TSINGHUA SCIENCE AND TECHNOLOGY
ISSN 1007-0214 06/17 pp36-40
Volume 16, Number 1, February 2011

Reliability Analysis of the Cayley Graphs of Dihedral Groups*

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Abstract: Cayley graphs have many good properties as models of communication networks. This study analyzes the reliability of the Cayley graph based on the dihedral graph. Graph theory and analyses show that almost all Cayley graphs of the dihedral graph D_{2n} are optimal super-λ. The number $N_i(G)$ of cutsets of size $i, \lambda \leqslant i \leqslant \lambda'$ is given as $N_i(G) = n\binom{(n-1)\delta}{i-\delta}$.

Key words: super-λ; reliability; Cayley graph; dihedral group

The Cayley graph is an important class of graphs which is vertex-transitive and is used for communication networks. Reliability analysis is important when studying communication networks. For a connected graph $G(V,E)$, we denote the number of cutsets of size i as $N_i(G)$. Assuming that all vertices are perfectly reliable and all edges fail independently with the same probability ρ, then the probabilistic reliability measure $R(G,\rho)$ of a connected graph G is defined as the probability that G remains connected when some edges fail,

$$R(G;\rho) = 1 - \sum_{i=\lambda}^{|E|} N_i(G)\rho^i (1-\rho)^{|E|-i}.$$

Bauer et al.[1] and Esfahanian[2] introduced some important definitions and a useful proposition.

Definition 1 A connected graph G is said to be super-λ if every cutset of size λ isolates a vertex with the minimum vertex degree δ of G.

Definition 2 A set S of edges of a connected graph G is called a restricted cutset (RC) if $G-S$ is disconnected and $G-S$ contains no trivial component K_1. The restricted edge connectivity $\lambda'(G)$ is the minimum size of an RC in G.

The following proposition[3] and definition[4] are useful in this article.

Proposition 1 If G is a connected graph with at least four vertices and it is not a star graph $K_{1,m}$, then $\lambda'(G)$ is well defined and $\lambda(G) \leqslant \lambda'(G) \leqslant \xi(G)$, where $\xi(G) = \min\{d(e) = d(x) + d(y) - 2 : e = (x,y) \in E(G)\}$.

Definition 3 A super-λ graph G is said to be optimal super-λ if $\lambda'(G) = \xi(G)$.

In general, all values N_i of a graph are difficult to determine. In fact, Ball[5] showed that this problem is NP-hard. However a special class of regular graphs has some results. Boesch and Wang[6] gave the necessary conditions and determined that $N_i(2k \leqslant i \leqslant 4k-3)$ for Harary graphs. These two results are as follows:

Proposition 2 Every connected circulant is super-λ unless it is $G(n;a)$ or $G(2j;\ 2,4,\cdots,j-1,j)$ for $j>1$ is odd.

Proposition 3 Let $H = G(n;1,2,\cdots,k), k<n/2$ and let U be a cutset of H with size i, $2k \leqslant i \leqslant 4k-3$; then U isolates exactly one vertex and

$$N_i(G) = n\binom{nk-2k}{i-2k}.$$

Furthermore Li and Li[4] proved the following proposition.

Proposition 4 Every connected circulant $G(n;a_1,$

Received: 2010-04-20; revised: 2010-05-15

* Supported by the National Key Basic Research and Development (973) Program of China (No. 2007CB311003)

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$a_2,\cdots,a_k$) is optimal super-λ unless $a_k=n/2$, g.c.d.(n, $a_1,a_2,\cdots,a_k$) >1 and $2k-1<n/2<4k-4$.

This article studies the reliability of another class of Cayley graph based on dihedral groups. This kind of Cayley graph is more complicated than the circulant. Some initial definitions and propositions are given below.

Definition 4 The dihedral group of order $2n$ is the group of symmetries of a regular polygon with n sides, usually denoted as D_{2n}, where $D_{2n}=\langle a,b|a^n=1, b^2=1,b^{-1}ab=a^{-1}\rangle$.

Since $b^{-1}ab=a^{-1}$, then $ba^s=a^{-s}b$. Thus all the elements in D_{2n} have the form a^ib^j ($1\leqslant i<n, j=0,1$).

Definition 5 Let G be a group and S be a subset $S\subset G$, then the Cayley graph Cay(G,S) is the digraph X with vertex set G and edge set $\{(x,y): yx^{-1}\in S\}$.

Denote the Cayley graph based on the dihedral group Cay(D_{2n},S) as $G(D_{2n},C)$ where $S=C\cup C^{-1}$ and $C=(a^{i_1},a^{i_2},\cdots,a^{i_{k_1}},a^{j_1}b,\cdots,a^{j_{k_2}}b)(1\leqslant i_s\leqslant n/2, a^{i_s}\in C)$. An edge of the form $(a^ib^j,a^{i\pm i_s}b^j)$ is called an a^{i_s}-edge and each edge of the form $(a^ib^j, a^{j_t-i}b^{j+1})$ is called a $a^{j_t}b$-edge.

A few lemmas for the Cayley graph $G(D_{2n},C)$ are given first followed by a theorem.

Some notations and propositions are given first.

In graph $G(D_{2n},C)$, let V_0 be the vertex set with the form a^i. Call the subgraph induced by V the a-part and let $V_1=V\setminus V_0$. Then call the subgraph induced by V_1 the b-part.

Since G is connected, then C is the generated subgroup of D_{2n}, and it is easy to find the following proposition.

Note 1 Let $C=(a^{i_1},a^{i_2},\cdots,a^{i_{k_1}},a^{j_1}b,\cdots,a^{j_{k_2}}b)$, $n_1=$ g.c.d.$(n,i_1,i_2,\cdots,i_{k_1})$, and $n_2=$ g.c.d.$(j_r-j_s(\bmod n):$ $a^{j_r}b,a^{j_s}b\in C, r\neq s)$, then $G=G(D_{2n},C)$ is connected if and only if $k_2=1, n_1=1$ or $k_2>1$, g.c.d.$(n_1,n_2)=1$.

Let $G=(V,E)$ be a graph. For $X\subseteq V$, $H=(X, V-X)$ denotes the set of edges with one end in X and the other not in X, and $\delta_G(X)$ denotes the number of edges in $(X,V-X)$. The following is a known result about $\delta_G(X)$[7]:

Proposition 5 $\delta_G(X\cap Y)+\delta_G(X\cup Y)\leqslant\delta_G(X)+\delta_G(Y)$ for any $X,Y\subseteq V(G)$.

Every cutset of size λ (or every restricted cutset of size λ') always has the form $H=(X,V-X)$ and both of the induced subgraphs $G[X]$ and $G[V-X]$ are connected.

Lemma 1 Let $G(D_{2n},C)$ be a connected Cayley graph, $C=\{a^{i_1},a^{i_2},\cdots,a^{i_{k_1}},a^{j_1}b,a^{j_2}b,\cdots,a^{j_{k_2}}b\}$ and $H=(X, V-X)$ be an RC of size λ'. If $|X|=m>3$, then there exists an $a^ib^j\in C$ such that the induced subgraph $G[X]$ contains at least two a^ib^j-edges.

Proof If not, there is at most one a^ib^j-edge in $G[X]$ for each $a^ib^j\in C$, then suppose there are δ_1 elements in C whose order is bigger than 2, and δ_2 elements in C with order 2 then $\lambda'=\delta_G(X)\geqslant|X|\cdot\delta-2(\delta_1+\delta_2)=m\cdot(2\delta_1+\delta_2)-2(\delta_1+\delta_2)$.

$$\lambda'\geqslant m(2\delta_1+\delta_2)-2(\delta_1+\delta_2)-2(2\delta_1+\delta_2)+2+\xi= (2m-6)\delta_1+(m-4)\delta_2+2+\xi>\xi.$$

Thus a contradiction occurs. This ends the proof of the lemma. □

Lemma 2 Let $G(D_{2n},C)$ be a connected graph and let $H=(X,V-X)$ be an RC with $\delta_G(X)=\lambda'$, $|X|\leqslant n$ and X is minimal. If for some $a^i(\text{or } a^ib)\in C$, there are at least two $a^i(\text{or } a^ib)\in C$ edges in $G[X]$, then $X=a^i\cdot X(\text{or } a^ib\cdot X)$.

Proof Let $X'=a^i\cdot X(\text{or } a^ib\cdot X)$. Since $G(D_{2n},C)$ is vertex symmetric, then X' is minimal RC with $\delta_G(X')=\lambda'$. So $|X'|=|X|\leqslant n$, and we have that $|V-(X\cap X')|\geqslant n\geqslant 2$. As there are at least two $a^i(\text{or } a^ib)\in C$ edges in $G[X]$, so $|X\cap X'|\geqslant 2$, $2\leqslant|X\cup X'|=|X|+|X'|-|X\cap X'|\leqslant 2n-2$ and $|V-(X\cup X')|\geqslant 2$.

Thus, if $G[X\cap X']$ or $G[V-X\cap X']$, say $G[X\cap X']$, has a trivial component K_1, one of these two cases will occur: if $G[X\cap X']$ has at least two trivial components, then $\delta_G(X\cap X')\geqslant\delta+\delta=2\delta>\xi\geqslant\lambda'$, else $G[X\cap X']$ has only one trivial component, then $\delta_G(X\cap X')\geqslant\lambda'+\delta>\lambda'$; otherwise, $\delta_G(X\cap X')\geqslant\lambda'$.

Similarly, if $G[V-(X\cup X')]$ has a trivial component K_1, then $\delta_G(X\cup X')>\lambda'$; otherwise $\delta_G(X\cap X')\geqslant\lambda'$.

Therefore, if $G[X\cap X']$ or $G[V-(X\cap X')]$ or $G[V-(X\cup X')]$ has a trivial component K_1, then $\delta_G(X\cap X')+\delta_G(X\cup X')>\lambda'+\lambda'=\delta_G(X)+\delta_G(X')$, which contradicts Proposition 5, so each of the three induced subgraphs $G[X\cap X']$ or $G[V-(X\cap X')]$ or $G[V-(X\cup X')]$ has no trivial component K_1. Hence, $\delta_G(X\cap X')\geqslant\lambda'$ and $\delta_G(X\cup X')\geqslant\lambda'$. By Proposition 5, we have

$$\lambda'+\lambda'\leqslant\delta_G(X\cap X')+\delta_G(X\cup X')\leqslant \delta_G(X)+\delta_G(X')=\lambda+\lambda'.$$

Thus, $\delta_G(X\cap X')=\delta_G(X\cup X')=\lambda'$. Since X is minimal, then $X\cap X'=X$, i.e., $X=X'$. The lemma is proved. □

Lemma 3 Let $G(D_{2n},C)$ be a connected graph and let $H=(X,V-X)$ be an RC with $\delta_G(X)=\lambda'$ and X is minimal. If $3<|X|\leqslant n$ and for some a^i(or a^ib)$\in C$ there is an a^i(or a^ib)-edge in $G[X]$, then $X=a^i\cdot X$(or $a^ib\cdot X$).

Proof To prove this lemma, three cases will occur:

Case 1 There is an a^ib-edge in $G[X]$. Let $(x,a^ib\cdot x)$ be an edge in $G[X]$, let $X'=a^ib\cdot X$, x and $a^ib\cdot x$ are both in X'. Then $|X'\cap X|\geqslant 2$. Similar to the proof of Lemma 2, we have $X=a^ib\cdot X$.

Case 2 There are only a^i-edges in $G[X]$. By Lemma 1 and Lemma 2, there exists an $a^i\in C$ such that $X=a^i\cdot X$, suppose $(x,a^j\cdot x)$ $(j\neq i)$ is an edge in $G[X]$. Since both $a^i\cdot x$ and $a^i\cdot a^j\cdot x$ are in $G[X]$, $(a^i\cdot x,a^j\cdot a^i\cdot x)$ $(a^i\cdot a^j\cdot x=a^j\cdot a^i\cdot x)$ is also an edge in $G[X]$, then by Lemma 2, $a^j\cdot X=X$.

Case 3 There are both a^i-edges and a^jb-edges in $G[X]$. By Case 1, $X=a^jb\cdot X$, suppose $(x,a^i\cdot x)\in G[X]$, then $a^jb\cdot x, a^jb\cdot a^i\cdot x\in G[X]$, and thus $(a^jb\cdot a^i\cdot x,a^jb\cdot x)$ is also an a^i-edge in $G[X]$, so $X=a^i\cdot X$. □

Lemma 4 Let $G(D_{2n},C)$ be a connected Cayley graph, $C=\{a^{i_1},a^{i_2},\cdots,a^{i_{k_1}},a^{j_1}b,a^{j_2}b,\cdots,a^{j_{k_2}}b\}$ and $H=(X,V-X)$ be an RC of size λ' with minimal X and $|X|=n'\leqslant n$. If $n'>3$, then $n'\,|\,2n$ and let $T=\{a^ib^j\in C$: there is at least one a^ib^j-edge in $G[X]\}$, then one of the following two cases will occur:

(1) If there is an a^jb-edge in $G[X]$, then $X=\{a^{id}:i=0,1,\cdots,n'/2-1\}\cup\{a^{id+l}b:i=0,1,\cdots,n'/2-1\}$ and the induced subgraph $G[X]$ is isomorphic to a connected graph $G(D_{n'},C')$ where $C'=\{a^{s/d}:a^s\in T\}\cup\{a^{\lfloor t/d\rfloor}b:a^tb\in T\}$;

(2) If there are only a^j-edges in $G[X]$, then $X=\{a^{id}:i=0,1,\cdots,n'-1\}$ and $G[X]$ is isomorphic to a circulant of order n' with $C'=\{l/d:a^l\in T\}$.

Proof Without loss of generality, suppose $1\in X$. First of all, prove that X is a subgroup of D_{2n}. Let $T=\{a^ib^j\in C$: there is at least one a^ib^j-edge in $G[X]\}$. Consider X and T as the subsets of D_{2n}. Let $\langle T\rangle$ be the subgroup of D_{2n} generated by T, since $1\in X$, then $\langle T\rangle\subseteq X$ by Lemma 3. On the other hand, for any $x\in X$, since $G[X]$ is connected, there is a path joining e and x which consists of a sequence of $a^lb^m\in T$, thus, $x\in\langle T\rangle$. Since $\langle T\rangle\subseteq X$, then $\langle T\rangle=X$. Therefore X is a subgroup of D_{2n}.

The following proves the rest of the lemma.

Case 1 There exists an a^ib edge in $G[X]$. For X as a subgroup of D_{2n}, then X is a group with order n' and $n'\,|\,2n$. Moreover, by Lemma 3, $X=a^ib\cdot X$, so $2\,|\,n'$. Let $d=2n/n'$, then $X=\{a^{id}:i=0,1,\cdots,n'/2-1\}\cup\{a^{id+l}b:i=0,1,\cdots,n'/2-1\}$(where a^lb is an arbitrary element in T). $E(G[X])=\{(x,a^{id}\cdot x):a^{id}\in T\}\cup\{(x,a^{jd+l}b\cdot x:a^{jd+l}b\in T)\}$. Let $G'=G(D_{n'},C')$ be a connected graph of order n' where $C'=\{a^{s/d}:a^s\in T\}\cup\{a^{\lfloor t/d\rfloor}b:a^tb\in T\}$. The bijection $f:V(G[X])\to V(G')$ defined by $f(a^{id})=a^i$, $f(a^{jd+l}b)=a^jb$ $(i=1,2,\cdots,n'/2-1)$ is an isomorphism between $G[X]$ and G'.

Case 2 There exists no a^ib edge in $G[X]$. For X as a subgroup of D_{2n}, then X is a group with order n' and $n'\,|\,2n$. Let $d=n/n'$, then $X=\{a^{id}: i=0,1,\cdots,n'-1\}$. $E(G[X])=\{(x,a^{id}\cdot x):a^{id}\in T\}$. Let G' be a circulant of order n' with $C'=\{l/d:a^l\in T\}$. Then, the bijection $f:V(G[X])\to V(G')$ defined by $f(a^{id})=i$ $(i=0,1,\cdots,n'-1)$ is an isomorphism between $G[X]$ and G'. The lemma is proved. □

Theorem 1 Let $G=G(D_{2n},C)$ be a connected graph with $C=\{a^{i_1},a^{i_2},\cdots,a^{i_{k_1}},a^{i_1}b,a^{i_2}b,\cdots,a^{i_{k_2}}b\}$ then

(1) G is optimal super-λ, namely, $\lambda'(G)=\xi(G)$.

(2) Every cutset of size i, $\delta\leqslant i\leqslant\xi-1$, isolates exactly one vertex and

$$N_i(G)=n\binom{(n-1)\delta}{i-\delta}.$$

Except for the four kinds of graphs given below:

(1) $k_1=0, k_2=2$;

(2) $C=\{a^{i_1},\cdots,a^{i_{k_1}},a^jb\}$, g.c.d.$(n,i_1,\cdots,i_{k_1})=1$ and $\delta>n/2$;

(3) $C=\{a^{n/2},a^{2i_1},a^{2i_2},\cdots,a^{2i_{k_1}},a^{j_1}b,a^{j_1+2j_2}b,\cdots,a^{j_1+2j_{k_2}}b\}$, $\delta-1>n/2$ and $n/2$ is an odd;

(4) $C=\{a^{n_1i_1},a^{n_1i_2},\cdots,a^{n_1i_{k_1}},a^{j_1}b,a^{j_1+n_1j_2}b,\cdots,a^{j_1+n_1j_{(k_2-1)}}b, a^jb\}$, such that g.c.d.$(j-j_1,n_1)=1$, $\delta-1>n/n_1$.

Proof Since $\xi=2\delta-2$, we have $\xi>\delta$ unless $\delta\leqslant 2$. And $\delta\leqslant 2$ means one of the following cases happens:

- $k_1=0$ and $k_2=2$, which corresponds to the first kind of graph given in the theorem.
- $a^{n/2}\in C, k_1=1$, and $k_2=1$, which results in G being disconnected which contradicts the hypothesis.

In other cases $\xi>\delta\geqslant\lambda$. So if we can prove that $\lambda'=\xi$, then $\lambda'>\lambda$. By the definition of λ' and super-λ graph, we know $\lambda'>\lambda$ means that the graph is super-λ. Also, we know $\lambda' \leqslant \xi$. Thus to prove the first item of the theorem, we need only to prove that all the graphs are $\lambda' \geqslant \xi$ except for the four kinds of graphs given in the theorem. Let $H=(X, V-X)$ be an RC of size λ', $|X| \leqslant n/2$, and X is minimal.

If m=2 then $G[X]$ is isomorphic to K_2, so $\lambda'=\xi$.

If $m=3$ then there are at most 3 edges in $G[X]$, thus $\lambda' \geqslant m\delta-6$ and

$$\lambda' \geqslant (m-2)\delta-4+(2\delta-2)+\xi \geqslant \delta-4+\xi.$$

So $\lambda' \geqslant \xi$ unless $\delta<4$. A few cases with $\delta<4$ are:

- $a^{n/2} \in C$: $k_1=1, k_2=1,2$.
- $a^{n/2} \notin C$: $k_1=0, k_2=1,2,3$; or $k_1=1, k_2=1$.

Therefore if $a^{n/2} \in C, k_1=1, k_2=1$ or $a^{n/2} \notin C, k_1=0$, $k_2=1$ then G is disconnected, and $k_1=0$ and $k_2=2$ is the first kind of graph.

Next consider the remaining three cases. If $a^{n/2} \in C, k_1=1$, and $k_2=2$. Since $m=3$, there is at most one $a^j b$-edge in $G[X]$, and $a^{n/2} \in C$, so there is at most one a^i-edge in $G[X]$, thus there are at most two edges in $G[X]$. So $\lambda' \geqslant m\delta-4=3\times\delta-4 \geqslant \delta-2+\xi$. Therefore, since $\lambda'>\xi$ contradicts the fact that $\lambda' \leqslant \xi$, these cases can not occur. If $a^{n/2} \notin C, k_1=0$, and $k_2=3$ then there are at most two $a^j b$-edges in $G[X]$, so similarly, $\lambda'>\xi$ and this also can not occur. If $a^{n/2} \notin C, k_1=1$, and $k_2=1$ and there are three edges in $G[X]$, this implies that $n=3$ which belongs to the second kind of graph given in the theorem. If $n>3$ there are two edges in $G[X]$, then $\lambda'>\xi$ and this can not occur. Thus if $m=3$ then $\lambda' \geqslant \xi$ except for the first two kinds of graphs given in the theorem.

If $m>3$, by Lemma 4, $G[X]$ is isomorphic to a circulant or $G(D_m, C')$. The following discusses the problems in these two cases:

Case 1 $G[X]$ is isomorphic to a circulant of order m with $\delta_X=s$ (the valency of $G[X]$), thus $s \leqslant m-1$. For convenience, let δ_a be the number of the a^i-edges of a vertex and let $\delta_b=\delta-\delta_a=k_2$ be the kinds of $a^j b$-edges in G.

Then

$$\begin{aligned}\lambda' &= m(\delta_a+\delta_b-s)-2(\delta_a+\delta_b)+2+\xi= \\ &(m-2)(\delta_a+\delta_b-s)-2s+2+\xi \geqslant \\ &(m-2)(\delta_a+\delta_b-s)-2(m-2)+\xi= \\ &(m-2)(\delta_a+\delta_b-s-2)+\xi.\end{aligned}$$

Therefore $\lambda' \geqslant \xi$ unless $\delta_a=s$ and $\delta_b=1$ in which case $\lambda'=m-2(s+1)+2-\xi$. If $\lambda'<\xi$ then $s>m/2$. The a-part is connected because $\delta_b=k_2=1$, so $m=n$ for $\delta_a=s$. This corresponds to the second kind of graph given in the theorem.

Case 2 $G[X]$ is isomorphic to a Cayley graph of a dihedral graph with $\delta_X=s+l$ (s is the number of a^i-edges of a vertex and l is the number of $a^j b$-edges of a vertex in $G[X]$), $s \leqslant m/2-1$ and $l \leqslant m/2$. Therefore,

$$\begin{aligned}\lambda' &= m(\delta_a+\delta_b-s-l)-2(\delta_a+\delta_b)+2+\xi= \\ &(m-2)(\delta_a+\delta_b-s-l)-2(s+l)+2+\xi \geqslant \\ &(m-2)(\delta_a+\delta_b-s-l-2)+\xi.\end{aligned}$$

If $\delta_a+\delta_b-s-l \geqslant 2$, then $\lambda' \geqslant \xi$. Thus $\lambda'<\xi$ if and only if $\delta_a=s+1$, $\delta_b=l$ and $s+l>m/2$ or $\delta_a=s$, $\delta_b=l+1$ and $s+l>m/2$.

For the first instance, since $\delta_a=s+1$, $a^{n/2} \in C$, and $G'=\mathrm{Cay}(D_{2n}, C \setminus a^{n/2})$ is disconnected. Suppose there are t $(t \mid n)$ connected components in each part of G' then g.c.d.$(t, n/2)=1$ and as a consequence $n/2$ is odd and $t=2$. So $m=n$. Thus $C=\{a^{n/2}, a^{2i_1}, a^{2i_2}, \cdots, a^{2i_{k_1}}, a^{j_1}b, a^{j_1+2j_2}b, \cdots, a^{j_1+2j_{k_2}}b\}$, $\delta-1>n/2$, and $n/2$ is odd. This is the third kind of graph given in the theorem.

Similarly, in the second instance suppose there is no $a^j b$- edge in $G[X]$ $(a^j b \in C)$. Let $G'=\mathrm{Cay}(D_{2n}, C \setminus a^j b)$ then G' is disconnected too and if there are t connected components in G', then $m=2n/t$ and $C=\{a^{n_1 i_1}, a^{n_1 i_2}, \cdots, a^{n_1 i_{k_1}}, a^{j_1}b, a^{j_1+n_1 j_2}b, \cdots, a^{j_1+n_1 j_{(k_2-1)}}b, a^j b\}$, such that g.c.d.$(j-j_1, n_1)=1$, $\delta-1>n/n_1$ which is the fourth kind of graph given in the theorem.

In conclusion, $\lambda' \geqslant \xi$ in all the graphs except for the four kinds mentioned in the theorem, so they are all optimal super-λ. As a consequence, for each $\delta \leqslant i<\xi$,

$$N_i(G)=n\binom{(n-1)\delta}{i-\delta}.$$

This ends the proof. □

References

[1] Bauer D, Boesch F, Suffel C, et al. Connectivity extremal problems in the analysis and design of probabilistic networks. In: Theory and Applications of Graphs. New York, USA: Wiley, 1981: 45-54.

[2] Esfahanian A. Generalized measure of fault tolerance with application to N-cubes. *IEEE Trans. Comput.*, 1989, **38**:

1586-1590.

[3] Esfahanian A, Hakimi S. On computing a conditional edge connectivity of a graph. *Info*. *Proecss*. *Lett*., 1988, **27**: 195-199.

[4] Li Qiaoliang, Li Qiao. Reliability analysis of circulant graphs. *Networks*, 1998, **31**: 61-65.

[5] Ball M. Complexity of network reliability computation. *Networks*, 1980, **10**: 153-165.

[6] Boesch F T, Wang J F. Super line connectivity of circulant graphs. *SIAM J*. *Alg*. *Dis*. *Math*, 1986, **1**: 89-98.

[7] Lovasz L. Combinational Problems and Exercises. North-Holland, Amsterdam, the Netherlands, 1979.