, so that the total number of sensors in the array is given by $M = \sum_{k=1}^K M_k$. Assume also that the array receives $L < M$ narrow-band signals from multiple far-field sources. In this section, we consider the case when each subarray is fully calibrated, i.e., there are no subarray orientation errors, while the intersubarray parameters are unknown or uncertain.

The array snapshots can be modeled as [6], [7]

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\alpha})\mathbf{s}(t) + \mathbf{n}(t), \quad t = 1, \dots, N$$

where $\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\alpha}) \triangleq [\mathbf{a}(\theta_1, \boldsymbol{\alpha}), \mathbf{a}(\theta_2, \boldsymbol{\alpha}), \dots, \mathbf{a}(\theta_L, \boldsymbol{\alpha})]$ is the $M \times L$ direction matrix, $\mathbf{a}(\theta, \boldsymbol{\alpha})$ is the array steering vector, $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_L]^T$ is the $L \times 1$ vector of the source DOAs, $\boldsymbol{\alpha}$ is the $K \times 1$ vector that contains all unknown manifold parameters (i.e., intersubarray displacements, timing errors, and channel mismatches between subarrays, or some combination of these effects [6]), $\mathbf{s}(t)$ is the $L \times 1$ vector of signal waveforms, $\mathbf{n}(t)$ is the $M \times 1$ vector of white circular complex Gaussian noise, N is the number of snapshots, and $(\cdot)^T$ denotes the transpose. The basic idea of the RARE algorithm is to model $\mathbf{a}(\theta, \boldsymbol{\alpha})$ as the product of a known matrix $\mathbf{V}(\theta)$ and an unknown vector $\mathbf{h}(\theta, \boldsymbol{\alpha})$ associated with the unknown intersubarray parameters [6], [8] such that

$$\mathbf{a}(\theta, \boldsymbol{\alpha}) = \mathbf{V}(\theta)\mathbf{h}(\theta, \boldsymbol{\alpha}) \quad (1)$$

where

$$\mathbf{V}(\theta) = \begin{bmatrix} \mathbf{v}_1(\theta) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_2(\theta) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{v}_K(\theta) \end{bmatrix}$$

and $\mathbf{v}_k(\theta)$ is the $M_k \times 1$ steering vector of the k th subarray. Therefore, the $M \times 1$ complex vector

$$\mathbf{v} \triangleq [\mathbf{v}_1^T(\theta), \mathbf{v}_2^T(\theta), \dots, \mathbf{v}_K^T(\theta)]^T$$

characterizes the array response when the intersubarray imperfections are completely ignored, while $\mathbf{h}(\theta, \boldsymbol{\alpha})$ is the $K \times 1$ complex vector associated with such imperfections. The unknown vector $\mathbf{h}(\theta, \boldsymbol{\alpha})$ can take different forms, depending on the type of intersubarray imperfections considered (see [6] and [8] for details).

The eigendecomposition of the array covariance matrix

$$\mathbf{R} = \mathbb{E}\{\mathbf{x}(t)\mathbf{x}^H(t)\}$$

is given by [12]

$$\mathbf{R} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H + \mathbf{G}\boldsymbol{\Gamma}\mathbf{G}^H \quad (2)$$

where $\mathbb{E}\{\cdot\}$ and $(\cdot)^H$ denote the statistical expectation and Hermitian transpose, respectively. In (2), the $L \times L$ and $(M - L) \times (M - L)$ diagonal matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$ contain, respectively, the L and $M - L$ signal- and noise-subspace eigenvalues of \mathbf{R} , whereas the columns of the $M \times L$ and $M \times (M - L)$ matrices \mathbf{U} and \mathbf{G} contain, respectively, the corresponding signal- and noise-subspace eigenvectors of \mathbf{R} .

Substituting the steering vector model (1) to the MUSIC equation [12]

$$\mathbf{a}^H(\theta, \boldsymbol{\alpha})\mathbf{G}\mathbf{G}^H\mathbf{a}(\theta, \boldsymbol{\alpha}) = 0$$

we have [6]

$$\mathbf{h}^H(\theta, \boldsymbol{\alpha})\mathbf{C}(\theta)\mathbf{h}(\theta, \boldsymbol{\alpha}) = 0 \quad (3)$$

where $\mathbf{C}(\theta)$ is a $K \times K$ matrix defined as

$$\mathbf{C}(\theta) \triangleq \mathbf{V}^H(\theta)\mathbf{G}\mathbf{G}^H\mathbf{V}(\theta).$$

Since $\mathbf{h}(\theta, \boldsymbol{\alpha}) \neq \mathbf{0}$, (3) can be true only if $\mathbf{C}(\theta)$ drops rank (i.e., $\det\{\mathbf{C}(\theta)\} = 0$) and, hence, this property offers the basic rank-dropping criterion used in RARE [6], [7]. It is worth noting that $\mathbf{C}(\theta)$ does not depend on any of the unknown intersubarray parameters $\boldsymbol{\alpha}$. It has been shown in [6] and [7] that under certain mild conditions on subarrays, the rank of $\mathbf{C}(\theta)$ drops (i.e., $\text{rank}\{\mathbf{C}(\theta)\} < K$) if and only if θ coincides with one of the source DOAs $\{\theta_i\}_{i=1}^L$.

In practice, \mathbf{R} is unknown and is usually replaced by its sample estimate

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{t=1}^N \mathbf{x}(t)\mathbf{x}^H(t)$$

whose eigendecomposition can be written as [12]

$$\hat{\mathbf{R}} = \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^H + \hat{\mathbf{G}}\hat{\boldsymbol{\Gamma}}\hat{\mathbf{G}}^H$$

where the $L \times L$ and $(M - L) \times (M - L)$ diagonal matrices $\hat{\boldsymbol{\Lambda}}$ and $\hat{\boldsymbol{\Gamma}}$ contain, respectively, the L and $M - L$ signal- and noise-subspace eigenvalues of $\hat{\mathbf{R}}$, whereas the columns of the $M \times L$ and $M \times (M - L)$ matrices $\hat{\mathbf{U}}$ and $\hat{\mathbf{G}}$ contain, respectively, the corresponding signal- and noise-subspace eigenvectors of $\hat{\mathbf{R}}$.

In this case, the DOAs can be estimated from the L highest peaks of any of the following two alternative spectral RARE estimators [6], [8]:

$$f_1(\theta) = \frac{1}{\det\{\hat{\mathbf{C}}(\theta)\}} \quad (4)$$

$$f_2(\theta) = \frac{1}{\mathcal{L}\{\hat{\mathbf{C}}(\theta)\}} \quad (5)$$

where

$$\hat{\mathbf{C}}(\theta) = \mathbf{V}^H(\theta)\hat{\mathbf{G}}\hat{\mathbf{G}}^H\mathbf{V}(\theta)$$

is the estimate of $\mathbf{C}(\theta)$, and $\mathcal{L}\{\cdot\}$ is the operator whose output is the smallest eigenvalue of a Hermitian matrix.

The estimators (4) and (5) are based on a one-dimensional search over θ . In the particular case of linear identically oriented subarrays whose interelement spacings are integer multiples of the known shortest baseline d , a search-free polynomial rooting-based reformulation of the RARE estimator is possible [7], [8]. In the aforementioned particular case, rewriting the matrix $\hat{\mathbf{C}}(\theta)$ as a function of $z = e^{j(2\pi/\lambda)d \sin \theta}$ gives

$$\hat{\mathbf{C}}(z) = \mathbf{V}^T(1/z)\hat{\mathbf{G}}\hat{\mathbf{G}}^H\mathbf{V}(z) \quad (6)$$

where λ is the wavelength. Using (6), it can be shown that the DOAs can be obtained by rooting the following polynomial [8]:

$$f(z) = \det\{\hat{\mathbf{C}}(z)\}. \quad (7)$$

It should be emphasized here that the original root-RARE estimator presented in [7] is written in a somewhat different (but equivalent) form with respect to (7), because the approach of [7] uses the concept of *virtual array*, which leads to another way of indexing the array sensors.

III. ROBUST RARE ALGORITHMS

In this section, we assume that, in addition to unknown intersubarray parameters, the *subarray orientations* are not known precisely. In this case, the direct application of the RARE algorithm is not possible, as each subarray itself is no longer fully calibrated. In the presence of such subarray orientation errors, the model in (1) can be transformed as

$$\mathbf{a}(\theta, \boldsymbol{\zeta}) = \mathbf{V}(\theta, \boldsymbol{\delta}_\theta) \mathbf{h}(\theta, \boldsymbol{\zeta}) \quad (8)$$

where $\boldsymbol{\delta}_\theta = [\delta\theta_1, \dots, \delta\theta_K]^T$; $\delta\theta_k$ is the orientation error of the k th subarray; and $\boldsymbol{\zeta} = [\boldsymbol{\alpha}^T, \boldsymbol{\delta}_\theta^T]^T$ is the vector containing all unknown array parameters.

In this case, the $M \times K$ direction matrix $\mathbf{V}(\theta, \boldsymbol{\delta}_\theta)$ takes the form

$$\mathbf{V}(\theta, \boldsymbol{\delta}_\theta) = \begin{bmatrix} \mathbf{v}_1(\theta + \delta\theta_1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_2(\theta + \delta\theta_2) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{v}_K(\theta + \delta\theta_K) \end{bmatrix}.$$

Assuming that the orientation errors are small, each vector $\mathbf{v}_k(\theta + \delta\theta_k)$ can be expanded using the first two terms of the Taylor series as

$$\mathbf{v}_k(\theta + \delta\theta_k) \simeq \mathbf{v}_k(\theta) + \delta\theta_k \frac{d\mathbf{v}_k(\theta)}{d\theta}.$$

Using this approximation, (8) can be written as

$$\mathbf{a}(\theta, \boldsymbol{\zeta}) = \left[\mathbf{V}(\theta), \frac{d\mathbf{V}(\theta)}{d\theta} \right] \begin{bmatrix} \mathbf{h}(\theta, \boldsymbol{\zeta}) \\ \mathbf{Q}\mathbf{h}(\theta, \boldsymbol{\zeta}) \end{bmatrix} = \mathbf{P}(\theta) \mathbf{g}(\theta, \boldsymbol{\zeta}) \quad (9)$$

where

$$\begin{aligned} \mathbf{Q} &\triangleq \text{diag}\{\delta\theta_1, \dots, \delta\theta_K\} \\ \mathbf{P}(\theta) &\triangleq \left[\mathbf{V}(\theta), \frac{d\mathbf{V}(\theta)}{d\theta} \right] \\ \mathbf{g}(\theta, \boldsymbol{\zeta}) &\triangleq [\mathbf{h}^T(\theta, \boldsymbol{\zeta}), \mathbf{h}^T(\theta, \boldsymbol{\zeta}) \mathbf{Q}^T]^T. \end{aligned}$$

Note that the model in (9) is similar to that in (1) in the sense that the matrix $\mathbf{P}(\theta)$ depends only on θ , while all the unknown parameters $\boldsymbol{\zeta}$ are captured in the vector $\mathbf{g}(\theta, \boldsymbol{\zeta})$. However, an important difference between the models (1) and (9) is that (9) describes a more general case when both the intersubarray parameters and orientation errors are unknown, whereas (1) corresponds to the case when there are no orientation errors at all.

The aforementioned similarity between (9) and (1) allows us to apply the idea of the conventional RARE algorithm to estimate the source DOAs in the case of unknown subarray orientation errors. Substituting (9) to the MUSIC equation $\mathbf{a}^H(\theta, \boldsymbol{\zeta}) \mathbf{G} \mathbf{G}^H \mathbf{a}(\theta, \boldsymbol{\zeta}) = 0$, we have

$$\mathbf{g}^H(\theta, \boldsymbol{\zeta}) \mathbf{B}(\theta) \mathbf{g}(\theta, \boldsymbol{\zeta}) = 0 \quad (10)$$

where the $2K \times 2K$ matrix $\mathbf{B}(\theta)$ is defined as

$$\mathbf{B}(\theta) \triangleq \mathbf{P}^H(\theta) \mathbf{G} \mathbf{G}^H \mathbf{P}(\theta).$$

Since $\mathbf{g}(\theta, \boldsymbol{\zeta}) \neq \mathbf{0}$, (10) can hold true only if the matrix $\mathbf{B}(\theta)$ drops rank. Therefore, to estimate the signal DOAs in the finite sample case, the following spectral functions can be used:

$$\tilde{f}_1(\theta) = \frac{1}{\det\{\hat{\mathbf{B}}(\theta)\}} \quad (11)$$

$$\tilde{f}_2(\theta) = \frac{1}{\mathcal{L}\{\hat{\mathbf{B}}(\theta)\}} \quad (12)$$

where

$$\hat{\mathbf{B}}(\theta) = \mathbf{P}^H(\theta) \hat{\mathbf{G}} \hat{\mathbf{G}}^H \mathbf{P}(\theta).$$

Note that (11) and (12) have a certain similarity to (4) and (5), respectively. However, an important difference between these estimators is that (11) and (12) can be applied to scenarios with unknown subarray orientation errors, whereas (4) and (5) assume that there are no such errors.

In the specific case of linear subarrays whose interelement spacings are integer multiples of the shortest baseline d , let us reformulate robust RARE in a search-free form using the aforementioned similarity and the approach of [6]. Using (9), the steering vector can be rewritten as

$$\begin{aligned} \mathbf{a}(\theta, \boldsymbol{\zeta}) &= \left[\mathbf{V}(z), \frac{d\mathbf{V}(z)}{dz} \right] \begin{bmatrix} \mathbf{h}(\theta, \boldsymbol{\zeta}) \\ \mathbf{Q}\mathbf{h}(\theta, \boldsymbol{\zeta}) \end{bmatrix} \\ &= \left[\mathbf{V}(z), \frac{d\mathbf{V}(z)}{dz} \right] \begin{bmatrix} \mathbf{h}(\theta, \boldsymbol{\zeta}) \\ u(\theta) \mathbf{Q}\mathbf{h}(\theta, \boldsymbol{\zeta}) \end{bmatrix} \\ &= \mathbf{F}(z) \mathbf{p}(\theta, \boldsymbol{\zeta}) \end{aligned} \quad (13)$$

where

$$\mathbf{F}(z) \triangleq \left[\mathbf{V}(z), \frac{d\mathbf{V}(z)}{dz} \right]$$

$$\mathbf{p}(\theta, \boldsymbol{\zeta}) \triangleq [\mathbf{h}^T(\theta, \boldsymbol{\zeta}), u(\theta) \mathbf{h}^T(\theta, \boldsymbol{\zeta}) \mathbf{Q}^T]^T$$

and $u(\theta) = j(2\pi/\lambda)d \cos \theta$. In (13), we have taken into account that

$$\frac{d\mathbf{v}_k(\theta)}{d\theta} = \frac{d\mathbf{v}_k(\theta)}{dz} \frac{dz}{d\theta}.$$

Note that the matrix $\mathbf{F}(z)$ is a function of z only. This allows us to estimate the signal DOA's by means of rooting the polynomial

$$\tilde{f}(z) = \det\{\hat{\mathbf{E}}(z)\} \quad (14)$$

where

$$\hat{\mathbf{E}}(z) = \mathbf{F}^T(1/z) \hat{\mathbf{G}} \hat{\mathbf{G}}^H \mathbf{F}(z).$$

IV. CRAMÉR–RAO BOUND

In this section, we derive the stochastic CRB¹ for partly calibrated arrays with imperfectly oriented subarrays by extending the results of [6].

Let us introduce the following $(2KL - L + K) \times 1$ vector:

$$\boldsymbol{\eta} \triangleq [\boldsymbol{\theta}^T, \boldsymbol{\xi}_2^T, \dots, \boldsymbol{\xi}_K^T, \boldsymbol{\gamma}_2^T, \dots, \boldsymbol{\gamma}_K^T, \boldsymbol{\delta}_\theta^T]^T$$

where

$$\begin{aligned} \boldsymbol{\xi}_k &\triangleq [\text{Re}\{h_{1,k}\}, \dots, \text{Re}\{h_{L,k}\}]^T \\ \boldsymbol{\gamma}_k &\triangleq [\text{Im}\{h_{1,k}\}, \dots, \text{Im}\{h_{L,k}\}]^T \end{aligned}$$

and, for notational brevity, the arguments in $\mathbf{h}(\theta_l, \boldsymbol{\zeta})$ are omitted and it is hereafter written as $\mathbf{h}_l = [h_{l,1}, \dots, h_{l,K}]^T$. Also, following [6], we assume that the first element of each vector \mathbf{h}_l is fixed to avoid the scaling ambiguity in the computation of the CRB.

The snapshots are assumed to satisfy the stochastic model

$$\mathbf{x}(t) \sim \mathcal{N}\{\mathbf{0}, \mathbf{R}\}$$

where $\mathcal{N}\{\cdot, \cdot\}$ is the complex Gaussian distribution

$$\mathbf{R} = \mathbf{A}(\theta, \boldsymbol{\zeta}) \mathbf{S} \mathbf{A}^H(\theta, \boldsymbol{\zeta}) + \sigma^2 \mathbf{I}$$

is the data covariance matrix, $\mathbf{S} = \mathbf{E}\{\mathbf{s}(t) \mathbf{s}^H(t)\}$ is the source covariance matrix, σ^2 is the noise variance, and \mathbf{I} is the identity matrix. The unknown parameters of the problem include the elements of the vector $\boldsymbol{\eta}$, the noise variance σ^2 , and the parameters of the source covariance matrix $\{\mathbf{S}_{ll}\}_{l=1}^L$ and $\{\text{Re}\{\mathbf{S}_{lk}\}, \text{Im}\{\mathbf{S}_{lk}\}; k > l\}_{l,k=1}^L$.

After concentrating the problem with respect to the parameters of the source covariance matrix and the noise variance, the $(2KL - L + K) \times (2KL - L + K)$ Fisher information matrix can be written as [6], [15]

$$[\text{CRB}^{-1}(\boldsymbol{\eta})]_{kl} = \frac{2N}{\sigma^2} \text{Re} \left\{ \text{trace} \left(\mathbf{W} \frac{\partial \mathbf{A}^H}{\partial \eta_l} \mathbf{P}_A^{-1} \frac{\partial \mathbf{A}}{\partial \eta_k} \right) \right\} \quad (15)$$

¹Note that the stochastic CRB is more preferable than the deterministic one because the latter bound is unattainable [14].

where $\mathbf{P}_A^\perp \triangleq \mathbf{I} - \mathbf{A}(\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H$ and the $L \times L$ matrix $\mathbf{W} = \mathbf{S}(\mathbf{A}^H \mathbf{A} \mathbf{S} + \sigma^2 \mathbf{I})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{S}$.

After tedious albeit straightforward manipulations involving computing all the derivatives in (15) and combining them in a proper matrix form, the CRB matrix can be expressed as

$$\text{CRB}(\boldsymbol{\eta}) = \frac{\sigma^2}{2N} \left[\text{Re} \left\{ \mathbf{T}^T [(\mathbf{1}\mathbf{1}^T \otimes \mathbf{W}) \odot (\mathbf{D}^H \mathbf{P}_A^\perp \mathbf{D})^T] \mathbf{T} \right\} \right]^{-1} \quad (16)$$

where \otimes and \odot denote the Kronecker and Schur–Hadamard matrix products, respectively. Here, $\mathbf{1}$ is the $(3K - 1) \times 1$ vector of ones, \mathbf{T} is the $(3KL - L) \times (2KL - L + K)$ matrix given by

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{1}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{1}} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \tilde{\mathbf{1}} \end{bmatrix}.$$

$\tilde{\mathbf{1}}$ is the $L \times 1$ vector of ones, and \mathbf{D} is the $M \times (3KL - L)$ matrix given by

$$\mathbf{D} \triangleq [\mathbf{D}_\theta, \mathbf{D}_{\xi_2}, \dots, \mathbf{D}_{\xi_K}, \mathbf{D}_{\gamma_2}, \dots, \mathbf{D}_{\gamma_K}, \mathbf{D}_{\delta\theta_1}, \dots, \mathbf{D}_{\delta\theta_K}]$$

where

$$\begin{aligned} \mathbf{D}_\theta &\triangleq \left[\frac{\partial \mathbf{V}(\theta_1, \boldsymbol{\delta}_\theta) \mathbf{h}_1}{\partial \theta_1}, \dots, \frac{\partial \mathbf{V}(\theta_L, \boldsymbol{\delta}_\theta) \mathbf{h}_L}{\partial \theta_L} \right] \\ \mathbf{D}_{\xi_k} &\triangleq [\mathbf{V}(\theta_1, \boldsymbol{\delta}_\theta) \mathbf{e}_k, \dots, \mathbf{V}(\theta_L, \boldsymbol{\delta}_\theta) \mathbf{e}_k] \\ \mathbf{D}_{\gamma_k} &\triangleq j \mathbf{D}_{\xi_k} \\ \mathbf{D}_{\delta\theta_k} &\triangleq \left[\frac{\partial \mathbf{V}(\theta_1, \boldsymbol{\delta}_\theta) \mathbf{h}_1}{\partial \delta\theta_k}, \dots, \frac{\partial \mathbf{V}(\theta_L, \boldsymbol{\delta}_\theta) \mathbf{h}_L}{\partial \delta\theta_k} \right] \end{aligned}$$

and \mathbf{e}_k is the $K \times 1$ vector with one in the k th position and zeros elsewhere.

V. SIMULATION RESULTS

Throughout our simulations, an array composed of two subarrays and two uncorrelated sources with the DOAs $\theta_1 = 10^\circ$ and $\theta_2 = 20^\circ$ are assumed. The unknown intersubarray displacement vector is $[1.3\lambda, 1.2\lambda]^T$. The first subarray is assumed to be free of orientation errors, while the second subarray is assumed to suffer from an orientation error. This error is either fixed (in the first and third examples) or random (in the second and fourth examples). The number of snapshots is $N = 100$, and all results are averaged over a total of 200 simulation runs. In each figure, the conventional and robust RARE estimators are compared,² and the CRB of (16) is also displayed.

In our first example, we assume that the first subarray consists of six sensors at the locations

$$\{(0, 0), (0.4\lambda, 0.2\lambda), (0.9\lambda, 0.4\lambda), (1.4\lambda, 0.7\lambda), (1.7\lambda, 1.1\lambda), (2.1\lambda, 1.3\lambda)\}$$

relative to the first sensor of this subarray, while the second subarray has six sensors at the locations

$$\{(0, 0), (0.4\lambda, 0.3\lambda), (0.8\lambda, 0.5\lambda), (1.3\lambda, 0.7\lambda), (1.9\lambda, \lambda), (2.3\lambda, 1.3\lambda)\}$$

also relative to the first sensor. The second subarray is modeled to have an orientation error of 2° . Fig. 1 compares the conventional and robust spectral RARE algorithms where the DOA estimation root-mean-

²As the intersubarray displacements are assumed to be completely unknown, it is not possible to compare with MUSIC that requires at least approximate knowledge of these displacements.

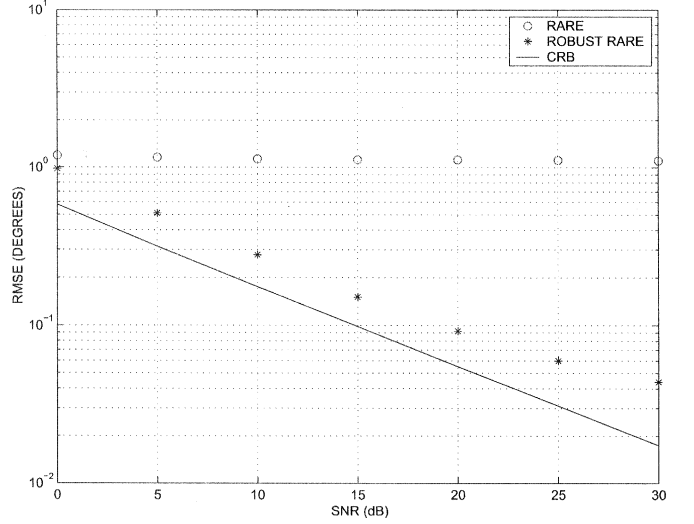


Fig. 1. RMSEs and CRB versus SNR. First example.

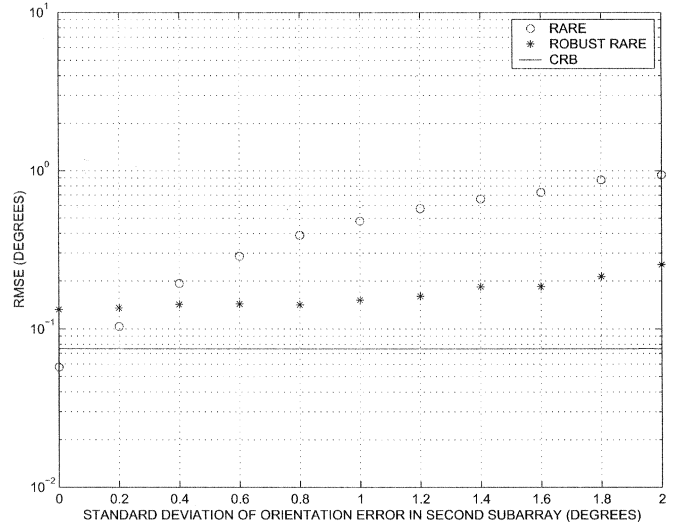


Fig. 2. RMSEs and CRB versus σ_v . Second example.

square errors (RMSEs) versus the signal-to-noise ratio (SNR) are displayed.

In the second example, it is assumed that the first subarray consists of four sensors with the locations

$$\{(0, 0), (0.3\lambda, 0.4\lambda), (0.7\lambda, 0.8\lambda), (1.2\lambda, 1.1\lambda)\}$$

relative to the first sensor, while the second subarray has six sensors with the locations

$$\{(0, 0), (0.4\lambda, 0.6\lambda), (0.91\lambda, \lambda), (1.3\lambda, 1.3\lambda), (1.6\lambda, 1.7\lambda), (1.9\lambda, 2\lambda)\}$$

relative to the first sensor. The second subarray is assumed to suffer from a random orientation error which changes from run to run and has Gaussian distribution with zero mean and the variance σ_v^2 . In Fig. 2, we display the DOA estimation RMSEs of the conventional and robust spectral RARE algorithms versus standard deviation σ_v for SNR = 20 dB. All the curves in this figure are averaged over random orientation error.

In the third example, the first subarray is a uniform linear array (ULA) of four sensors with the interelement spacing of 0.5λ , and the

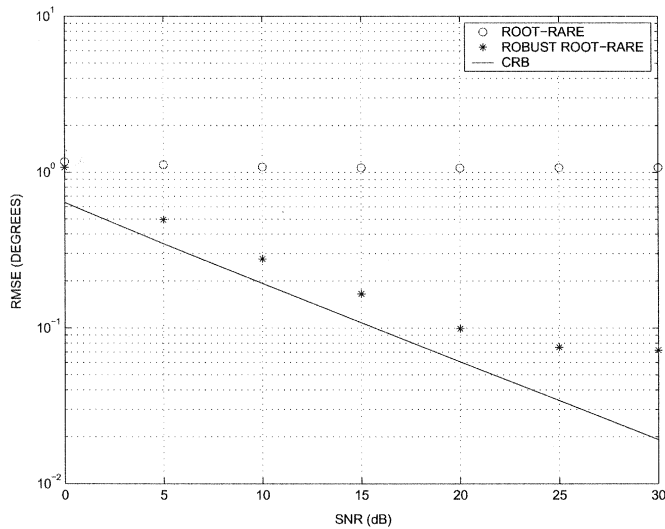


Fig. 3. RMSEs and CRB versus SNR. Third example.

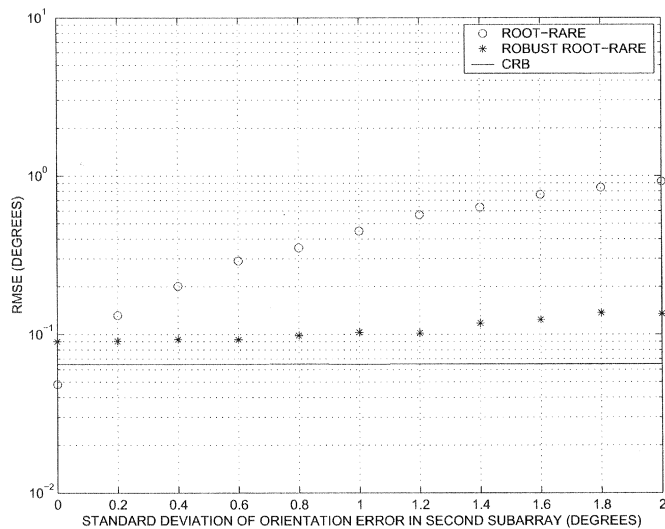


Fig. 4. RMSEs and CRB versus σ_v . Fourth example.

second subarray is a ULA of six sensors with the interelement spacing of 0.4λ . The first subarray is assumed to have no orientation error, while the second subarray has a fixed orientation error of 2° . The conventional and robust root-RARE algorithms are compared in Fig. 3, where the DOA estimation RMSEs are displayed versus the SNR.

In our last example, both subarrays are assumed to have the same configuration as in the previous example. The second subarray has a random Gaussian orientation error with zero mean and the variance σ_v^2 . Fig. 4 shows the DOA estimation RMSEs of the conventional and robust root-RARE algorithms versus standard deviation σ_v at SNR = 20 dB. All the curves in this figure are averaged over random orientation error.

All examples clearly demonstrate that in the presence of subarray orientation errors, the robust RARE algorithms have substantially better performance than the conventional RARE techniques, both in the case of spectral and root-RARE estimators. Interestingly, the RMSE of the robust RARE approach remains close enough to the corresponding CRB, whereas the RMSE of conventional RARE may be far away from this bound because the performance of the conventional RARE technique is corrupted by the orientation errors. It can be seen from Figs. 2 and 4 that performance improvements of robust RARE

with respect to conventional RARE in the case of subarray misorientations are achieved at the price of a moderate performance degradation in the ideal case (where no misorientations occur). It is also worth noting that in the latter case, the conventional RARE estimator can be shown to outperform the CRB (16). A simple explanation of this fact is that, if there is no subarray orientation errors, then the CRB in (16) uses redundant (nonparsimonious) parameterization. In this case, the model becomes overparameterized and, obviously, the corresponding CRB may be outperformed by methods that avoid such redundant parameterization.

VI. CONCLUSION

The problem of DOA estimation in partly calibrated arrays composed of multiple subarrays with unknown intersubarray parameters and imperfectly known subarray orientations has been addressed. Robust modifications of the RARE algorithm have been found that substantially improve its DOA estimation performance in the case of subarray orientation errors, and an appropriate Cramér–Rao bound has been derived. Simulation results demonstrate substantial performance improvements achieved by the new robust RARE DOA estimation technique relative to the conventional RARE algorithm.

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