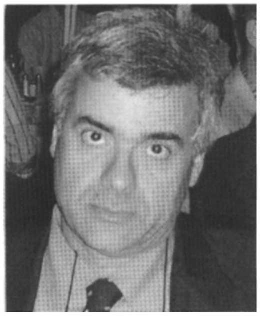

Giuseppe Pelosi
Dept. of Electronics and Telecommunications
University of Florence
Via di Santa Marta, 3
I-50139 Florence, Italy
Tel: +39 055-4796-759
Fax: +39 055-4797-767
E-mail: giuseppe.pelosi@unifi.it, g.pelosi@ieee.org

Foreword by the Associate Editor

Hermann Günter Grassmann (the correct German spelling would be Graßmann, with the German "β") (Figure 1), whose biography we host in the Historical Corner in this issue, was a mathematician and linguistic who devoted particular attention to Sanskrit and Veda books. I will not go in detail about his biography, since this is the matter of our hosted article by Prof. Yilmaz. However, I wish to point out that even if – as our author points out Grassmann's contributions were neglected and very slowly accepted in the scientific community, and his work lately acknowledged, there was a Grassmann Bicentennial Conference held September 16-19, 2009, jointly in Potsdam and Szczecin (Germany and Poland, respectively), to celebrate him. In this framework, two books were published: *Hermann Graßmann – Biography*, an English translation of the German Graßmann biography published in 2006, and a book of sources, *Hermann Graßmann – Roots and Traces. Autographs and Unknown Documents* (in German and English), which was meant to complement the biography, both from Birkhäuser Verlag AG of Basel.

Figure 1. Hermann Günter Grassmann (April 15, 1809, Stettin, Prussia, at that time to September 26, 1877, Stettin, Germany, at that time, now Szczecin, Poland).

Indeed, Grassmann's works were so obscure as to be nearly unreadable, and this played a major role in the above-mentioned slowness with which his ideas spread out. Some of those who hold Grassmann in esteem indeed see him as a martyr of mathematics, treated as marginal by the community (F. Klein and R. Hermann *Development of Mathematics in the 19th Century*, Brookline, MA, Math Sci Press, 1979 – English translation of the original 1928 German book *Vorlesungen uber die Entwicklung der Mathematik im 19 Jahrhundert*).

Grassmann and His Contributions to Electromagnetics

Asim Egemen Yilmaz

Ankara University Department of Electronics Engineering
06100, Tandogan, Ankara, Turkey
Tel: +90 312 203 35 00; Fax: +90 312 212 54 80; E-mail: aeyilmaz@eng.ankara.edu.tr; asimegemenyilmaz@yahoo.com

Abstract

The main aim of this essay is to revisit and to remind the reader of the direct and indirect contributions of Hermann Günther Grassmann to electromagnetics. Being ahead of his time, Grassmann did not have a chance to see that his mathematical ideas and studies were accepted and acclaimed. Acceptance and reception of his publications in applied sciences took more than a century. In this essay, we will try to revisit three major contributions of this genius scientist: his exterior algebra leading to the differential forms, his force law, and his law of optics.

Keywords: History; electromagnetic theory; Maxwell's equations; geometrical calculus; differential forms; electrodynamic force law; law of optics

1. Introduction

Hermann Günther Grassmann (1809-1877) was born as the third of twelve children of a teacher at the Gymnasium (equivalent to a high school, particularly a secondary school) of the town of Stettin (in East Prussia). Grassmann developed a lifelong interest in philology, and a desire to become a Lutheran minister. He studied theology and philology for six semesters in Berlin. He then returned to Stettin, and started a career as a teacher, like his father. After his unsuccessful attempts to obtain a position in the university in the 1840s (due to a poor reception of his mathematical papers by that time), he continued working as a teacher for his whole life.

Due to his position in the Gymnasium, Grassmann had to deal with a wide range of subjects throughout his life. Like his father, he was supposed to lecture at all levels on various subjects, from religion to biology through Latin, mathematics, physics, and chemistry, about 30 hours per week. Under these circumstances, with the help of his enthusiasm for writing and publishing, he published papers on various subjects such as the theory of colors, the theory of sound, musical harmonization, as well as a book on elementary arithmetic. Especially after being disappointed over the poor reception of his mathematical ideas in the 1840s, his main interest in philology dominated the rest of his life. His expertise in Sanskrit and Indo-European languages attained for him the recognition that he could not get via his mathematical works [1]. In addition to his Law of Phonology, he is famous for his 1873 dictionary, *Wörterbuch zum Rig-Veda*, which is still considered to be one of the most important tools for studying old Indian texts.

Even though Grassmann was very well known as a philologist, his contributions in the applied sciences were respected more than a century later. The applied sciences society recognized him, especially after the publications of Dieudonné [1] and Fearnley-Sandler [2] in the late 1970s. After the 1980s, his studies about the representation of vectorial quantities became recognized among engineers. His expression of the electrodynamic force – which used to be misinterpreted and considered to be wrong – has been appreciated since the middle 1990s. His law of optics was shadowed by the names of Helmholtz and Maxwell, even though these two great scientists acknowledged the strong influence of Grassmann on their work during the construction of the Young-Helmholtz-Maxwell theory of vision.

The details of all of the studies and the life of Grassmann can be found in [3], together with a huge list of other relevant references. His scientific contributions and influence are summarized in a more compact form in [4]. The scope of this essay is limited to the studies of Grassmann that made direct and indirect contributions to electromagnetic theory and applications. As will be seen throughout the following sections, in all areas in which he had a contribution, Grassmann had the misfortune to have to compete with a very popular and respected scientist dealing with that topic. This misfortune followed him more than a century after his death, as Dieudonné mentioned in [1] in 1979: "...and the shroud of ignorance and uncertainty still surrounds his [Grassmann's] life and works in the minds of most mathematicians of our time, even when they put his original ideas to daily use."

2. Exterior Algebra and Differential Forms (Grassmann vs. Hamilton)

In 1844, Hermann Günter Grassmann published his book, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik* (republished in the 1990s in an English translation [5]). In this book, he developed the idea of an algebra where the symbols representing geometric entities – such as points, lines, and planes – are manipulated using certain rules. Grassmann introduced the so-called exterior algebra, which was based upon the exterior product, $\wedge$, with the definition

$$a \wedge b = -b \wedge a\,, \quad (1)$$

$$a \wedge a = 0\,. \quad (2)$$

In the early 1900s, Élie Cartan (1869-1951) applied Grassmann's algebra to the theory of "differential forms" in his book, *Leçons sur les invariants intégraux*. After a negligence of more than a half-century, the advantages of the differential forms have become much appreciated. This was especially true after its introduction to the electromagnetics community by Deschamps in [6], where the notation is acclaimed due to its convenience, compactness, and completeness. After [6], differential forms found applications in numerical methods, boundary conditions, Green's functions, and anisotropic media. In some references (such as that of Russer [7]), Grassmann's exterior calculus and differential forms are referred to as "allowing for the solution of field theoretical problems easily and directly," and "establishing a direct connection to geometrical images and supplying additional physical insight." Again, according to Deschamps [6], the notation "obeys simple rules that are easy to memorize, and leads to a most elegant formulation of Stokes' theorem."

Let us now try to illustrate how the differential forms simplify electromagnetic theory. First, we will start the ordinary formulation, which is required to describe and to understand the scalar and vectorial quantities of electromagnetic theory.

Let Ω be a conducting domain of interest, and let Γ be its boundary. The symbols $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ denote the spaces of all square-integrable scalar and vector functions on Ω, respectively. As usual, $\mathbf{n}$ denotes the unit normal vector outward from Γ. The vector spaces $H_0(\Omega,\mathrm{curl})$, $H_0(\Omega,\mathrm{div})$ can be defined as

$$H_0(\Omega,\text{grad}) = \{\phi \in H(\Omega,\text{grad}) | \phi = 0 \text{ on } \Gamma\} H_0(\Omega,\text{grad}), \quad (3)$$

$$H_0(\Omega,\text{curl}) = \{\mathbf{u} \in H(\Omega,\text{curl}) | \mathbf{u} \times \mathbf{n} = 0 \text{ on } \Gamma\}, \quad (4)$$

$$H_0(\Omega,\text{div}) = \{\mathbf{u} \in H(\Omega,\text{div}) | \mathbf{u} \bullet \mathbf{n} = 0 \text{ on } \Gamma\}, \quad (5)$$

where

$$H(\Omega,\text{grad}) = \{\phi \in L^2(\Omega) | \nabla\phi \in L^2(\Omega)\}, \quad (6)$$

$$H(\Omega,\text{curl}) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) | \nabla \times \mathbf{u} \in \mathbf{L}^2(\Omega)\}, \quad (7)$$

$$H(\Omega,\text{div}) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) | \nabla \bullet \mathbf{u} \in L^2\Omega\}. \quad (8)$$

With this information, the domains and ranges of our well-known differential operators can be listed as in Table 1. In other words, the four Hilbert spaces $H(\Omega,\text{grad})$, $H(\Omega,\text{curl})$, $H(\Omega,\text{div})$, $L^2(\Omega)$, and the three operators ∇, $\nabla\times$, and $\nabla\bullet$ form a deRham cohomology relative to Γ:

$$H(\Omega,\text{grad}) \xrightarrow{\nabla} H(\Omega,\text{curl}) \xrightarrow{\nabla\times} H(\Omega,\text{div}) \xrightarrow{\nabla\bullet} L^2(\Omega) \quad (9)$$

With this notation, we can state that the scalar electric potential is an element of $H(\Omega,\text{grad})$, the electric and magnetic field intensities belong to $H(\Omega,\text{curl})$, and the electric and magnetic fluxes are elements of $H(\Omega,\text{div})$.

We will now consider Grassmann's differential forms. Detailed information and the relevant formulation can be found in some other references (such as those of Deschamps [6], Russer and Warnick [7-9], Engl [10], Burke [11], Baldomir [12], Lindell and Jancewicz [13, 14], Koning [15], Tonti [16], Pezzaglia [17], and Gross and Kotiuga [18]). We will just give the essential points here.

The differential-form calculus is based on the concept of four entities called p-forms in three-dimensional space. The 0-form and 3-form are both scalar quantities in curvilinear geometry, while the 1-form and 2-form are vector quantities in curvilinear geometry.

- The 0-form takes a zero-dimensional vector – a point – and returns a scalar, which corresponds to the evaluation of the scalar function at that point. These entities are useful for describing physical quantities that are continuous across a material interface, such as potentials. The electric potential is a 0-form quantity.
- 1-forms correspond to quantities with tangential continuity across a material interface, such as the electric field.
- The 2-forms have normal continuity, and represent fluxes, such as the magnetic flux density.
- The 3-forms are defined within a specific volume, and therefore have no imposed continuity between adjacent volumes, which allows them to represent discontinuous fields, such as charge density.

Moreover, a single "exterior derivative" (or exterior differential operator) d relates the p-forms to each other. Hence, the deRham cohomology in Equation (9) simplifies to

$$\text{0-form} \xrightarrow{d} \text{1-form} \xrightarrow{d} \text{2-form} \xrightarrow{d} \text{3-form}. \quad (10)$$

Table 2 summarizes the p-forms, together with their features, and which important quantities of electromagnetics belong to which class.

Table 1. The domains and ranges of differential operators.

		Domain			
		$H(\Omega,\text{grad})$	$H(\Omega,\text{curl})$	$H(\Omega,\text{div})$	$L^2(\Omega)$
Range	$H(\Omega,\text{grad})$		$\nabla\bullet$		
	$H(\Omega,\text{curl})$	∇		$\nabla\times$	
	$H(\Omega,\text{div})$		$\nabla\times$		∇
	$L^2(\Omega)$			$\nabla\bullet$	

Table 2. Differential forms and significant quantities of electromagnetic theory.

	0-form	1-form	2-form	3-form
Physical types	Scalar Potentials	Fields, vector potentials	Fluxes, vector densities, Poynting vector	Scalar densities
Alternative terminology	Scalars	Polar vectors	Axial vectors	Pseudoscalars
Minimum continuity	Total	Tangential	Normal	None
Integrated over	Point	Line	Surface	Volume
Applicable derivative	Grad	Curl	Div	None
EM theory quantities	ϕ	**A, E, H**	**B, D, J, S**	ρ, W_e, W_m
Corresponding Hilbert space	$H(\Omega,\text{grad})$	$H(\Omega,\text{curl})$	$H(\Omega,\text{div})$	$L^2(\Omega)$

Unfortunately, the formulation of differential forms did not receive much attention and interest by the time it was published. As pointed out by several authors, there were several reasons for this. In [19], Lawvere listed the following reasons:

- Grassmann's German writing style, which was found very difficult even by native-German-speaking mathematics students;

- Grassmann's arbitrarily unclear philosophical discussions of mathematical issues, which constituted almost the first half of the relevant study;

- Grassmann's mathematical misconceptions and unusual terminology, such as speaking of laws rather than axioms.

These factors degraded the readability of the work, and limited its short-term influence and impact. Another factor, which might be the most important of all, was the presence of another study on that subject in those days, with a different perspective. This was nothing but the "quaternion formulation" by William Rowan Hamilton (1805-1865). Hamilton was famous due to his brilliant mind and success stories starting from childhood. He carried lots of titles and honors, including a knighthood, and had fanatic supporters and followers [20]. It was thus natural that Hamilton and his work had a greater chance for attention, acclaim, and acceptance compared to that of Grassmann, who was an unknown secondary-school teacher from the town of Stettin.

At this point, let us remember the most important event of electromagnetic theory. In 1873, James Clerk Maxwell (1831-1879) published his *Treatise on Electricity and Magnetism.* Maxwell had not used quaternionic methods at all in working out his four famous papers on electricity and magnetism. On the other hand, by 1870, partly under the influence of his childhood friend Peter Guthrie Tait (1831-1901) (the most energetic supporter of Hamilton), Maxwell had begun to read about quaternions. Moreover, Maxwell expressed many of the results presented in his *Treatise* not only in Cartesian form, but also in their quaternionic equivalent. This happened frequently enough that the common impression was as follows: Maxwell himself preferred these methods, but he had decided not to force them upon readers of his book. However, according to the authors of more recent publications, Maxwell's position was somewhat different. In [18], Gross and Kotiuga stated that Maxwell consciously avoided both Grassmann's exterior algebra and Hamilton's quaternions as a formalism for electromagnetism, in order to avoid ideological debates, especially a possible polemic with his friend Tait. In [20], Crowe stated that Maxwell's attitude towards quaternions used to be misinterpreted that he was advertising them. With some supportive quotations from Maxwell, Crowe claimed that Maxwell considered the quaternions a useful attempt for the representation of space-related quantities (an innovation comparable to that of René Descartes (1596-1650)), but that Maxwell found the approach unsatisfactory.

As a matter of fact, in later decades, some great mathematicians tried to get rid of the inconsistencies in Hamilton's and/or Grassmann's works. Three major figures were William Kingdon Clifford (1845-1879), Josiah Willard Gibbs (1839-1903), and Oliver Heaviside (1850-1925). Being aware of both Hamilton's and Grassmann's works, Clifford was the first to combine the two approaches (with a bias towards Grassmann's approach), in order to come up with a more-reasonable system, in 1877. In a paper dated 1878, paper appearing in the *American Journal of Mathematics*, with the title "Applications of Grassmann's Extensive Algebra," Clifford demonstrated the usefulness of Grassmann's approach and notation. Gibbs and Heaviside were the ones who completed the evolution, and who defined the system yielding the modern geometric calculus now in use. Heaviside is the one who is usually credited for expressing Maxwell's equations in the current form with which we engineers are familiar. It should be noted that Tait heavily criticized Gibbs along this evolution, by referring to the new hybrid notation as "a sort of hermaphrodite monster, compounded of the Hamiltonian and Grassmannian notations" [20].

As a conclusion to this section, it should be emphasized that Grassmann's efforts had significant impact throughout the development of the modern geometric calculus (as seen in Figures 1 and 2), as well as the current formulation of Maxwell's equations. His formulation yielding the differential forms constituted an alternative and simple approach for understanding, visualizing, and categorizing the quantities of electromagnetic theory.

3. The Electrodynamic Force Law (Grassmann vs. Ampère)

Since 1820, it has been known that there is a ponderomotive force between two metallic circuits when an electric current flows inside them [21]. There are two main expressions for the calculation of this force: those of Grassmann and of Ampère.

According to André-Marie Ampère (1775-1836), the force $d^2\mathbf{F}_{ji}$ exerted by a current element $I_j d\mathbf{l}_j$, located at $\mathbf{r}_j$, on another current element $I_i d\mathbf{l}_i$, located at $\mathbf{r}_i$, is given by

$$d^2\mathbf{F}_{ji}^A = -\frac{\mu_0}{4\pi} I_i I_j \frac{\hat{\mathbf{r}}}{r^2}\left[2\left(d\mathbf{l}_i \cdot d\mathbf{l}_j\right) - 3\left(\hat{\mathbf{r}} \cdot d\mathbf{l}_i\right)\left(\hat{\mathbf{r}} \cdot d\mathbf{l}_j\right)\right], \quad (11)$$

where $\mu_0 = 4\pi \times 10^{-7}$ kgm/C^2 is the permeability of free space, $r = \left|\mathbf{r}_i - \mathbf{r}_j\right|$ is the distance between the elements, and $\hat{\mathbf{r}} = \left|\mathbf{r}_i - \mathbf{r}_j\right|$ is the unit vector pointing from *j* to *i*.

In 1845, Grassmann published his study about the derivation of the force expression in volume 64 of *Annalen der Physik und Chemie.* Based on the Biot-Savart law, Grassmann's expression stated that the force of *j* on *i* is given by

$$\begin{aligned} d^2\mathbf{F}_{ji}^G &= I_i d\mathbf{l}_i \times dB_j^{B-S} = I_i d\mathbf{l}_i \times \left(\frac{\mu_0}{4\pi} I_j \frac{d\mathbf{l}_i \times \hat{\mathbf{r}}}{r^2}\right) \\ &= -\frac{\mu_0}{4\pi}\frac{I_i I_j}{r^2}\left[\left(d\mathbf{l}_i \cdot d\mathbf{l}_j\right)\hat{\mathbf{r}} - \left(d\mathbf{l}_i \cdot \hat{\mathbf{r}}\right)d\mathbf{l}_j\right]. \end{aligned} \quad (12)$$

Ampère's expression follows the action-reaction (Isaac Newton's (1643-1727) third law) in the strong form, as the force is always directed along the line joining the elements. Grassmann's expression between current elements does not obey the action-reaction principle, with the exception of some particular situations.

During the 1980s, many experiments (such as [22-29]) were performed in order to validate and compare these two expressions. As a matter of fact, Maxwell himself was aware of both expressions while he was writing his *Treatise*. Moreover, he tried to identify which expression was better (additionally considering two

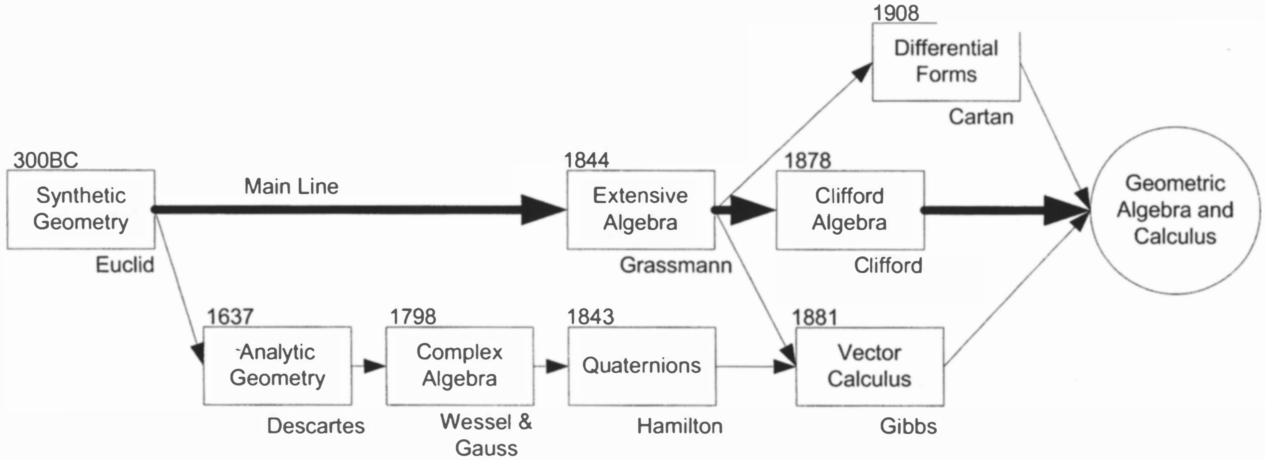

Figure 1. The evolution of geometric calculus and Grassmann's position on the main line (extracted from [38] and [39]).

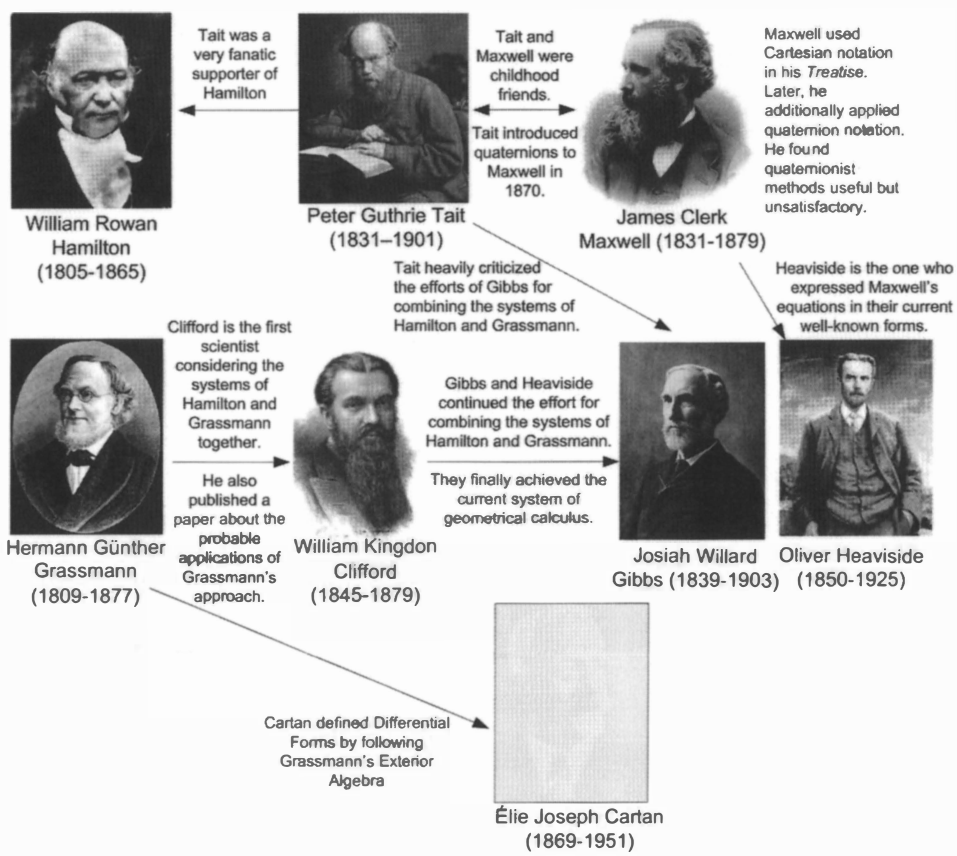

Figure 2. The key players in the construction of the geometric calculus and Maxwell's equations in their current form (depicted from [20]; public images taken from Wikipedia [40-46]; a photograph of Cartan can be found at [47]).

other expressions, derived by himself). He concluded as follows [30]: "Of these four different assumptions, that of Ampère is undoubtedly the best, since it is the only one which makes the forces on the two elements not only equal and opposite, but in the straight line which joins them."

Starting from Maxwell's work to the studies in the 1980s, almost all of the experimental studies of these two expressions favored Ampère's in terms of validity and accuracy. The common impression was thus that Grassmann's expression was just an approximate but erroneous version. However, in 1996, Assis and Bueno [21, 31] showed that Grassmann's expression is equivalent to Ampère's. In other words, their study showed that both expressions yield the same results when considering the net force on any current element of a closed circuit of arbitrary shape. Interested readers might proceed to other references (such as that of Lucas [32] and Radović [33]) for the details of the derivations and experimental setups.

4. The Law of Optics (Grassmann vs. Helmholtz)

Grassmann published his seminal paper, entitled "Zur Theorie der Farbenmischung," in *Annalen der Physik* in 1853. In it, he criticized an 1852 paper of Hermann von Helmholtz (1821-1894), the *Wunderkind* of German natural science. In the relevant paper, Helmholtz attacked the theory of color advanced by the English scientist, David Brewster (1781-1868). By that time, Helmholtz rejected the ideas of Thomas Young (1773-1829), who claimed that there should be three primary colors in nature from which all other colors can be constructed without exception. Helmholtz claimed that his own experiments indicated the contrary; in the conclusion of his paper, Helmholtz eliminated the theories of both Young and Brewster.

Reformulating the color-mixing procedure of Newton, Grassmann pointed out the fact that every color pair should have a complement. He argued that Helmholtz should have failed to detect a pair of complementary colors during the experiments for some particular reason. He insisted that Helmholtz had come up with a totally wrong conclusion due to a mistake.

During the preparation of a response to Grassmann's paper, Helmholtz reconsidered all his studies and thoughts. He noticed that Grassmann was correct, and that his own objections to Young's thoughts were inappropriate. After revising his experiments, Helmholtz corrected his results. This yielded the well-known Young-Helmholtz-Maxwell theory of vision, which is still valid and in use. Details of this story can be found in various references (such as that of Lenoir [34], Sherman [35], and Turner [36]).

The consequence of Grassmann's seminal paper was the phenomenon called color matching, which can be expressed as [37]

$$\mathrm{C} = R\mathrm{R} + G\mathrm{G} + B\mathrm{B}. \qquad (13)$$

This should be interpreted as

"color stimulus C is matched by:

- R units of primary stimulus R, mixed with
- G units of primary stimulus G, and
- B units of primary stimulus B."

Consequently, colors having the same stimulus values will match the same resulting color. According to Grassmann's law, color matches obey the rules of linearity and additivity. Mathematically speaking, if two color stimuli

$$\mathrm{C}_1 = R_1\mathrm{R} + G_1\mathrm{G} + B_1\mathrm{B} \qquad (14)$$

$$\mathrm{C}_2 = R_2\mathrm{R} + G_2\mathrm{G} + B_2\mathrm{B} \qquad (15)$$

are mixed, then the resulting stimulus will be

$$\mathrm{C}_1 + \mathrm{C}_2 = (R_1 + R_2)\mathrm{R} + (G_1 + G_2)\mathrm{G} + (B_1 + B_2)\mathrm{B}. \qquad (16)$$

Even though Grassmann modestly stated that he only applied Newton's principle in his paper, his formulations and cautions forced Helmholtz to reconsider his studies. This eventually yielded the so-called "Young-Helmholtz-Maxwell" theory of vision, for which Grassmann once again received no credit.

5. Conclusion

After ignorance and negligence for more than a century – and especially after the 1980s – Grassmann has received the respect he deserves. Recently, his seminal works have been the subject of numerous proceedings, theses, papers, and book chapters. He has been inducted to the *Encyclopedia of Mathematics* (though a late induction). Moreover, the 150th anniversary of the publication of his book, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, was celebrated in 1994 via international conferences and meetings. The main aim of this essay was to introduce this great scientist with a brilliant mind to those who have not heard of him before, and to once more remind readers of his importance in the history of science.

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