Correction

Corrections to "Basins of Attraction in Fully Asynchronous Discrete-Time Discrete-State Dynamic Networks"

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***Abstract*—This paper brings a correction to the formulation of the basins of fixed-point states of fully asynchronous discrete-time discrete-state dynamic networks presented in our paper that appeared in the IEEE TRANSACTIONS ON NEURAL NETWORKS, vol. 17, no. 2, pp. 397–408, March 2006. In our subsequent works on totally asynchronous systems, we have discovered that the formulation given in that previous paper lacks an additional condition. We present in this paper why the previous formulation is incomplete and give the correct formulation.**

***Index Terms*—Asynchronism, convergence, networks dynamic.**

I. INTRODUCTION

The result presented in our previous paper [1] deals with the basins of attraction of fully asynchronous discrete-time discrete-state networks. Such networks are usually described as a collection of n neurons such that each neuron i takes a finite number of discrete values. If the value of neuron i is noted x_i, $i \in \{1,\ldots,n\}$, the global state of the system is then described by $x = (x_1,\ldots,x_n)$ and the set of global states is

$$E = \prod_{i=1}^{n} E_i$$

where E_i is the finite set of values which can be taken by neuron i. The dynamic of the network is given by the activation function f such that

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

where each f_i is the activation function of neuron i. Those f_i are supposed to be general and are not restricted to threshold networks. The global state of the network at the discrete time t (also called iteration t) is denoted by

$$x^t = \left(x_1^t, \ldots, x_n^t\right).$$

A global state x^* for which $f(x^*) = x^*$ is called a fixed point of the system and its associated basin of attraction is the set of initial states of the network which surely lead to it. Furthermore, the most general execution mode of those networks is the fully asynchronous one. That model implies the definition of the strategy $J(t)$, which corresponds to the set of neurons updated at time t, as well as the state of neuron j available for neuron i at time t: $a_j^i(t) = t - r_j^i(t) \le t$, where $r_j^i(t)$ denotes the delay of neuron j with respect to neuron i. The dynamic of the system is then described by Algorithm 1, in which it is generally assumed (see [2]) that $\forall t$, $a_i^i(t) = t$ (the delay $r_i^i(t) = 0$) for all $i \in \{1,\ldots,n\}$.

Algorithm 1: Asynchronous iteration

Given an initial state $x^0 = (x_1^0, \ldots, x_n^0)$
for each time step $t = 0, 1, \ldots$ **do**
  for each neuron $i = 1, \ldots, n$ **do**
    if $i \in J(t)$ **then**
      $x_i^{t+1} = f_i(x_1^{a_1^i(t)}, \ldots, x_n^{a_n^i(t)})$
    else
      $x_i^{t+1} = x_i^t$
    end if
  end for
end for

Also, the following conditions are assumed to ensure a representative evolution.

Definition 1.1: Let us consider an n-neuron network and the strategy $J = \{J(t)\}_{t\in\mathbb{N}}$, a sequence of subsets of the n neurons. For $i \in \{1,\ldots,n\}$, let $\{a_1^i(t), \ldots, a_n^i(t)\}_{t\in\mathbb{N}}$ be a sequence of $\mathbb{N}^n$, such that:

c1) $a_j^i(t) = t - r_j^i(t)$ with $0 \le r_j^i(t) \le t$, $r_j^i(t)$ being the delay of neuron j according to neuron i at the discrete time t.
c2) $\forall i, j \in \{1,\ldots,n\}$, $\lim_{t\to\infty} a_j^i(t) = \infty$, i.e., although the delays associated with neuron i are unbounded, they follow the evolution of the system.
c3) No neuron is neglected by the updating rule. This condition is called *fair sampling* condition and is equivalent to: $\forall i \in \{1,\ldots,n\}, |\{t, i \in J(t)\}| = \infty$.

As pointed out in [1], the description of the complete basins of attraction of the fixed points of such systems is a critical step towards the complete understanding and control of their behavior. In that previous paper, we proposed a formulation of the attraction basin in fully asynchronous mode of a fixed point of a given network whose activation function is known. Unfortunately, in our subsequent works, we have discovered that our formulation is not restrictive enough and an additional condition is necessary.

In the following section, we briefly recall the formulation of the basin of a fixed point given in [1]. In Section III, the problem with that formulation is described. Then, the new formulation is presented in Section IV.

II. PREVIOUS DESCRIPTION OF THE BASINS OF ATTRACTION

In this section, we recall the result presented in [1] and the reader should refer to that paper for further details.

Definition 2.1: The *asynchronous sets sequence* $(S_i(x^*)$, $0 \le i < |E|)$ of a fully asynchronous network associated to the

Manuscript received March 25, 2008; revised May 13, 2009; accepted May 21, 2009. First published July 21, 2009; current version published August 05, 2009.

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Digital Object Identifier 10.1109/TNN.2009.2027016

set of states E and the activation function f is recursively defined as follows:

$$\begin{aligned} S_0(x^*) &= \{x^*\} \\ S_i(x^*) &= \left\{ \begin{array}{l} x \in E / x \notin \bigcup_{j=0}^{i-1} S_j(x^*), x \notin \mathrm{Cycle}(E, f), \\ \text{let } A = CP\,(x, f(x)) \setminus p\_\mathrm{Cyc}(x, E, f) \\ \forall y \in A, y \in \bigcup_{j=0}^{i-1} S_j(x^*) \end{array} \right\}, \qquad 0 < i < |E|. \end{aligned}$$

In this definition, $\mathrm{Cycle}(E, f)$ describes the set of states which belong to a cycle in chaotic mode. Such cycles are defined according to *pseudoperiods* (see [3]), which corresponds to a time interval in which all the neurons of the system are updated at least one time. Hence, a state x is in $\mathrm{Cycle}(E, f)$ if there exists a sequence of successive global states of the system containing a pseudoperiod and starting and ending with state x. It can be noticed that this formulation also includes the fixed points of the system since they form a cycle of length one. Symmetrically, the set $p\_\mathrm{Cyc}(x, E, f)$ describes the pseudocycles containing the state x. Pseudocycles are not actual cycles as the sequence of states involved in a pseudocycle does not induce the presence of any pseudoperiod. Thus, they do not correspond to an actual evolution of the system but rather to a transient behavior. Finally, $CP(x, y)$ is the set of vectors obtained by the Cartesian product of vectors x and y from E

$$CP(x, y) = \prod_{i=1}^{n} z_i, \qquad \text{where } \begin{cases} z_i = \{x_i\}, & \text{if } x_i = y_i \\ z_i = \{x_i, y_i\}, & \text{if } x_i \neq y_i. \end{cases}$$

Theorem 2.2: The basin of a fixed point x^* [denoted by $CV(x^*)$] of a fully asynchronous system can be described by the union of the corresponding asynchronous sets sequence.

Theorem 2.2 formulates that the basin of a fixed point x^* can be recursively built by using the property that a state x is in the basin of x^* if and only if all its possible successors at the next iteration, deduced from the differences between x and its image $f(x)$, are also in that basin. In the next section, we show why this property is correct in the chaotic mode (without delays) but not sufficient in the fully asynchronous one.

III. Problem in the Previous Formulation

The problem with Definition 2.2 comes from the fact that in some cases, the delays may imply that the updating of a neuron uses a global state of the system which has never been actually reached. Let us consider the example of a three-neurons boolean system whose activation function is given in Fig. 1(a) and the particular asynchronous execution given in Fig. 1(b). It can be seen that at times 5 and 6, x_2 and x_1 are, respectively, updated using the global state $(0, 0, 1)$, which has never been actually taken by the system in the previous times. It is interesting to note that such a behavior is specifically due to the delays between the neurons of the system induced by the fully asynchronous mode.

In that context, the updating of x_1 leads to a global state of the form $(1, ., .)$ which is not reachable from $(0, 1, 1)$ in sequential or even chaotic modes. Moreover, if the images by f of all the states of the form $(1, ., .)$ are of the form $(1, ., .)$, which is the case in our example, then it is not possible to reach back a state of the form $(0, ., .)$ since there is no delay between the first neuron and itself. This implies on the one hand that the state $(0, 1, 1)$ does not always lead to the fixed point $(0, 0, 0)$, and on the other hand, that it does not belong to any cycle. However, our previous formulation gives $S_0(000) = \{000\}$ and $S_1(000) = \{010\}$ and since $CP(011, f(011)) = \{010\} \subset \bigcup_{j=0}^{1} S_j(000)$, we have $S_2(000) = \{011\}$ although the state $(0, 1, 1)$ should not be included in $CV(000)$.

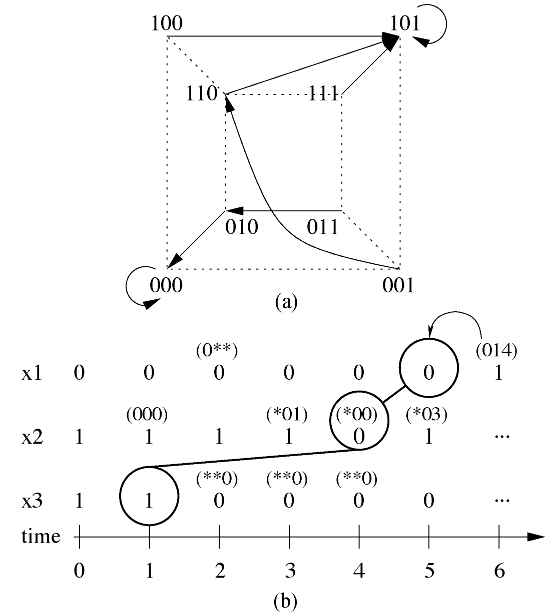

Fig. 1. (a) Activation function of a three-neurons boolean system. The vertices are the global states of the system (x_1, x_2, x_3) and the arrows are the transitions. (b) Example of its fully asynchronous evolution at the bottom. The numbers between parentheses indicate the delays used for the updating of the neuron when it occurs, and the $*$ means any delay.

This is why only considering the difference between x and $f(x)$ to deduce all the possible successors of a state is not sufficient in the fully asynchronous mode. The problem in Definition 2.2 comes precisely from not taking into account the possible successors generated by some particular configurations of delays. Thus, although Definition 2.2 is correct for all systems in chaotic mode, it is only correct for some systems in fully asynchronous mode, but not all. In some fully asynchronous systems, it may include more states than the actual basin of attraction of a given fixed point. That problem has not been detected in the proof of Theorem 3.2 given in [1] because the condition (A.3) is not restrictive enough. That condition ensures that any state in $CV(x^*)$ may lead to x^* in the asynchronous mode but does not prevent that those states may also lead to other fixed points or cycles.

Thus, to obtain a correct formulation of the basins for any fully asynchronous system, it is necessary to take into account the possible successors induced by the updatings of each neuron performed with any possible configuration of delays.

IV. New Formulation

The new formulation does not imply any modification of the conditions in the previous formulation of the basin but only adds a new one whose role is to discard the spurious states which do not actually lead to the fixed point x^* for some configurations of delays. That additional condition requires three new definitions.

Definition 4.1: For a given state x, the set of states reachable from x in the synchronous mode is defined by

$$SP(x) = \left\{ y / \exists k \in \mathbb{N} \text{ such that } f^k(x) = y \right\}$$

where $f^0(x) = x$. The name SP stands for *sequential path*.

Definition 4.2: For a given state x and a fixed point x^* such that $x^* \in SP(x)$, the set $DI_k(x, x^*)$ is defined as follows:

$$DI_k(x, x^*) = \prod_{i=1}^{n} \begin{cases} \{x_i^*\}, & \text{if } i = k \\ \{y_i / y = (y_1, \ldots, y_i, \ldots, y_n) \in SP(x)\}, & \text{otherwise.} \end{cases}$$

It represents the set of possible vectors that may be used to update the component k once the state x^* has been reached, starting from state x. DI stands for *differential images*.

Definition 4.3: For a given state x and a fixed point x^* such that $x^* \in SP(x)$, the set $AE(x, x^*)$ is defined by

$$AE(x, x^*) = \prod_{i=1}^{n} \{f_i(y)/y \in DI_i(x, x^*)\}.$$

That last definition describes the set of states which correspond to the possible *asynchronous evolutions* of the system initialized with x, after having reached x^* before the convergence. It has to be pointed out that in the asynchronous mode, other states may be reached after a fixed point due to the delays.

Those definitions allow us to give the new formulation of the basin of attraction of a given fixed point x^*.

Definition 4.4: Let us define the set $CV(x^*, E, f)$ of a fully asynchronous network associated to the set of states E and the activation function f as the union of the *asynchronous sets sequence* $S_i(x^*), 0 \leq i < |E|$, recursively defined by

$$\begin{aligned} S_0(x^*) &= \{x^*\} \\ S_i(x^*) &= \Bigg\{ x \in E / x \notin \bigcup_{j=0}^{i-1} S_j(x^*), x \notin \mathrm{Cycle}(E, f), \\ &\qquad \text{let } A = CP\,(x, f(x)) \setminus p\_\mathrm{Cyc}(x, E, f) \\ &\qquad \forall y \in A, y \in \bigcup_{j=0}^{i-1} S_j(x^*), \forall y \in AE(x, x^*), y \in \bigcup_{j=0}^{i-1} S_j(x^*) \Bigg\}, \\ &\qquad 0 < i < |E|. \end{aligned}$$

Theorem 4.5: The set $CV(x^*, E, f)$ represents the basin of the fixed point x^* of a fully asynchronous network with the set of states E and the activation function f.

It should be noted that in [1], the word *eventually* in Theorem 3.2 is to be replaced by the word *potentially* and, as the last two subcases may not be distinct, the word *either* should be suppressed.

A. Proof

The major part of the proof of Theorem 4.5 is common to the one of Theorem 3.2 in our initial paper [1]. For clarity, we recall here the different steps of that proof.

A) $\forall x \in CV(x^*)$, $x \overset{a.cv}{\Longrightarrow} x^*$:
  1) $CV(x^*)$ does not contain any other fixed point than x^*;
  2) $CV(x^*)$ does not contain any element of a cycle other than x^*;
  3) $\forall x \in CV(x^*)$, $x \overset{a.i}{\rightsquigarrow} x^*$.

B) $\forall y \notin CV(x^*)$, $y \underset{a.cv}{\not\Longrightarrow} x^*$.

The only difference takes place in condition (A.3). Only this modified condition is described here and the reader should refer to [1] to get more details on the other parts of the proof.

The new condition (A.3) is as follows.

(A.3) $\forall x \in CV(x^*)$, the set $AI(x)$ of states, different from x, reached by any asynchronous evolution of the system starting from x is included in $CV(x^*)$.

Proof of (A.3): We proceed by recurrence on the $S_i(x^*)$ subsets of $CV(x^*)$. The condition (A.3) is obvious for $x \in S_0(x^*) = \{x^*\}$. Let us consider that the condition (A.3) is verified for all the subsets $S_i(x^*), 0 \leq i \leq k$, and show that the condition is still verified for the subset $S_{k+1}(x^*)$.

Let us consider $x \in S_{k+1}(x^*)$. The proof of the previous formulation has already shown that x is actually in $CV(x^*)$ when there is no delay. So, we only focus here on the implication of the presence of delays. As already mentioned, it induces different possible evolutions after having reached x^* when the system is initialized with x. As all those possible evolutions are contained in $AE(x, x^*)$ and, by definition, that set is included in the previous $S_i(x^*)$, this means that all the subsequent evolutions will surely converge toward x^*.

On the other hand, if we consider $x \notin CV(x^*)$, this implies that either x is in a cycle, or it leads to different fixed points from an execution to the other. In the first case, the condition on $\mathrm{Cycle}(E, f)$ in Definition 4.4 prevents x from being included in any $S_i(x^*)$. In the second case, the fact that x may lead to different fixed points is detected in the set $AE(x, x^*)$ which would then contain states outside $CV(x^*)$, and thus outside any $S_i(x^*)$. In such a case, x could not be included in $CV(x^*)$.

V. Conclusion

The lack of another restraining condition has been pointed out in the theoretical formulation of the basins of attraction given in [1]. That former version is correct in the case of chaotic networks without any delays but is not sufficient in the more general case of totally asynchronous networks. The source of the problem has been identified and illustrated by a counterexample. It comes from the influence of the history of the system evolution on its current evolution by implying updatings of neurons with global states which have never actually been reached by the system. Finally, a new theoretical formulation has been given and proved which fixes that problem.

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