Correspondence

Algebraic Two-Satellite TOA/FOA Position Solution on an Ellipsoidal Earth

A direct algebraic solution is given for estimates of the ambiguous locations of an RF emitter on the surface of an ellipsoidal Earth given time-of-arrival (TOA) and frequency-of-arrival (FOA) measurements at two satellites.

I. INTRODUCTION

Geolocation systems employing a constellation of satellite-based receivers or bent-pipe relays can use time-of-arrival (TOA) and frequency-of-arrival (FOA) measurements to locate RF emitters. These systems can compute position "fixes" using iterative linearization techniques based on Newton's method to solve, in a least-squares sense, overdetermined systems of the governing equations [1]. When the emitter is observed by only two satellites, however, a three-dimensional fix cannot be computed. In this case the emitter location can be estimated ambiguously by assuming the emitter is stationary on the surface of the Earth. The solution is ambiguous because generally more than one location on Earth can produce the same TOA and FOA at the two satellites. The ambiguous source location makes Newton's iteration an unsatisfactory approach, as this procedure converges to just one of the solutions.

When ambiguous solutions are expected, a direct algebraic procedure for all solutions is desirable. Schmidt [2] provided the first algebraic solution for the two-satellite TOA/FOA problem with a procedure that requires a one-dimensional search to produce solutions of the desired altitude. More recently Ho and Chan [3] have derived a direct algebraic procedure for all solutions on a spherical Earth and provide an iterative procedure to correct these solutions for an ellipsoidal Earth. Note that the solutions would need to be iterated separately since they would, generally, be at different latitudes.

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We derive a new direct algebraic procedure for all emitter locations on the surface of an ellipsoidal Earth that are consistent with the TOA and FOA measurements observed at two Earth-orbiting satellites of known position and velocity. Note that the algorithm could also be used to locate a stationary receiver on the Earth using two transmitter-equipped satellites, such as GPS.

In Section II we derive a system of five nonlinear equations that define the two-satellite ellipsoidal-Earth problem. Simultaneous solution of these equations will produce an estimate of the emitter's position, transmit time, and frequency. In Section III we solve the system by eliminating the position variables giving two polynomials in transmit time and frequency. These two polynomials are solved by finding the roots of their resultant [4]. This resultant is an eighth-order polynomial whose roots provide all solutions of the system plus extraneous solutions. In Section IV we show the results of a numerical example. In Section V we derive the position error variance.

II. PROBLEM STATEMENT

The equation that relates the emitter position coordinates in an inertial frame at the time of transmission $\mathbf{x} = [x\ \ y\ \ z]^T$, a column vector, the ith satellite position $\mathbf{s}_i = [x_i\ \ y_i\ \ z_i]^T$ at the TOA, the transmit time, t, and the TOA at the ith satellite, t_i, is

$$R_i(\mathbf{x},\mathbf{s}_i) = c(t_i - t) \equiv \tau_i - \tau \tag{1}$$

where c is the speed of light, τ is scaled transmission time, τ_i is the scaled TOA defined by inspection of (1), and R_i is the emitter-to-satellite range given by

$$R_i = |\mathbf{x} - \mathbf{s}_i| \equiv \sqrt{(\mathbf{x} - \mathbf{s}_i)^T(\mathbf{x} - \mathbf{s}_i)}. \tag{2}$$

Squaring and expanding (1) for $i = 1$ gives TOA1

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{s}_1^T\mathbf{x} + r_1^2 = \tau_1^2 + \tau^2 - 2\tau_1\tau \tag{3}$$

and for $i = 2$ gives TOA2

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{s}_2^T\mathbf{x} + r_2^2 = \tau_2^2 + \tau^2 - 2\tau_2\tau \tag{4}$$

where

$$r_i^2 \equiv \mathbf{s}_i^T\mathbf{s}_i. \tag{5}$$

Subtracting (3) from (4) gives the time difference of arrival (TDOA) equation which is linear in $\mathbf{x}$

$$2(\mathbf{s}_1 - \mathbf{s}_2)^T\mathbf{x} = r_1^2 - r_2^2 - \tau_1^2 + \tau_2^2 + 2(\tau_1 - \tau_2)\tau. \tag{6}$$

The equation relating the transmitted frequency f, the Doppler-shifted FOA f_i, and range rate $\dot{R}_i$, the time derivative of R_i, is

$$\dot{R}_i = \lambda(f - f_i) \equiv \nu - \nu_i \tag{7}$$

where λ is the nominal RF wavelength and ν and ν_i are scaled emitter frequency and scaled FOA, respectively, defined by inspection of (7). Expanding $\dot{R}_i$ in (7) using (2) gives

$$\dot{R}_i = \frac{(\mathbf{x} - \mathbf{s}_i)^T(\dot{\mathbf{x}} - \dot{\mathbf{s}}_i)}{R_i} = \nu - \nu_i \tag{8}$$

where $\dot{\mathbf{x}}$ is the emitter velocity, and $\dot{\mathbf{s}}_i$ is the ith satellite velocity. If we assume that the emitter is stationary on the surface of the Earth then, in an Earth-centered inertial (ECI) frame, the emitter velocity is given by the laws of circular motion as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv \mathbf{P}\mathbf{x} \tag{9}$$

where ω is the rotation rate of the Earth, and the matrix $\mathbf{P}$ is defined by inspection of (9).

Putting (1), the TOA equation, into (8), the FOA equation, and expanding, gives

$$\mathbf{x}^T\dot{\mathbf{x}} - \mathbf{s}_i^T\dot{\mathbf{x}} - \mathbf{x}^T\dot{\mathbf{s}}_i + \mathbf{s}_i^T\dot{\mathbf{s}}_i = (\tau_i - \tau)(\nu - \nu_i). \tag{10}$$

Using (9) and $\mathbf{x}^T\dot{\mathbf{x}} = 0$ in (10) gives the equations FOATOA1

$$(\dot{\mathbf{s}}_1^T + \mathbf{s}_1^T\mathbf{P})\mathbf{x} = \nu\tau - \nu_1\tau - \tau_1\nu + \tau_1\nu_1 + \mathbf{s}_1^T\dot{\mathbf{s}}_1 \tag{11}$$

and FOATOA2

$$(\dot{\mathbf{s}}_2^T + \mathbf{s}_2^T\mathbf{P})\mathbf{x} = \nu\tau - \nu_2\tau - \tau_2\nu + \tau_2\nu_2 + \mathbf{s}_2^T\dot{\mathbf{s}}_2. \tag{12}$$

We digress momentarily to note that we can write the TOA equation, (1), explicitly showing a time epoch t_0 from which the TOAs are measured, giving

$$R_1(\mathbf{x},\mathbf{s}_1) = c((t_1 - t_0) - (t - t_0)) \tag{13}$$

$$R_2(\mathbf{x},\mathbf{s}_2) = c((t_2 - t_0) - (t - t_0)). \tag{14}$$

We are free to choose any convenient epoch from which the TOAs t_1 and t_2 are measured. By choosing $t_0 = t_1$ or $t_0 = t_2$ the TOA data appears in the equations as a TDOA and as a reference for t, the transmission time. The computed location therefore depends on the TDOA, while the estimated transmission time depends on the TOAs.

Explicitly showing a frequency reference f_0 in (7) gives

$$\dot{R}_i = \lambda((f - f_0) - (f_1 - f_0)) \tag{15}$$

$$\dot{R}_i = \lambda((f - f_0) - (f_2 - f_0)). \tag{16}$$

A similar argument shows that the computed location depends on the frequency difference of arrival (FDOA) while the frequency of transmission f depends on the FOAs. I do not explicitly show the reference time and frequency hereafter.

We now add an emitter radius, or altitude (ALT), equation to the system of governing equations. The ALT equation is

$$\mathbf{x}^{\mathbf{T}}\mathbf{Q}\mathbf{x} = r_e^2 \tag{17}$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r_e^2/r_p^2 \end{bmatrix} \tag{18}$$

where r_e and r_p are the equatorial and polar radii of the ellipsoid, or oblate spheroid, on which we assume the emitter lies.

We now have five independent equations, TOA1 (3), TDOA (6), FOATOA1 (11), FOATOA2 (12), and ALT (17), which we desire to solve for emitter position and the scaled time and frequency of transmission, x, y, z, τ, and ν.

III. SOLUTION METHOD

Our approach will follow [3] in the use of matrix notation to compute coefficients of polynomials whose roots will provide our solutions. We begin the solution of the system of model equations by putting TDOA (6), FOATOA1 (11), and FOATOA2 (12), into matrix form

$$\begin{bmatrix} 2(\mathbf{s}_1 - \mathbf{s}_2)^T \\ (\dot{\mathbf{s}}_1 + \mathbf{P}^T\mathbf{s}_1)^T \\ (\dot{\mathbf{s}}_2 + \mathbf{P}^T\mathbf{s}_2)^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \nu\tau + \begin{bmatrix} 2(\tau_1 - \tau_2) \\ -\nu_1 \\ -\nu_2 \end{bmatrix} \tau + \begin{bmatrix} 0 \\ -\tau_1 \\ -\tau_2 \end{bmatrix} \nu + \begin{bmatrix} r_1^2 - r_2^2 - \tau_1^2 + \tau_2^2 \\ \tau_1\nu_1 + \mathbf{s}_1^T\dot{\mathbf{s}}_1 \\ \tau_2\nu_2 + \mathbf{s}_2^T\dot{\mathbf{s}}_2 \end{bmatrix}. \tag{19}$$

We solve this system for $\mathbf{x}$ which we put into TOA1 and ALT giving two equations in τ and ν which we solve simultaneously.

The matrix on the left side of (19) and the four vectors on the right side are given the names $\mathbf{A}$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{p}$, and $\mathbf{q}$, respectively, giving

$$\mathbf{A}\mathbf{x} = \mathbf{g}\nu\tau + \mathbf{h}\tau + \mathbf{p}\nu + \mathbf{q} \tag{20}$$

where the new variables are defined by equating terms in (19) and (20). Solving for $\mathbf{x}$ gives

$$\begin{aligned} \mathbf{x} &= \mathbf{A}^{-1}\mathbf{g}\nu\tau + \mathbf{A}^{-1}\mathbf{h}\tau + \mathbf{A}^{-1}\mathbf{p}\nu + \mathbf{A}^{-1}\mathbf{q} \\ &\equiv \mathbf{z}\nu\tau + \mathbf{u}\tau + \mathbf{v}\nu + \mathbf{d} \end{aligned} \tag{21}$$

where vectors $\mathbf{z}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{d}$ are defined by equating terms in (21).

Putting (21) in to TOA1 (3) gives

$$\begin{aligned} &(\mathbf{z}\nu\tau + \mathbf{u}\tau + \mathbf{v}\nu + \mathbf{d})^T(\mathbf{z}\nu\tau + \mathbf{u}\tau + \mathbf{v}\nu + \mathbf{d}) \\ &\quad - 2\mathbf{s}_1^T(\mathbf{z}\nu\tau + \mathbf{u}\tau + \mathbf{v}\nu + \mathbf{d}) + r_1^2 - \tau_1^2 - \tau^2 + 2\tau_1\tau = 0. \end{aligned} \tag{22}$$

Expanding (22) gives

$$\begin{aligned} &a_1\nu^2\tau^2 + a_2\nu\tau^2 + a_3\nu^2\tau + a_4\tau^2 + a_5\nu^2 \\ &\quad + a_6\nu\tau + a_7\tau + a_8\nu + a_9 = 0 \end{aligned} \tag{23}$$

where

$$\begin{aligned} &a_1 = \mathbf{z}^T\mathbf{z} \qquad a_2 = 2\mathbf{z}^T\mathbf{u} \qquad a_3 = 2\mathbf{z}^T\mathbf{v} \\ &a_4 = \mathbf{u}^T\mathbf{u} - 1 \qquad a_5 = \mathbf{v}^T\mathbf{v} \qquad a_6 = 2\mathbf{v}^T\mathbf{u} + 2\mathbf{z}^T(\mathbf{d} - \mathbf{s}_1) \\ &a_7 = 2\mathbf{u}^T(\mathbf{d} - \mathbf{s}_1) + 2\tau_1 \qquad a_8 = 2\mathbf{v}^T(\mathbf{d} - \mathbf{s}_1) \\ &a_9 = \mathbf{d}^T(\mathbf{d} - 2\mathbf{s}_1) + r_1^2 - t_1^2. \end{aligned}$$

Putting (21) into ALT, (17), gives

$$(\mathbf{z}\nu\tau + \mathbf{u}\tau + \mathbf{v}\nu + \mathbf{d})^T\mathbf{Q}(\mathbf{z}\nu\tau + \mathbf{u}\tau + \mathbf{v}\nu + \mathbf{d}) - r_e^2 = 0 \tag{24}$$

which can be written as

$$\begin{aligned} &b_1\nu^2\tau^2 + b_2\nu\tau^2 + b_3\nu^2\tau + b_4\tau^2 + b_5\nu^2 \\ &\quad + b_6\nu\tau + b_7\tau + b_8\nu + b_9 = 0 \end{aligned} \tag{25}$$

where

$$\begin{aligned} &b_1 = \mathbf{z}^T\mathbf{Q}\mathbf{z} \qquad b_2 = 2\mathbf{z}^T\mathbf{Q}\mathbf{u} \qquad b_3 = 2\mathbf{z}^T\mathbf{Q}\mathbf{v} \\ &b_4 = \mathbf{u}^T\mathbf{Q}\mathbf{u} \qquad b_5 = \mathbf{v}^T\mathbf{Q}\mathbf{v} \qquad b_6 = 2\mathbf{v}^T\mathbf{Q}\mathbf{u} + 2\mathbf{z}^T\mathbf{Q}\mathbf{d} \\ &b_7 = 2\mathbf{u}^T\mathbf{Q}\mathbf{d} \qquad b_8 = 2\mathbf{v}^T\mathbf{Q}\mathbf{d} \qquad b_9 = \mathbf{d}^T\mathbf{Q}\mathbf{d} - r_e^2. \end{aligned}$$

In order to simultaneously solve (23) and (25) using the resultant [4] we write them as polynomials in ν with coefficients that depend on τ giving

$$\begin{aligned} &(a_1\tau^2 + a_3\tau + a_5)\nu^2 + (a_2\tau^2 + a_6\tau + a_8)\nu \\ &\quad + (a_4\tau^2 + a_7\tau + a_9) = 0 \end{aligned} \tag{26}$$

and

$$\begin{aligned} &(b_1\tau^2 + b_3\tau + b_5)\nu^2 + (b_2\tau^2 + b_6\tau + b_8)\nu \\ &\quad + (b_4\tau^2 + b_7\tau + b_9) = 0. \end{aligned} \tag{27}$$

We write (26) and (27) compactly as

$$c_1\nu^2 + c_2\nu + c_3 = 0 \tag{28}$$

and

$$d_1\nu^2 + d_2\nu + d_3 = 0 \tag{29}$$

where the new variables are defined by comparing terms with the original equations.

By multiplying (28) and (29) by ν, we create two new polynomials that have the same roots as the original equations plus the additional root $\nu = 0$. Writing the old and new equations in matrix form gives

$$\begin{bmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & 0 \\ 0 & d_1 & d_2 & d_3 \\ d_1 & d_2 & d_3 & 0 \end{bmatrix} \begin{bmatrix} \nu^3 \\ \nu^2 \\ \nu \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{30}$$

The system of equations in (30) can have solutions only if the determinant of the matrix is 0. The determinant of the matrix in (30) is called the resultant of the polynomials $P_1(\tau,\nu)$ in (28) and $P_2(\tau,\nu)$ in (29), and is itself a polynomial in τ. The vanishing of the

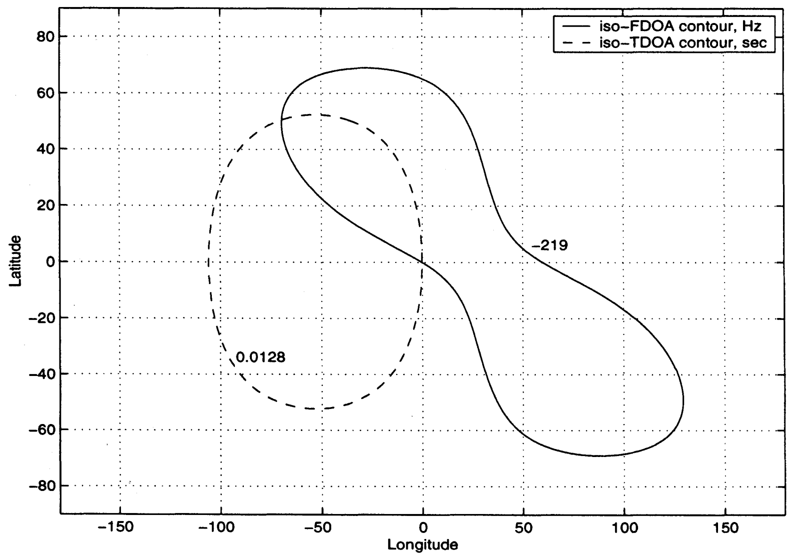

Fig. 1. iso-TDOA and iso-FDOA contours that pass through transmitter location.

TABLE I
TOA (s) and FOA (Hz) Data for Numerical Example

TOA 1	FOA 1	TOA 2	FOA 2	TDOA	FDOA
0.067319449	−0.02860	0.080104907	−218.84675	0.012785458	−218.81814

TABLE II
Time, Frequency, Location Solutions and Total Errors for Numerical Example

t (sec)	$f - f_0$ (Hz)	Latitude (deg)	Longitude (deg)	Altitude (km)	Total Error
0.15702186	1955.945	46.09915	84.44358	0.000	70871.1
	8516.116	34.48395	37.08321	29073.9	76943.2
0.175988360	574.288	14.77693	164.16476	0.000	86895.4
	9127.911	40.91563	38.34555	33954.0	93637.8
−0.019026354	−1862.320	50.47372	−69.44022	0.000	2.4637e-11
	7495.202	−27.87497	23.81673	35754.4	35780.7
0.000000000	0.000	0.00000	0.00000	0.000	6.2898e-12
	6840.019	−23.27652	24.49375	29796.3	29812.0

resultant is shown in [4] to be both necessary and sufficient that polynomials P_1 and P_2 have a common root. So to find all the τ and ν that solve both P_1 and P_2 we can find all the τ that make the determinant of the matrix in (30) vanish and then compute the corresponding ν from (28) or (29).

Setting the resultant equal to zero gives

$$\begin{vmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & 0 \\ 0 & d_1 & d_2 & d_3 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix} = 0. \tag{31}$$

Expanding this determinant gives an eighth-order polynomial in τ

$$w_1\tau^8 + w_2\tau^7 + w_3\tau^6 + w_4\tau^5 + w_5\tau^4 + w_6\tau^3 + w_7\tau^2 + w_8\tau + w_9 = 0 \tag{32}$$

where w_1 to w_9 are long expressions in a_1 to a_9 and b_1 to b_9, best calculated using a symbolic algebra or computer algebra system (CAS). The eight roots of (32) are candidate emitter transmission times scaled by the speed of light. We are only interested in real transmission times so for each real τ we calculate ν from (28) or (29). Solving one of these quadratic equations will produce both a desired and an extraneous solution for ν. Only real frequencies are of interest so the solution can be discarded if the discriminant is negative. From each real τ and real ν we calculate $\mathbf{x}$ using (21). Eliminating spurious solutions is now accomplished by plugging the trial solutions into the original system of equations. Trial solutions, τ, ν, and $\mathbf{x}$, that satisfy (1), (7), and (17) are the solutions we seek. Solutions that do not solve the original equations are spurious and are discarded. A numerical example of the procedure is given in the next section.

IV. NUMERICAL EXAMPLE

We now show results of a numerical example. A 1 GHz transmitter is simulated at 0° latitude and 0° longitude on the surface of a WGS84 [6] ellipsoidal Earth. At the time of transmission $t = 0$, one satellite is at 0° latitude and 0° longitude and the other is at 0° latitude and 60° longitude. Both satellites are in circular orbits of radius 26560 km in planes inclined 55°. Both satellites are at their ascending nodes at the time of transmission. This geometry produces the TDOA and FDOA contours shown in Fig. 1 which pass through the emitter location at 0° latitude and 0° longitude.

All points on the dashed contour have a TDOA of approximately 0.0128 s and all points on the solid contour have an FDOA of approximately −219 Hz. More accurate values are given in Table I where times are relative to the transmission time $t = 0$ and frequencies are relative to the nominal system frequency, and in this example the actual transmission frequency $f = f_0 = 1$ GHz.

To estimate the position of the emitter, assumed to lie on the surface of the WGS84 ellipsoid, the polynomial in (32) is formed. Time variables t, t_1, and t_2, are measured with respect to the first TOA, t_1, and frequency variables are measured with respect to the nominal system frequency. Satellite positions and velocities are expressed in a convenient nonrotating frame. I use the inertial frame that is coincident with the rotating Earth-centered Earth-fixed (ECEF) frame at the time of the first TOA. The eight roots of (32), divided by the speed of light, are the candidate emitter transmission times. Four of these roots are complex and four are real. The real roots are shown referenced to the original time epoch in column 1 of Table II.

Each of the real candidate transmission times are put into (28) which then gives a quadratic equation in the scaled transmission frequency v. Solving this equation gives the two emitter frequency solutions shown in column 2 of Table II as an offset from the nominal system frequency in Hz. Putting the scaled time and frequency solutions into (21) gives the emitter location estimate $\mathbf{x}$ which is converted to WGS84 geodetic latitude, longitude, and altitude, and shown in columns 3-5 of Table II.

Total square model error is the root of the sum of the squares (RSS) of TOA1, TOA2, FOA1, FOA2, and ALT model errors. Model error equations are obtained by slightly modifying (1), (7), and (17) such that all terms of the equations are moved to one side. Evaluating the modified equations at the candidate t, f, and $\mathbf{x}$ solutions gives the error in the fit of the measured data and the candidate solution. Column 6 of Table II is the RSS of the modified (1) and (17) in units of kilometers and the modified (7) in units of Hertz. Since the numbers in column 6 are sums of numbers with different units we attach only the following meaning to these sums. When the RSS error is small, as in lines 5 and 7, we are assured that each of the individual terms are small. Using an acceptance threshold of 0.001, for example, guarantees that accepted solutions produce the observed TDOA, or equivalently the range difference, and the assumed altitude to within a meter and the observed FDOA to within a milli-Hertz.

The solutions in lines 5 and 7 of Table II have RSS errors many orders of magnitude less than this threshold. The small residual RSS error in the desired solutions is the result of rounding - not measurement noise. For example, adding 1 μs to TOA1 and 0.5 Hz to FOA1 to simulate measurement noise increases the error in the computed solution manifold to 5.8 km while the residual errors in column 6 of Table II do not change significantly. The residual errors indicate only the consistency of the equations being evaluated. Similarly, a 10 km error in the assumed emitter altitude gives a 15 km location error but no increase in RSS error. The two solutions in lines 5 and 7 of Table II are the two intersections of the TDOA and FDOA contours in Fig. 1.

It is possible for measurement noise to create a system of equations that has no real solutions. The contours in Fig. 1 in this case do not intersect. When this happens, one can increase the assumed altitude which generally increases the arc lengths of the two contours and can create real solutions. For example, a TDOA defines a hyperbolic sheet that may not even intersect the Earth ellipsoid due to measurement noise. The sheet would however intersect concentric ellipsoids that are large enough. Increasing the radii of the ellipsoid that we assume the emitter lies on, therefore, can create real solutions. The intersection of the TDOA hyperboloid and the ellipsoid is the dashed curve in Fig. 1. By increasing the assumed altitude we have changed the equations to a system having real solutions, where the original did not. Another approach is to use only the real parts of the roots of (32) and then accept candidate solutions whose transmission time, frequency, and altitude are plausible. Note that the spurious solutions in Table II could have been rejected using these tests as they have either a noncausal transmission time (later than the TOAs) or violate the altitude assumption. RSS error can be used here but the thresholds would need to be set higher than if only numerical rounding were determining the total error.

V. SOLUTION ERROR VARIANCE

We now derive the expression for the solution error variance as in [3] but with some new terms involving $\dot{x}$ reflecting our use of an ECI frame. The solution error variances along the three orthogonal axes can be used to convey the orientation of the spatial distribution of expected error in a computed location. See [5] for an analysis of the effects of uncertainties in the satellite positions and velocities, emitter motion, and geometric dilution of precision (GDOP). We include the effect of errors in our estimate of the emitter altitude rather than consider this an exactly known constraint.

Solution error variance can be evaluated by linearizing the model equations about the solution point, assumed near the true emitter location. We begin by eliminating the time and frequency of transmission leaving three equations in the three principle solution variables, the emitter position coordinates. Subtracting (1), with $i = 1$ from (1) with $i = 2$ gives

$$R_2(\mathbf{x},\mathbf{s}_2) - R_1(\mathbf{x},\mathbf{s}_1) = \tau_2 - \tau_1. \tag{33}$$

Similarly subtracting (7), with from (7) with gives

$$\dot{R}_2 - \dot{R}_1 = \nu_2 - \nu_1. \tag{34}$$

We also have

$$\sqrt{\mathbf{x}^T\mathbf{Q}\mathbf{x}} = r_e \tag{35}$$

from (17).

Linearizing these governing equations to relate small variations in the TDOA, FDOA, and assumed emitter equatorial radius r_e, to small solution displacements $\Delta\mathbf{x}$, gives

$$\mathbf{J}\Delta\mathbf{x} = \begin{bmatrix} \Delta(\tau_2 - \tau_1) \\ \Delta(\nu_2 - \nu_1) \\ \Delta r_e \end{bmatrix} \tag{36}$$

where $\mathbf{J}$ is the matrix whose rows are gradients of (33), (34), and (35), respectively,

$$\mathbf{J} = \begin{bmatrix} \nabla R_2 - \nabla R_1 \\ \nabla\dot{R}_2 - \nabla\dot{R}_1 \\ \nabla\sqrt{\mathbf{x}^T\mathbf{Q}\mathbf{x}} \end{bmatrix}. \tag{37}$$

Computing the required gradients evaluated at the solution point $\mathbf{x}_0$ we obtain

$$\nabla R_i = \frac{(\mathbf{x}_0 - \mathbf{s}_i)^T}{R_i} \tag{38}$$

from (2). Using (8) and (9) gives

$$\begin{aligned}\nabla\dot{R}_i &= \frac{R_i(\mathbf{x}_0^T\mathbf{P} + \mathbf{x}_0^T\mathbf{P}^T - \mathbf{s}_i^T\mathbf{P} - \dot{\mathbf{s}}^T) - (R_i\dot{R}_i)(\nabla R_i)}{R_i^2} \\ &= \frac{(\mathbf{x}_0 - \mathbf{s}_i)^T\mathbf{P}}{R_i} + \frac{(\dot{\mathbf{x}}_0 - \dot{\mathbf{s}}_i)^T}{R_i} - \frac{\dot{R}_i(\mathbf{x}_0 - \mathbf{s}_i)^T}{R_i^2}\end{aligned} \tag{39}$$

and from (35) we have

$$\nabla\sqrt{\mathbf{x}^T\mathbf{Q}\mathbf{x}} = \frac{\mathbf{x}_0^T\mathbf{Q}}{|\mathbf{x}_0^T\mathbf{Q}\mathbf{x}_0|}. \tag{40}$$

Using (36) the covariance of the emitter position is then the expectation

$$\Sigma_{\mathbf{x}} = E\{\mathbf{x}\mathbf{x}^T\} = \mathbf{J}^{-1}\Sigma_{\mathbf{d}}(\mathbf{J}^{-1})^T = \mathbf{K}\Sigma_{\mathbf{d}}\mathbf{K}^T \tag{41}$$

where $\mathbf{K} = \mathbf{J}^1$ and $\Sigma_{\mathbf{d}}$ is the covariance of the errors in the data, which are the components on the right-hand side of (36). These data errors are usually assumed to be zero mean and independent, making $\Sigma_{\mathbf{d}}$ diagonal. Note that the first and second elements on the diagonal of $\Sigma_{\mathbf{d}}$ are the variances of the TDOA and the FDOA rather than of the TOAs and FOAs.

VI. CONCLUSIONS

An algorithm for the computation of the ambiguous emitter location on an ellipsoidal Earth has been given for the two-satellite TOA/FOA geolocation problem. Prior art [3] was to determine all solutions on a spherical Earth each of which could then be iterated to produce a solution on the ellipsoidal Earth. The algorithm presented here finds all solutions on the ellipsoidal Earth.

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