FAST RECONSTRUCTION IN PERIODIC NONUNIFORM SAMPLING OF DISCRETE-TIME BAND-LIMITED SIGNALS

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ABSTRACT

We develop a fast direct reconstruction scheme that reduces the computational effort in solving the periodic nonuniform sampling problem for discrete-time band-limited signals. This is achieved by exploiting the periodic structure of the samples and of the **DFT** matrix.

1. INTRODUCTION

The irregular sampling problem consists of reconstructing a discrete-time signal of length N given K irregularly spaced samples. Fast iterative methods in 1D [1] and 2D [7] exist which are independent of the sampling pattern. Consider an irregular set of samples obtained by taking multiple copies of uniform sets but which differ by some shifts, i.e. a periodic nonuniform set of samples. Periodic nonuniform sampling for multi-band signals has been studied in [2, 3]. In [2] the problem is considered in terms of an M-channel filter bank and is solved using a POCs method. In [3] a well-conditioned universal sampling pattern is determined for the reconstruction of multi-band signals.
We first define the periodic nonuniform sampling problem of discrete-time band-limited signals. We show how the problem can be reduced by exploiting the periodic structure of the samples and of the **DFT** matrix. We develop a fast direct reconstruction scheme and compare its complexity to the unstructured direct solving method. The motivation in developing a fast direct method is to speed up the search when the shifts are unknown [5].

2. PERIODIC NONUNIFORM SAMPLING OF DISCRETE-TIME BAND-LIMITED SIGNALS

2.1. Problem definition

We begin by recalling the definition of a band-limited discrete-time periodic signal.

Definition 1 *A discrete-time signal* $\mathbf{x} = (x_0, \dots, x_{N-1})^T$ *of length N is band-limited to L (in the low-pass sense) if the last $N - L$ components of the Discrete Fourier Transform*

$$\hat{\mathbf{x}} = \mathbf{DFT}_N \cdot \mathbf{x} \qquad (1)$$

are zero, i.e. $\hat{\mathbf{x}} = (\hat{x}_0, \hat{x}_1, \dots, \hat{x}_{L-1}, 0, \dots, 0)^T$.

From Eq. (1) the signal is obtained by inverting the $\mathbf{DFT}_N$ matrix where $\mathbf{DFT}_N^{-1} = \frac{1}{N}\mathbf{DFT}^*$. Moreover the $N - L$ last columns of the $\mathbf{DFT}_N^{-1}$ are irrelevant to the signal and can therefore be omitted. We obtain the following system of equations

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & W_N & \dots & W_N^{L-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & W_N^{N-1} & \dots & W_N^{(N-1)(L-1)} \end{bmatrix} \cdot \begin{pmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \vdots \\ \hat{x}_{L-1} \end{pmatrix}.$$

Equivalently in a more compact form

$$\mathbf{x}(\mathcal{N}) = \frac{1}{N}\mathbf{DFT}_N^*(\mathcal{N}, \mathcal{L}) \cdot \hat{\mathbf{x}}(\mathcal{L}) \qquad (2)$$

where $\mathbf{DFT}_N^*(\mathcal{N}, \mathcal{L}) = \{W_N^{nl}\}_{n\in\mathcal{N}, l\in\mathcal{L}}$, $(W_N = e^{i2\pi/N})$, $\mathcal{N} = \{0, \dots, N-1\}$ is the index set of the signal $\mathbf{x}$ and $\mathcal{L} = \{0, \dots, L-1\}$ corresponds to the index set of the nonzero components of the spectrum $\hat{\mathbf{x}}$.
The irregular sampling problem for discrete-time band-limited signals consists in recovering the signal $\mathbf{x}$ from K samples $\mathbf{x}(\mathcal{N}_K)$ where $\mathcal{N}_K = \{n_k\}_{k=1}^K$ is an irregularly spaced set of indices and a subset of $\mathcal{N}$. This is equivalent to solving the following system of K equations and L unknowns $\hat{\mathbf{x}}(\mathcal{L})$

$$\mathbf{x}(\mathcal{N}_K) = \frac{1}{N}\mathbf{DFT}_N^*(\mathcal{N}_K, \mathcal{L}) \cdot \hat{\mathbf{x}}(\mathcal{L}) \qquad (3)$$

where K must be greater or equal to L.
In this paper, we are interested in a particular irregular set namely the periodic nonuniform set. The definition follows and an example is illustrated in Fig.1.

Definition 2 *A periodic nonuniform set of samples $\mathcal{N}_K$ is a union of C uniform sets of size $K_i = N/T$ differing by a shift s_i*

$$\mathcal{N}_K = \bigcup_{i=1}^{C} \{\mathcal{N}_{K_i} = \{nT + s_i\}_{n=0}^{K_i - 1}\}$$

where N is the length of the signal and T is the discrete-time uniform sampling interval.

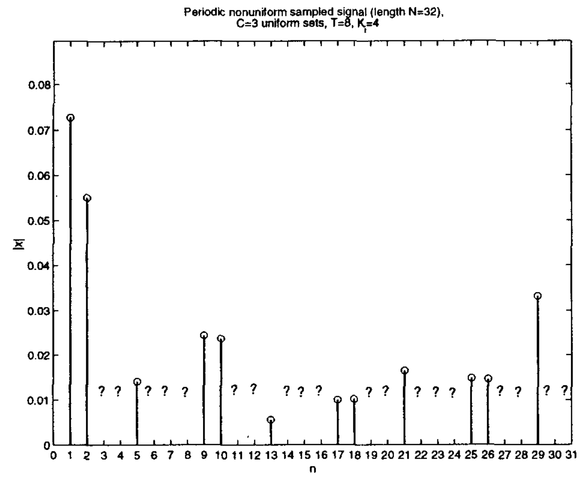

Figure 1: Example of a periodic nonuniform sampled discrete-time signal of length $N = 32$ obtained from $C = 3$ uniform sets with interval $T = 8$, $\mathcal{N}_{K_1} = \{1, 9, 17, 25\}$, $\mathcal{N}_{K_2} = \{2, 10, 18, 26\}$, $\mathcal{N}_{K_3} = \{5, 13, 21, 29\}$.

2.2. Direct solving method

As mentioned in the last subsection to recover the signal $\mathbf{x}(\mathcal{N})$ it suffices to solve the system of equations in (3) for $\hat{\mathbf{x}}(\mathcal{L})$ and then substitute in Eq. (2). Given that the signals are band-limited the $K \times L$ matrix in (3) is a Vandermonde matrix which assures the existence of a solution. A solution $\hat{\mathbf{x}}(\mathcal{L})$ may be obtained in the least squares sense using the generalized inverse of the matrix $\mathbf{F} = \frac{1}{N}\mathbf{DFT}_N^*(\mathcal{N}_K, \mathcal{L})$, i.e.

$$\hat{\mathbf{x}}(\mathcal{L}) = (\mathbf{F}^*\mathbf{F})^{-1}\mathbf{F}^* \cdot \mathbf{x}(\mathcal{N}_K). \tag{4}$$

This calculation requires matrix multiplication and inversion and may be costly for large values of K and L. Iterative methods such as the Papoulis Gerchberg method [6] or adapted weights conjugate gradient method [1, 7] can also be applied but these do not take into account the periodic structure of the samples. We are interested in exploiting the periodic nonuniform sampling pattern so as to reduce the dimension of the problem and speed up the direct method.

3. FAST RECONSTRUCTION

In this section we show how the dimension of the problem can be reduced by exploiting the periodic structure of the samples. We present the fast direct method by means of a small example and then describe the general case. Finally we compare the computations between the structured and the unstructured direct method.

3.1. Example

Consider a discrete-time signal $\mathbf{x}$ of length $N = 8$, band-limited to $L = 4$, i.e.

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_8 & W_8^2 & W_8^3 \\ 1 & W_8^2 & W_8^4 & W_8^6 \\ 1 & W_8^3 & W_8^6 & W_8 \\ 1 & W_8^4 & 1 & W_8^4 \\ 1 & W_8^5 & W_8^2 & W_8^7 \\ 1 & W_8^6 & W_8^4 & W_8^2 \\ 1 & W_8^7 & W_8^6 & W_8^5 \end{bmatrix} \cdot \begin{pmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} \tag{5}$$

where $W_8 = e^{i2\pi/8}$.

Suppose the discrete-time uniform sampling interval is equal to $T = 4$ then the number of samples in the uniform set is $K = 2 < L$ which is insufficient to reconstruct the signal (the number of samples to recover the signal must be at least L). If we take $C = 3$ uniform sets of samples, for example, at locations $\mathcal{N}_{K_1} = \{0, 4\}$, $\mathcal{N}_{K_2} = \{1, 5\}$ and $\mathcal{N}_{K_3} = \{2, 6\}$ then we obtain a periodic nonuniform set of locations $\mathcal{N}_K = \bigcup_{i=1}^{3} \mathcal{N}_{K_i} = \{0, 1, 2, 4, 5, 6\}$. We reformulate the problem by partitioning the system in (5) according to the uniform sets $N_{K_i}, (i = 1, 2, 3)$,

$$\begin{pmatrix} x_0 \\ x_4 \\ x_1 \\ x_5 \\ x_2 \\ x_6 \end{pmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_8^4 & 1 & W_8^4 \\ 1 & W_8 & W_8^2 & W_8^3 \\ 1 & W_8^5 & W_8^2 & W_8^7 \\ 1 & W_8^2 & W_8^4 & W_8^6 \\ 1 & W_8^6 & W_8^4 & W_8^2 \end{bmatrix} \cdot \begin{pmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} \tag{6}$$

Notice that $\begin{bmatrix} 1 & 1 \\ 1 & W_8^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} = 2 \cdot \mathbf{DFT}_2^{-1}$. By multiplying Eq. (6) on each side by the following diagonal block matrix,

$$\frac{1}{2} \begin{bmatrix} \mathbf{DFT}_2 & \mathbf{O}_2 & \mathbf{O}_2 \\ \mathbf{O}_2 & \mathbf{DFT}_2 & \mathbf{O}_2 \\ \mathbf{O}_2 & \mathbf{O}_2 & \mathbf{DFT}_2 \end{bmatrix} \tag{7}$$

where $\mathbf{O}_2$ is a 2×2 zero matrix we obtain the following partitioned system

$$\frac{8}{2}\begin{pmatrix} \mathbf{DFT}_2\begin{pmatrix} x_0 \\ x_4 \end{pmatrix} \\ \mathbf{DFT}_2\begin{pmatrix} x_1 \\ x_5 \end{pmatrix} \\ \mathbf{DFT}_2\begin{pmatrix} x_2 \\ x_6 \end{pmatrix} \end{pmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & W_8 \end{bmatrix} & \begin{bmatrix} W_8^2 & 0 \\ 0 & W_8^3 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & W_8^2 \end{bmatrix} & \begin{bmatrix} W_8^4 & 0 \\ 0 & W_8^6 \end{bmatrix} \end{bmatrix} \cdot \begin{pmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} \quad (8)$$

Each block of the partitioned matrix in Eq. (8) is a diagonal matrix whose values are given by the rows associated to x_0, x_1, x_2 and columns $\{0,1\}$,$\{2,3\}$ of the matrix in Eq. (6). The fact that the blocks are diagonal matrices hints that the number of operations to calculate the generalized inverse of the partitioned system will be less than the unstructured one.

3.2. General fast reconstruction scheme

The key step in the example of Sec. 3.1 which reduces the problem is the multiplication of Eq. (6) by the diagonal block $\mathbf{DFT}_2$ matrix in Eq. (7). We generalize by the following

$$\frac{N}{K_i}\begin{pmatrix} \mathbf{DFT}_{K_i}\mathbf{x}(\mathcal{N}_{K_1}) \\ \mathbf{DFT}_{K_i}\mathbf{x}(\mathcal{N}_{K_2}) \\ \vdots \\ \mathbf{DFT}_{K_i}\mathbf{x}(\mathcal{N}_{K_C}) \end{pmatrix} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} & \dots & \mathbf{D}_{1R} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \dots & \mathbf{D}_{2R} \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{D}_{C1} & \mathbf{D}_{C2} & \dots & \mathbf{D}_{CR} \end{bmatrix} \cdot \begin{pmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \vdots \\ \hat{x}_{L-1} \end{pmatrix}$$
$$\mathbf{y} = \mathbf{D} \cdot \hat{\mathbf{x}} \quad (9)$$

where

- N is the length of the signal,
- T is the discrete-time uniform sampling interval,
- C is the number of uniform sets,
- K_i is the size of one uniform set of locations $(= N/T)$,
- $\mathcal{N}_{K_i}$ is the $i-th$ uniform set of locations, $i = 1, \dots, C$ $(= \{nT + s_i\}_{n=0,K_i-1})$,
- s_i is the $i{-}th$ shift from the uniform set $\{nT\}_{n=0,K_i-1}$ $(0 \leq s_i \leq T-1)$,
- K is the size of the periodic nonuniform set $(= CK_i)$,
- L is the band-limit $(= RK_i, 1 \leq R \leq C)$ and
- $\mathbf{D}_{ij}$ are diagonal matrices, $i = 1, \dots, C, j = 1, \dots R$ $(= diag(\{W_N^{s_i l}\}_{l \in \{(j-1)K_i, \dots, jK_i-1\}})$.

As described in Section 2.2 we obtain a solution in the least squares sense by means of the generalized inverse of the partitioned matrix $\mathbf{D}$ in Eq. (9). Hence we obtain,

$$\hat{\mathbf{x}}(\mathcal{L}) = (\mathbf{D}^*\mathbf{D})^{-1}\mathbf{D}^*\mathbf{y} \quad (10)$$

where $\mathbf{y} = [\mathbf{DFT}(\mathbf{x}(\mathcal{N}_{K_1}), \dots, \mathbf{DFT}(\mathbf{x}(\mathcal{N}_{K_C})]^T$ and $(\mathbf{D}^*\mathbf{D})^{-1}\mathbf{D}^*$ is also a partitioned matrix whose blocks are diagonal matrices. The fast reconstruction scheme is illustrated in Fig.2.

3.3. Computational complexity

In this section we compare the complexity of the fast reconstruction scheme with the direct unstructured one. The inverse of a partitioned matrix is obtained from Eq. (11) in the Appendix 5.1 where we let $\mathbf{A} = \mathbf{D}^*\mathbf{D}$. Note that $\mathbf{A}$ is a partitioned $R \times R$ matrix where each block $\mathbf{A}_{mn}$ is $K_i \times K_i$ diagonal matrix. Define $op_{\mathbf{A}}(R)$ as the number of operations required to invert $\mathbf{A}$. We suppose that R is a power of 2 and use a divide and conquer approach to determine $\mathbf{A}^{-1} = (\mathbf{D}^*\mathbf{D})^{-1}$ (i.e. let $\alpha = \{1, \dots, R/2\}$). We obtain the following recurrence equation

$$\begin{aligned} op_{\mathbf{A}}(R) &= 10 op_{\mathbf{A}}(R/2) + 12(R/2)^3 K_i + 4(R/2)^2 K_i \\ op_{\mathbf{A}}(1) &= K_i \end{aligned}$$

and conclude that $op_{\mathbf{A}}(R) \in O(R^{log_2 10} K_i)$. In Tab. 1 we summarize the number of operations (multiplication, inversion) required for each scheme. By substituting $K = CK_i$ and $L = RK_i$ in Tab. 1, keeping the variables C and R constant and letting K_i vary, we compare the two schemes in Fig. 3.

Complexity			
Direct scheme		Fast scheme	
$\mathbf{F}^*\mathbf{F}$	L^2K	$\mathbf{D}^*\mathbf{D}$	R^2CK_i
$(\mathbf{F}^*\mathbf{F})^{-1}$	L^3	$(\mathbf{D}^*\mathbf{D})^{-1}$	$R^{log_2 10}K_i$
$\mathbf{F}^* \cdot \mathbf{x}(\mathcal{N}_K)$	LK	$\mathbf{D}^*\mathbf{y}$	$RCK_i log K_i$

Table 1: Summary of complexity $O(operations)$ for both schemes.

4. CONCLUSION

We developed a fast direct method that reconstructs a periodic non-uniformly sampled discrete-time band-limited signal. An extension to the 2D case with known and unknown shifts is the topic of on-going research.

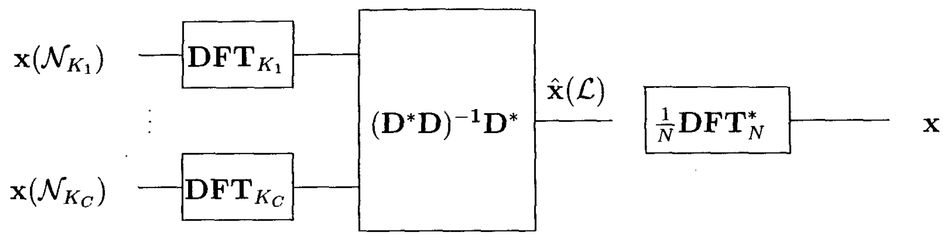

Figure 2: Fast reconstruction scheme for periodic nonuniform sampling

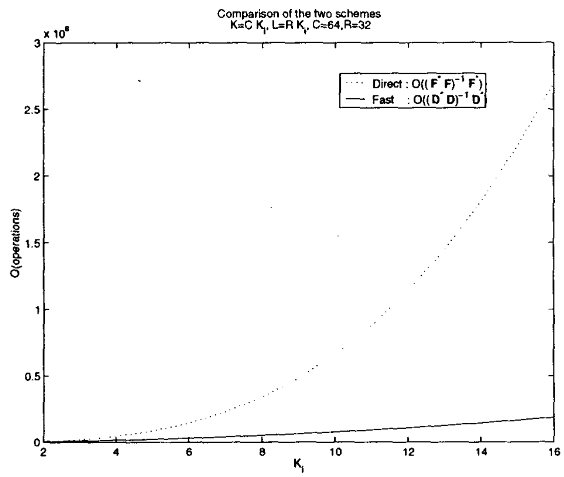

Figure 3: Comparison of the unstructured direct method and the fast structured scheme.

5. APPENDIX

5.1. Inverse of a Partitioned Matrix

Suppose $\mathbf{A}$ is an $R \times R$ partitioned matrix,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \dots & \mathbf{A}_{1R} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \dots & \mathbf{A}_{2R} \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{A}_{R1} & \mathbf{A}_{R2} & \dots & \mathbf{A}_{RR} \end{bmatrix}$$

where $\mathbf{A}_{mn}$ are square matrices. The inverse $\mathbf{B} = \mathbf{A}^{-1}$ of a partitioned matrix is defined by [4]

$$\begin{aligned} \mathbf{B}(\alpha,\alpha) &= \left[\mathbf{A}(\alpha,\alpha) - \mathbf{A}(\alpha,\alpha')\mathbf{A}(\alpha',\alpha')^{-1}\mathbf{A}(\alpha',\alpha)\right]^{-1} \\ \mathbf{B}(\alpha,\alpha') &= \mathbf{A}(\alpha,\alpha)^{-1}\mathbf{A}(\alpha,\alpha') \qquad (11) \\ &\quad \cdot \left[\mathbf{A}(\alpha',\alpha)\mathbf{A}(\alpha,\alpha)^{-1}\mathbf{A}(\alpha,\alpha')\mathbf{A}(\alpha',\alpha')\right]^{-1} \end{aligned}$$

where α is a subset of $\{1,\dots,R\}$ and α' is the complement index set of α. For instance if $R = 4$ and $\alpha = \{1,2\}$, $\alpha' = \{3,4\}$ then $\mathbf{A}(\alpha,\alpha) = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$ and $\mathbf{A}(\alpha,\alpha') = \begin{bmatrix} \mathbf{A}_{13} & \mathbf{A}_{14} \\ \mathbf{A}_{23} & \mathbf{A}_{24} \end{bmatrix}$.

6. REFERENCES

[1] H. G. Feichtinger and K. Grochenig, "Theory and practice of irregular sampling," in *Wavelets: Mathematics and Applications*, pp.305-363. CRC Press, Boca Raton, Florida, 1994.

[2] C. Herley and P.W. Wong, "Minimum rate sampling of signals with arbitrary frequency support," in *Proc. IEEE ICIP*, Lausanne, Sept. 1996.

[3] P. Feng and Y. Bresler, "Spectrum-blind minimum-rate sampling and reconstruction of multi-band signals," in *Proc. IEEE ICASSP*, Atlanta, May 1996

[4] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge university press, New York, 1985.

[5] P. Marziliano and M. Vetterli, "Irregular sampling with unknown locations", in *Proc. IEEE ICASSP*, Phoenix, March 1999.

[6] H. Stark, *Image Recovery: Theory and Application*, Academic press, San Diego, 1987.

[7] T. Strohmer, "Computationally attractive reconstruction of band-limited images from irregular samples", *IEEE Trans. Image Processing,* vol. 6, pp. 540-548, 1997.