

Steady-State Performance of Kalman Filter for DPLL

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Abstract: For certain system models, the structure of the Kalman filter is equivalent to a second-order variable gain digital phase-locked loop (DPLL). To apply the knowledge of DPLLs to the design of Kalman filters, this paper studies the steady-state performance of Kalman filters for these system models. The results show that the steady-state Kalman gain has the same form as the DPLL gain. An approximate simple form for the steady-state Kalman gain is used to derive an expression for the equivalent loop bandwidth of the Kalman filter as a function of the process and observation noise variances. These results can be used to analyze the steady-state performance of a Kalman filter with DPLL theory or to design a Kalman filter model with the same steady-state performance as a given DPLL.

Key words: Kalman filter; digital phase-locked loop (DPLL); steady-state performance

Introduction

Digital phase-locked loops (DPLLs) are widely used for carrier or spread-code tracking, symbol synchronization, or timing recovery in various communication systems. Since these problems are essentially phase estimation problems, Kalman filter theory can also be used in these applications^[1-6]. For some system models, the Kalman filter has the same structure as second-order variable gain DPLLs^[1,2]. The use of Kalman filters instead of DPLLs makes it possible to simultaneously achieve fast acquisition time, wide acquisition range, and low tracking jitter, without the tradeoffs associated with conventional fixed-gain DPLLs^[1,2,6].

For precise system models, the Kalman filter is a linear minimum mean square error (LMMSE) estimation, which gives both the estimate and the estimate error. However, in practice, the precise system model is impossible to determine. Thus, the Kalman filter performance is difficult to evaluate. In contrast, DPLLs are widely used and the appropriate loop bandwidth

can be easily chosen for various schemes. Since the structures of Kalman filters and second-order DPLLs are the same, the steady-state Kalman gain should be equivalent to the second-order DPLL gain. To apply the knowledge of DPLLs to the design of Kalman filters, the relationship between the Kalman filter model and DPLL loop bandwidth should be studied. Pataoutian gave an exact solution for the steady-state Kalman gains with respect to the noise variances and initial parameters^[5]. However, the form is too complex to illustrate the relationships. Therefore, a simple approximate solution is needed to clarify the relationships.

This paper shows that the steady-state Kalman gain has the same form as the DPLL gain. This paper also presents an approximate simple solution of the steady-state Kalman gain with an expression for the equivalent loop bandwidth of the Kalman filter as a function of the process and observation noise variances. These results can be used to analyze the steady-state performance of Kalman filters with DPLL theory. The result can also be used in reverse to determine the noise variances in the Kalman filter model for a given equivalent loop bandwidth to achieve rapid acquisition without loss of tracking reliability.

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1 Kalman Filter Model for DPLL

Denote ϕ_n and f_n as the phase and Doppler shift of the signal to track at sample n , respectively, and T as the sample period. Define the state vector as $\mathbf{s}_n = (\phi_n \ Tf_n)^T$ and x_n as the observed signal phase.

The dynamic system can then be modeled as

$$\mathbf{s}_n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{s}_{n-1} + \begin{pmatrix} 0 \\ u_n \end{pmatrix} = \mathbf{A}\mathbf{s}_{n-1} + \mathbf{U}_n \quad (1)$$

$$x_n = (1 \ 0)\mathbf{s}_n + \omega_n = \mathbf{h}\mathbf{s}_n + \omega_n \quad (2)$$

where $u_n \sim N(0, \sigma_Q^2)$ and $\omega_n \sim N(0, \sigma_n^2)$ represent the process and observation noise^[1].

Define $\mathbf{G}_n = (G_{0,n} \ G_{1,n})^T$ as the Kalman gain, \mathbf{K} as the error variance matrix, $\hat{\mathbf{s}}_n = (\hat{\phi}_n \ T\hat{f}_n)^T$ as the n -th estimate, and $\hat{\mathbf{s}}_{n|n-1}$ as the n -th prediction from sample $n-1$. Note that $\varepsilon_n = x_n - \mathbf{h}\hat{\mathbf{s}}_{n|n-1}$. The Kalman equations with the initial conditions $\hat{\mathbf{s}}_{-1}$ and $\mathbf{K}_{-1|1}$ ^[7] are as follows:

$$\mathbf{K}_{n|n-1} = \mathbf{A}\mathbf{K}_{n-1|n-1}\mathbf{A}^T + \mathbf{Q} \quad (3)$$

$$\mathbf{G}_n = \frac{\mathbf{K}_{n|n-1}\mathbf{h}^T}{\mathbf{h}\mathbf{K}_{n|n-1}\mathbf{h}^T + \sigma_n^2} \quad (4)$$

$$\mathbf{K}_{n|n} = \mathbf{K}_{n|n-1} - \mathbf{G}_n\mathbf{h}\mathbf{K}_{n|n-1} \quad (5)$$

$$\hat{\mathbf{s}}_{n|n-1} = \mathbf{A}\hat{\mathbf{s}}_{n-1} \quad (6)$$

$$\hat{\mathbf{s}}_n = \hat{\mathbf{s}}_{n|n-1} + \mathbf{G}_n\varepsilon_n \quad (7)$$

where $\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_Q^2 \end{pmatrix}$.

Equation (7) can then give the following expression for the signal phase prediction:

$$\hat{\phi}_{n+1|n} = \hat{\phi}_{n|n-1} + T\hat{f}_{n|n-1} + G_{0,n}\varepsilon_n + G_{1,n}\varepsilon_n \quad (8)$$

The block diagram representation of Eq. (8) shown in Fig. 1 is equivalent to a second-order DPLL with a proportional integral, except for the time-varying Kalman gains instead of the fixed gains in the DPLL^[1,2].

2 Steady-State Equivalent Loop Bandwidth of Kalman Filter

This section proves the equivalence between the converged Kalman gain and the DPLL gain and then develops a relationship between the noise variances for the Kalman filter and the DPLL loop bandwidth.

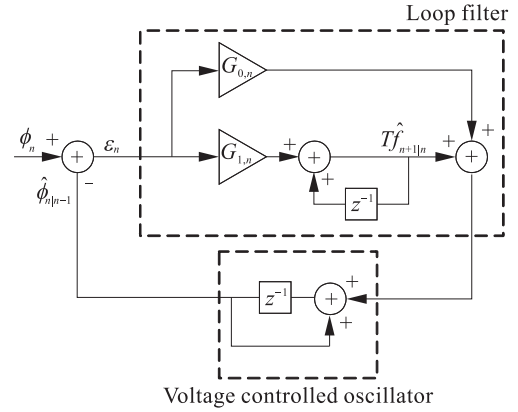


Fig. 1 Kalman filter block diagram

When $n \rightarrow \infty$, the Kalman filter converges to the steady state, which means that $\mathbf{K}_{n+1|n} = \mathbf{K}_{n|n-1} = \mathbf{K} = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}$, $\mathbf{G}_{n+1} = \mathbf{G}_n = \mathbf{G}$. These equations can be substituted into Eqs. (3)-(5) to get the followings (as shown in the Appendix):

$$K_{00}^4 = \sigma_Q^2(K_{00} + \sigma_n^2)(K_{00} + 2\sigma_n^2)^2 \quad (9)$$

$$\mathbf{G} = \begin{pmatrix} \frac{K_{00}}{K_{00} + \sigma_n^2} & \frac{\sigma_Q}{\sqrt{K_{00} + \sigma_n^2}} \end{pmatrix}^T \quad (10)$$

Defining $\omega = \frac{\sqrt{2}K_{00}}{T(K_{00} + 2\sigma_n^2)}$ as the equivalent natural frequency, the steady-state Kalman gain has the form,

$$\mathbf{G} = \begin{pmatrix} \frac{8\frac{1}{\sqrt{2}}\omega T}{4 + 4\frac{1}{\sqrt{2}}\omega T} & \frac{4\omega^2 T^2}{4 + 4\frac{1}{\sqrt{2}}\omega T} \end{pmatrix}^T \quad (11)$$

According to Ref. [8], the gain of second-order DPLL is

$$(C_1 \ C_2) = \frac{1}{k_0 k_1} \begin{pmatrix} \frac{8\zeta\omega T}{4 + 4\zeta\omega T + (\omega T)^2} & \frac{4(\omega T)^2}{4 + 4\zeta\omega T + (\omega T)^2} \end{pmatrix} \quad (12)$$

where ζ denotes the damping ratio, and k_0 and k_1 denote the gains of the discriminator and the numerical controlled oscillator (NCO). Comparison of Eqs. (11) and (12) shows that the steady state Kalman gain is equivalent to the DPLL gain with $\zeta = 1/\sqrt{2}$ and $k_0 k_1 = 1$, except for the $(\omega T)^2$ term in the denominator. In the DPLL, the sampling time and the natural frequency should fulfill $\omega T \ll 1$ to function properly^[8,9]. Thus, $(\omega T)^2$ is a very small quantity, which

can be neglected.

The relationship between σ_Q, σ_n , and ω is based on a closed-form expression for the steady state Kalman gain. The exact solution is given in Ref. [5], which is

$$\begin{cases} \mathbf{G} = (1 - z_0 z_1 \quad (1 - z_0)(1 - z_1))^T, \\ z_i = (\gamma_i - \sqrt{\gamma_i^2 - 4}) / 2, \\ \gamma_i = 2 + (-1)^i \sqrt{-\frac{\sigma_Q^2}{\sigma_n^2}}, \quad i = 0, 1 \end{cases} \quad (13)$$

However, this form is too complex to obtain the expression for ω , so an approximate solution is used.

Equation (9) is solved to get the expression for K_{00} .

The right side of Eq. (9) can be expanded to give

$$\left(\frac{K_{00}}{\sigma_n^2}\right)^4 = \left(\frac{\sigma_Q}{\sigma_n}\right)^2 \left[\left(\frac{K_{00}}{\sigma_n^2}\right)^3 + 5\left(\frac{K_{00}}{\sigma_n^2}\right)^2 + 8\frac{K_{00}}{\sigma_n^2} + 4 \right] \quad (14)$$

With the assumption that $K_{00} \ll \sigma_n^2$, the first-order to third-order terms for K_{00} in the right side of Eq. (14) can be neglected. Then, the approximate solution for K_{00} is

$$K_{00} \approx \sigma_n^2 \sqrt{2\sigma_Q/\sigma_n} \quad (15)$$

Substituting Eq. (15) into Eqs. (9) and (10) gives the approximate solutions for the steady-state Kalman gain and the equivalent natural frequency,

$$\mathbf{G} \approx \left(\frac{\sqrt{2\sigma_Q/\sigma_n}}{1 + \sqrt{2\sigma_Q/\sigma_n}} \quad \frac{\sigma_Q/\sigma_n}{\sqrt{1 + \sqrt{2\sigma_Q/\sigma_n}}} \right)^T \quad (16)$$

$$\omega \approx \frac{2\sqrt{\sigma_Q/\sigma_n}}{T(2 + \sqrt{2\sigma_Q/\sigma_n})} \quad (17)$$

For a second-order DPLL with $\zeta = 1/\sqrt{2}$, the loop bandwidth $B_L = \frac{3\sqrt{2}}{8}\omega$. Thus, the relationship between σ_Q, σ_n , and the equivalent loop bandwidth can be obtained from Eq. (17) as

$$B_L = \frac{3\sqrt{2\sigma_Q/\sigma_n}}{4T(2 + \sqrt{2\sigma_Q/\sigma_n})} \quad (18)$$

or

$$\frac{\sigma_Q}{\sigma_n} = \left(\frac{4\sqrt{2TB_L}}{3 - 4TB_L} \right)^2 \quad (19)$$

This derivation assumed that $K_{00} \ll \sigma_n^2$. From

Eqs. (15) and (17), this is equivalent to $\omega T \ll 1$, which is a reasonable assumption for the DPLL. To illustrate the result, Fig. 2 shows the exact and approximate solutions for \mathbf{G} and B_L when $T = 0.001$. The approximate solution is accurate when the equivalent loop bandwidth is small. Equation (17) also shows that the steady-state performance of the Kalman filter is only associated with the ratio of σ_Q to σ_n .

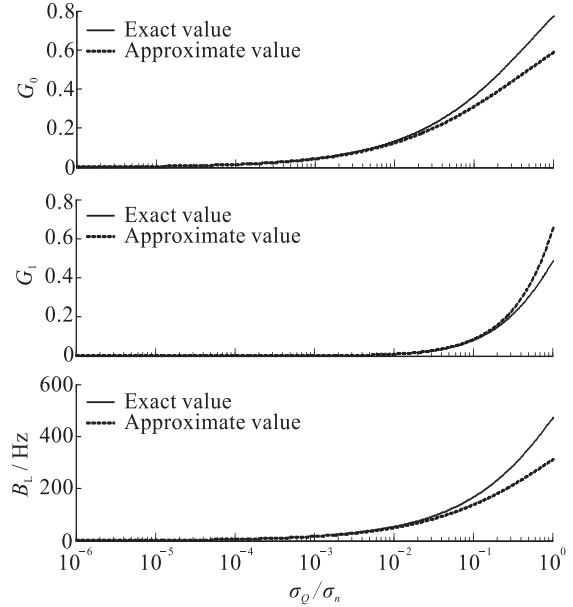


Fig. 2 Exact and approximate solutions for the steady-state Kalman gain and equivalent loop bandwidth for $T = 0.001$

3 Example Application

Consider the problem of the pseudo-noise (PN) code tracking in global positioning system (GPS) receivers. Most GPS receivers employ a delay-locked loop (DLL), which is a DPLL with an early-late discriminator to synchronize the PN code in the received signal and the local generated code. A frequency of 1 Hz is often considered appropriate for the DLL loop bandwidth in GPS receivers^[10]. With Eq. (19), we can find that a Kalman filter can have the same steady-state behavior as a 1-Hz bandwidth DLL as long as the noise variances of the Kalman filter model fulfills $\sigma_Q/\sigma_n = 3.6 \times 10^{-6}$. For example, the noise variances can be set to $\sigma_n = 1$ and $\sigma_Q = 3.6 \times 10^{-6}$. The performance of the Kalman filter is compared to that of a 1-Hz bandwidth DLL tracking the same signal in Fig. 3, which plots the tracking error over time. The Kalman

filter has the same tracking jitter as the DLL at steady state, with the Kalman filter tracking error converging to zero much faster than the DLL tracking error.

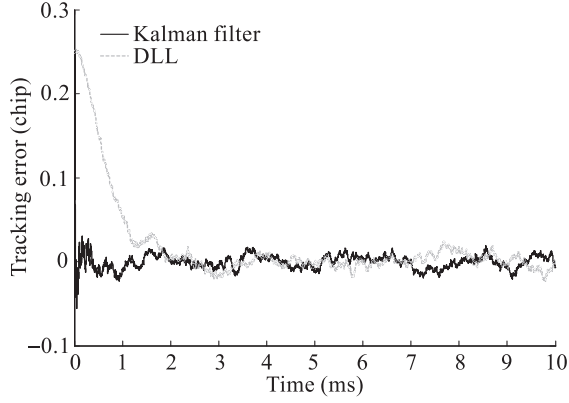


Fig. 3 Tracking errors for Kalman filter and 1 Hz bandwidth DLL. SNR = 14 dB, $\phi_0 = 0.25$ chips, and $T = 0.001$. For the Kalman filter, $\hat{s}_{-1} = [0 \ 0]$ and

$$K_{-1|1} = \begin{bmatrix} 0.25^2/3 & 0 \\ 0 & 10^2 T^2/3 \end{bmatrix}.$$

4 Conclusions

The steady-state performance of a Kalman filter is related to that of a second-order DPLL. The steady-state Kalman gain is proved to have the same form as the DPLL gain. An approximate simple form of the steady-state Kalman gain is used to obtain a closed-form expression for the equivalent loop bandwidth. Simulations show that the approximate solution is accurate when the equivalent loop bandwidth is small. These results can be used to analyze the steady-state performance of a Kalman filter with DPLL theory or to design a Kalman filter with the same steady-state performance as a DPLL.

Appendix

Denote $K_{n|n} = K_{n-1|n-1} = P = \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix}$. Substituting this equation along with the steady-state equations for K and G into Eqs. (3)-(5) gives

$$K = \begin{pmatrix} P_{00} + P_{10} + P_{01} + P_{11} & P_{01} + P_{11} \\ P_{10} + P_{11} & P_{11} + \sigma_Q^2 \end{pmatrix} \quad (20)$$

$$G = \begin{pmatrix} \frac{K_{00}}{K_{00} + \sigma_n^2} & \frac{K_{10}}{K_{00} + \sigma_n^2} \end{pmatrix}^T \quad (21)$$

$$P = \begin{pmatrix} (1 - G_0)K_{00} & (1 - G_0)K_{01} \\ K_{10} - G_1K_{00} & K_{11} - G_1K_{01} \end{pmatrix} \quad (22)$$

Substituting Eqs. (21) and (22) into Eq. (20) and expanding the matrix yield

$$K_{00} = K_{00} + K_{01} + K_{10} + K_{11} - (G_0 + G_1)(K_{00} + K_{01}) \quad (23)$$

$$K_{01} = K_{01} + K_{11} - G_0K_{01} - G_1K_{01} \quad (24)$$

$$K_{10} = K_{10} + K_{11} - G_1K_{00} - G_1K_{01} \quad (25)$$

$$K_{11} = K_{11} - G_1K_{01} + \sigma_Q^2 \quad (26)$$

Since $G_0K_{10} = G_1K_{00}$, Eqs. (24)-(26) can be simplified to

$$K_{01} = K_{10} = \sigma_Q \sqrt{K_{00} + \sigma_n^2} \quad (27)$$

$$K_{11} = \sigma_Q^2 + \frac{\sigma_Q K_{00}}{\sqrt{K_{00} + \sigma_n^2}} \quad (28)$$

Substituting Eqs. (27) and (28) into Eqs. (23) and (21) gives Eqs. (9) and (10).

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