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Fault-Tolerant Control for a Class of Uncertain Systems with Actuator Faults

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Abstract: The problem of fault-tolerant controller design for a class of polytopic uncertain systems with actuator faults is studied in this paper. The actuator faults are presented as a more general and practical continuous fault model. Based on the affine quadratic stability (AQS), the stability of the polytopic uncertain system is replaced by the stability at all corners of the polytope. For a wide range of problems including H_∞ and mixed H_2/H_∞ controller design, sufficient conditions are derived to guarantee the robust stability and performance of the closed-loop system in both normal and fault cases. In the framework of the linear matrix inequality (LMI) method, an iterative algorithm is developed to reduce conservativeness of the design procedure. The effectiveness of the proposed design is shown through a flight control example.

Key words: fault-tolerant control (FTC); affine quadratic stability (AQS); continuous actuator fault; multi-objective synthesis; linear matrix inequality (LMI)

Introduction

With the growing complexity of modern control systems, research on fault-tolerant control (FTC) has received great attention over the past several years. FTC is a control technique that provides the ability to maintain overall system stability and acceptable performance in the event of component failures[1]. FTC methods can be broadly classified into two types: passive and active. The active FTC (AFTC), including fault detection and diagnosis (FDD) and control reconfiguration, is generally complicated for safety-critical systems. Passive FTC (PFTC) exploits the inherent redundancy of the controlled system. Consequently, PFTC without on-line FDD and control reconfiguration implements easily as compared with AFTC. Therefore, PFTC becomes a popular method for accommodating the component failures, and some works[2-5] have been carried out in recent years. However, the component failures considered in the above literature are all described as a discrete fault model, which is the simplest case of component failures. Based on a more practical continuous fault model, which consists of a scaling factor with upper and lower bounds to the signal to be measured or to the control action, Yang et al.[6] designed reliable H_∞ controllers for sensor and actuator faults, respectively. Subsequently, problems of reliable tracking controller design against actuator faults are studied by using the continuous fault model in Refs. [7,8]. However, polytopic uncertainties and multi-objective synthesis have not been considered simultaneously in these papers. Moreover, it yields a conservative result by employing a fixed quadratic Lyapunov function to deal with all cases.

This paper studies the fault-tolerant controller

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design for a class of uncertain systems with actuator faults. Furthermore, a more general and practical continuous fault model is considered for actuator failures. Sufficient conditions for the existence of fault-tolerant controllers are derived based on the concept of affine quadratic stability (AQS), which guarantees the robust stability and system performance in both normal and fault cases. Finally, an iterative algorithm separating Lyapunov function variables from controller gain is developed via additional variables to obtain the controller with less conservativeness. Simulations show that our methods give better performance than the standard design method.

1 Problem Statement

Consider a polytopic uncertain system described by

$$\begin{cases}\dot{\boldsymbol{x}}(t)=\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t)+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{u}(t)+\boldsymbol{G}(\boldsymbol{\theta})\boldsymbol{w}(t),\\ \boldsymbol{y}(t)=\boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{x}(t)\end{cases} \tag{1}$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$ is the state, $\boldsymbol{u}(t)\in\mathbb{R}^m$ is the control input, $\boldsymbol{w}(t)\in\mathbb{R}^h$ is the disturbance input, and $\boldsymbol{y}(t)\in\mathbb{R}^p$ is the measured output. We adopt the following polytopic uncertainties in the system matrices $\boldsymbol{A}(\boldsymbol{\theta})$, $\boldsymbol{B}(\boldsymbol{\theta})$, $\boldsymbol{C}(\boldsymbol{\theta})$, and $\boldsymbol{G}(\boldsymbol{\theta})$:

$$\Omega\triangleq\{[\boldsymbol{A}(\boldsymbol{\theta}),\boldsymbol{B}(\boldsymbol{\theta}),\boldsymbol{C}(\boldsymbol{\theta}),\boldsymbol{G}(\boldsymbol{\theta})]\,|\,[\boldsymbol{A}(\boldsymbol{\theta}),\boldsymbol{B}(\boldsymbol{\theta}),\boldsymbol{C}(\boldsymbol{\theta}),\boldsymbol{G}(\boldsymbol{\theta})]=\sum_{i=1}^{N}\theta_i(\boldsymbol{A}_i,\boldsymbol{B}_i,\boldsymbol{C}_i,\boldsymbol{G}_i);\theta_i\geqslant 0,\ \sum_{i=1}^{N}\theta_i=1;i=1,\ldots,N\} \tag{2}$$

The matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $\boldsymbol{C}_i$, and $\boldsymbol{G}_i$ are known constant matrices with appropriate dimensions, which correspond to different vertices of the polytope. $\boldsymbol{\theta}=[\theta_1,\theta_2,\ldots,\theta_N]^{\mathrm{T}}$ is a uncertain constant parameter vector.

Actuator fault is the most frequent failure in control systems. In this paper, we study the problem of fault-tolerant controller design for polytopic uncertain systems with actuator faults. Moreover, a continuous fault model of the actuator is adopted here.

Set $\boldsymbol{u}(t)$ and $\boldsymbol{u}^{\mathrm{F}}(t)$ present actuator outputs in normal case and fault case, respectively. Then,

$$\boldsymbol{u}^{\mathrm{F}}(t)=\boldsymbol{F}\boldsymbol{u}(t) \tag{3}$$

where $\boldsymbol{F}$ is the matrix of actuator effectiveness factors and satisfies

$$\boldsymbol{F}\in\Theta\triangleq\{\boldsymbol{F}=\mathrm{diag}[f_1,f_2,\ldots,f_m],\ f_j\in[f_{lj},f_{uj}],f_{uj}\geqslant 1,j=1,2,\ldots,m\} \tag{4}$$

By introducing the following matrices:

$$\boldsymbol{F}_0=\mathrm{diag}[f_{01},f_{02},\ldots,f_{0m}],\ \boldsymbol{W}=\mathrm{diag}[w_1,w_2,\ldots,w_m],\ \boldsymbol{L}=\mathrm{diag}[l_1,l_2,\ldots,l_m],\ |\boldsymbol{L}|=\mathrm{diag}[|l_1|,|l_2|,\ldots,|l_m|] \tag{5}$$

where

$$f_{0j}=\frac{1}{2}(f_{lj}+f_{uj}),\ w_j=\frac{f_{uj}-f_{lj}}{f_{lj}+f_{uj}},$$

$$l_j=\frac{f_j-f_{0j}}{f_{0j}},\ j=1,2,\ldots,m.$$

Then, we get the continuous fault model as follows:

$$\boldsymbol{F}=\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L}),\ |\boldsymbol{L}|\leqslant\boldsymbol{W}\leqslant\boldsymbol{I} \tag{6}$$

Remark 1 $f_j=0$ means total outage of the j-th actuator channel and $f_j=1$ means a healthy actuator channel. Partial loss of the j-th actuator channel is given by $0\leqslant f_{lj}<f_j<f_{uj}$. It is worth mentioning that the above continuous fault model includes the discrete fault model.

Hence, the system (1) with actuator faults is given by

$$\begin{cases}\dot{\boldsymbol{x}}(t)=\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t)+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{u}(t)+\boldsymbol{G}(\boldsymbol{\theta})\boldsymbol{w}(t),\\ \boldsymbol{y}(t)=\boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{x}(t)\end{cases} \tag{7}$$

2 Fault-Tolerant Controller Design

In this section, the problem under consideration is to design a state feedback fault-tolerant controller of the following form:

$$\boldsymbol{u}(t)=\boldsymbol{K}\boldsymbol{x}(t) \tag{8}$$

The closed-loop system with actuator faults is represented as the following by substituting Eq. (8) into Eq. (7):

$$\begin{cases}\dot{\boldsymbol{x}}(t)=[\boldsymbol{A}(\boldsymbol{\theta})+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]\boldsymbol{x}(t)+\boldsymbol{G}(\boldsymbol{\theta})\boldsymbol{v}(t),\\ \boldsymbol{y}(t)=\boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{x}(t)\end{cases} \tag{9}$$

Then, we get a new expression of the closed-loop system by introducing a controlled output $z(t)$ as measurement,

$$\begin{cases}\dot{\boldsymbol{x}}(t)=[\boldsymbol{A}(\boldsymbol{\theta})+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]\boldsymbol{x}(t)+\boldsymbol{G}(\boldsymbol{\theta})\boldsymbol{v}(t),\\ \boldsymbol{y}(t)=\boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{x}(t),\\ \boldsymbol{z}(t)=[\boldsymbol{C}_z+\boldsymbol{D}_z\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]\boldsymbol{x}(t)\end{cases} \tag{10}$$

Before giving our main results, we first present the following definition and lemmas which play important roles in demonstrating the results.

Definition 1 AQS[9] Consider the polytopic uncertain system (1). We call this system AQS if there exists an affine quadratic Lyapunov function,

$$\boldsymbol{V}(\boldsymbol{x},\boldsymbol{\theta})=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\theta})\boldsymbol{x}>0,\ \dot{\boldsymbol{V}}(\boldsymbol{x},\boldsymbol{\theta})=\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}<0 \tag{11}$$

where affine quadratic Lyapunov variable $\boldsymbol{P}(\boldsymbol{\theta})=\sum_{i=1}^{N}\theta_i\boldsymbol{P}_i$, $\sum_{i=1}^{N}\theta_i=1$.

Lemma 1[9] Consider a scalar quadratic function of uncertain parameter $\boldsymbol{\theta}$,

$$f(\theta_1,\ldots,\theta_K)=\alpha_0+\sum_i \alpha_i\theta_i+\sum_{i<j}\beta_{ij}\theta_i\theta_j+\sum_i \delta_i\theta_i^2 \quad (12)$$

And assume that $f(\cdot)$ is multi-convex, that is,

$$2\delta_i=\frac{\partial^2 \boldsymbol{f}}{\partial\theta_i^2}(\boldsymbol{\theta})\geqslant 0,\ \text{for } i=1,\ldots,N \quad (13)$$

Then $f(\cdot)$ is negative in the hyper-rectangle $\theta_i\in[\underline{\theta}_i,\ \overline{\theta}_i]$ if and only if it takes negative values at the corners of this hyper-rectangle.

Lemma 2[7] Set $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ as known real constant matrices with appropriate dimensions, $\boldsymbol{U}$ as a positive definite diagonal matrix, and $\boldsymbol{\Sigma}$ as a time-varying matrix satisfying $|\boldsymbol{\Sigma}|\leqslant \boldsymbol{U}$. Then

$$\boldsymbol{R}_1\boldsymbol{\Sigma}\boldsymbol{R}_2+\boldsymbol{R}_2^{\mathrm T}\boldsymbol{\Sigma}^{\mathrm T}\boldsymbol{R}_1^{\mathrm T}\leqslant \alpha\boldsymbol{R}_1\boldsymbol{U}\boldsymbol{R}_1^{\mathrm T}+\alpha^{-1}\boldsymbol{R}_2^{\mathrm T}\boldsymbol{U}\boldsymbol{R}_2 \quad (14)$$

where $\alpha>0$, $\boldsymbol{\Sigma}=\mathrm{diag}[\sigma_1,\sigma_2,\ldots,\sigma_q]$.

2.1 H_∞ fault-tolerant controller synthesis

This subsection concentrates on the robust stability and H_∞ performance of the closed-loop system (10) and explicates the advantages of the proposed AQS.

Theorem 1 The polytopic uncertain system (10) has affine quadratic H_∞ performance (AQP) γ if there exist symmetric matrices $\boldsymbol{P}_i$ such that

$$\boldsymbol{P}(\boldsymbol{\theta})=\sum_{i=1}^{N}\theta_i\boldsymbol{P}_i,\ \sum_{i=1}^{N}\theta_i=1,\ i=1,\ldots,N \quad (15)$$

$$\begin{bmatrix}[\boldsymbol{A}(\boldsymbol{\theta})+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]^{\mathrm T}\boldsymbol{P}(\boldsymbol{\theta})+\boldsymbol{P}(\boldsymbol{\theta})[\boldsymbol{A}(\boldsymbol{\theta})+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}] & * & * \\ \boldsymbol{G}^{\mathrm T}(\boldsymbol{\theta})\boldsymbol{P}(\boldsymbol{\theta}) & -\gamma\boldsymbol{I} & * \\ \boldsymbol{C}_z+\boldsymbol{D}_z\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K} & 0 & -\gamma\boldsymbol{I}\end{bmatrix}<0 \quad (16)$$

hold for all uncertain parameters θ_i, where '*' denotes entries that can be deduced from the symmetry of the matrix.

Theorem 1 can be readily obtained based on Definition 4.1 in Ref. [9] and is omitted here for brevity.

Although this result shows that AQP can be dependent on the Lyapunov variable $\boldsymbol{P}(\boldsymbol{\theta})$, it does not yield an implementable controller design method for all faults. To overcome this problem, the following theorem is derived by invoking the proposed preliminaries.

Theorem 2 Consider the closed-loop augmented system (10). For given positive scalars γ_i and μ_i, if there exist symmetric positive definite matrices $\boldsymbol{P}_j$ satisfying

$$\begin{bmatrix}(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})^{\mathrm T}\boldsymbol{P}_i+\boldsymbol{P}_i(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K}) & * & * & * & * \\ \boldsymbol{G}_i^{\mathrm T}\boldsymbol{P}_i & -\gamma_i\boldsymbol{I} & * & * & * \\ \boldsymbol{C}_z+\boldsymbol{D}_z\boldsymbol{F}_0\boldsymbol{K} & 0 & -\gamma_i\boldsymbol{I}+\alpha_2\boldsymbol{D}_z\boldsymbol{F}_0\boldsymbol{W}\boldsymbol{F}_0\boldsymbol{D}_z^{\mathrm T} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm T}\boldsymbol{P}_i & 0 & 0 & -\alpha_1^{-1}\boldsymbol{I} & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & 0 & -(\alpha_1^{-1}+\alpha_2^{-1})^{-1}\boldsymbol{I}\end{bmatrix}<0 \quad (17)$$

$$\begin{bmatrix}(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})^{\mathrm T}\boldsymbol{P}_i+\boldsymbol{P}_i(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})+\mu_i\boldsymbol{I} & * & * & * \\ \boldsymbol{G}_i^{\mathrm T}\boldsymbol{P}_i & \mu_i\boldsymbol{I} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm T}\boldsymbol{P}_i & 0 & -\alpha_3^{-1}\boldsymbol{I} & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & -\alpha_3\boldsymbol{I}\end{bmatrix}>0 \quad (18)$$

the controller stabilizes the closed-loop system and the upper bound of the H_∞ performance index is γ_i.

Proof Applying the Schur complement lemma[10] to the above inequality (17), we have

$$\begin{bmatrix}(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})^{\mathrm T}\boldsymbol{P}_i+\boldsymbol{P}_i(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K}) & * & * \\ \boldsymbol{G}_i^{\mathrm T}\boldsymbol{P}_i & -\gamma_i\boldsymbol{I} & * \\ \boldsymbol{C}_z+\boldsymbol{D}_z\boldsymbol{F}_0\boldsymbol{K} & 0 & -\gamma_i\boldsymbol{I}\end{bmatrix}+\begin{bmatrix}\alpha_1\boldsymbol{P}_i\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{W}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm T}\boldsymbol{P}_i+\alpha_1^{-1}\boldsymbol{K}^{\mathrm T}\boldsymbol{W}\boldsymbol{K} & * & * \\ 0 & 0 & * \\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}\alpha_2^{-1}\boldsymbol{K}^{\mathrm T}\boldsymbol{W}\boldsymbol{K} & * & * \\ 0 & 0 & * \\ 0 & 0 & \alpha_2\boldsymbol{D}_z\boldsymbol{F}_0\boldsymbol{W}\boldsymbol{F}_0\boldsymbol{D}_z^{\mathrm T}\end{bmatrix}<0 \quad (19)$$

It is easy to see that inequality (19) is equivalent to the following expression by using Lemma 2:

$$\begin{bmatrix} [\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]^{\mathrm{T}}\boldsymbol{P}_i+ \boldsymbol{P}_i[\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}] & * & * \\ \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_i & -\gamma_i\boldsymbol{I} & * \\ \boldsymbol{C}_z+\boldsymbol{D}_z\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K} & 0 & -\gamma_i\boldsymbol{I} \end{bmatrix} < 0 \quad (20)$$

Based on the bounded real lemma[11], expression (20) guarantees stability and H_∞ performance of the closed-loop system at the corners of the polytope (2). Adopting the concept of Lemma 1, we can guarantee the robust stability and H_∞ performance of the system if and only if the AQP formulation described in Theorem 1 is multi-convex. Therefore, assume the AQP formulation

$$\boldsymbol{f}(\boldsymbol{\theta})=\begin{bmatrix} [\boldsymbol{A}(\boldsymbol{\theta})+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\theta})+ \boldsymbol{P}(\boldsymbol{\theta})[\boldsymbol{A}(\boldsymbol{\theta})+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}] & * & * \\ \boldsymbol{G}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{P}(\boldsymbol{\theta}) & -\gamma\boldsymbol{I} & * \\ \boldsymbol{C}_z+\boldsymbol{D}_z\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K} & 0 & -\gamma\boldsymbol{I} \end{bmatrix} \quad (21)$$

is multi-convex, namely,

$$\frac{\partial^2 \boldsymbol{f}(\boldsymbol{\theta})}{\partial\theta_i^2}=\begin{bmatrix} [\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]^{\mathrm{T}}\boldsymbol{P}_i+ \boldsymbol{P}_i[\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}] & \boldsymbol{P}_i\boldsymbol{G}_i \\ \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_i & 0 \end{bmatrix} \geqslant 0 \quad (22)$$

The inequality (22) is hard to solve via LMI since it is not a strict positive condition. A simple remedy consists of replacing inequality (22) by

$$\begin{bmatrix} [\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]^{\mathrm{T}}\boldsymbol{P}_i+ \boldsymbol{P}_i[\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}] & \boldsymbol{P}_i\boldsymbol{G}_i \\ \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_i & 0 \end{bmatrix} + \mu_i\begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} > 0 \quad (23)$$

Resembling the equivalence of formulas (17) and (20), inequality (18) is equivalent to inequality (23). This completes the proof. □

Remark 2 Theorem 2 transforms the construction of the parameter-dependent Lyapunov function into an LMI problem. Moreover, the stability of the polytopic uncertain system with continuous faults can be reduced to the stability of the systems corresponding to the upper bound of the continuous fault. However, it is noticed that LMIs in Theorem 2 are numerically in tractable due to the coupling between Lyapunov function variables and controller gain. Hence, some important additional variables are introduced into the following theorem to avoid the coupling.

Theorem 3 Consider the closed-loop augmented system (10). For given positive scalars γ_i, if there exist symmetric positive definite matrices P_j satisfying

$$\begin{bmatrix} \boldsymbol{\Phi}(\boldsymbol{K},\boldsymbol{P}_i) & * & * & * & * & * \\ \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_i & -\gamma_i\boldsymbol{I} & * & * & * & * \\ \boldsymbol{C}_z+\boldsymbol{D}_z\boldsymbol{F}_0\boldsymbol{K} & 0 & -\gamma_i\boldsymbol{I}+\alpha_1\boldsymbol{D}_z\boldsymbol{F}_0W\boldsymbol{F}_0\boldsymbol{D}_z^{\mathrm{T}} & * & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_i & 0 & 0 & -\alpha_1^{-1}\boldsymbol{I} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & 0 & -(\alpha_1^{-1}+\alpha_2^{-1})^{-1}\boldsymbol{I} & * \\ \boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{F}_0\boldsymbol{K} & 0 & 0 & 0 & 0 & -\boldsymbol{I} \end{bmatrix} < 0 \quad (24)$$

$$\begin{bmatrix} \boldsymbol{\Phi}(\boldsymbol{K},\boldsymbol{P}_i)+\mu_i\boldsymbol{I} & * & * & * \\ \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_i & \mu_i\boldsymbol{I} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_i & 0 & -\alpha_3^{-1}\boldsymbol{I} & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & -\alpha_3\boldsymbol{I} \end{bmatrix} > 0 \quad (25)$$

where

$$\boldsymbol{\Phi}(\boldsymbol{K},\boldsymbol{P}_i)=\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}_i-\boldsymbol{P}_i\boldsymbol{B}_i\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{i0}-\boldsymbol{P}_{i0}\boldsymbol{B}_i\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_i+ \boldsymbol{P}_{i0}\boldsymbol{B}_i\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{i0}-\boldsymbol{K}^{\mathrm{T}}\boldsymbol{F}_0\boldsymbol{F}_0\boldsymbol{K}_0-\boldsymbol{K}_0^{\mathrm{T}}\boldsymbol{F}_0\boldsymbol{F}\boldsymbol{K}_0+\boldsymbol{K}_0^{\mathrm{T}}\boldsymbol{F}_0\boldsymbol{F}_0\boldsymbol{K}_0.$$

The controller stabilizes the closed-loop system and the upper bound of H_∞ performance index is γ_i.

Proof LMI expression (24) can be represented as follows using the Schur complement lemma[11],

$$\begin{bmatrix} (\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K}) & * & * & * & * \\ \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_i & -\gamma_i\boldsymbol{I} & * & * & * \\ \boldsymbol{C}_z+\boldsymbol{D}_z\boldsymbol{F}_0\boldsymbol{K} & 0 & -\gamma_i\boldsymbol{I}+\alpha_2\boldsymbol{D}_z\boldsymbol{F}_0W\boldsymbol{F}_0\boldsymbol{D}_z^{\mathrm{T}} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P} & 0 & 0 & -\alpha_1^{-1}\boldsymbol{I} & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & 0 & -(\alpha_1^{-1}+\alpha_2^{-1})^{-1}\boldsymbol{I} \end{bmatrix} +$$

$$\begin{bmatrix}(P_i-P_{i0})B_i\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}(P_i-P_{i0})B_i\\0\\0\\0\\0\end{bmatrix}^{\mathrm{T}}+\begin{bmatrix}(K-K_0)^{\mathrm{T}}F_0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}(K-K_0)^{\mathrm{T}}F_0\\0\\0\\0\\0\end{bmatrix}^{\mathrm{T}}<0 \tag{26}$$

Formula (24) is a strict condition of formula (17) since the left-hand side of inequality (26) includes left-hand side of formula (17) and two positive definite items. Therefore, it is apparent that the condition (17) holds if formula (24) is satisfied. Similar to the above-mentioned demonstration, we can also explain the relation between expressions (18) and (25). This completes the proof. □

Remark 3 The non-convex optimization problem in Theorem 2 can be solved by giving the additional variables K_0 and P_{i0}. The conservativeness of this transformation lies in the differences between $K-K_0$ and P_i-P_{i0}. Thus, the following iterative algorithm will be developed to minimize this conservativeness.

Algorithm 1

(1)[12] Select proper upper bounds $\gamma_i>0$, then obtain the initial controller gain $K^0_{\mathrm{opt}}=Z_{\mathrm{opt}}X^{-1}_{\mathrm{opt}}$ via the optimization problem: minimize tr(Y), subject to X>0 and

$$\begin{bmatrix}XA_i^{\mathrm{T}}+A_iX+B_iF_0Z+Z^{\mathrm{T}}F_0B_i^{\mathrm{T}}+\alpha_1B_iF_0WF_0B_i^{\mathrm{T}} & * & * & *\\ G_i^{\mathrm{T}} & -\gamma_iI & * & *\\ C_z+D_zF_0K & 0 & -\gamma_iI+\alpha_1D_zF_0WF_0D_z^{\mathrm{T}} & *\\ W^{1/2}Z & 0 & 0 & -(\alpha_1^{-1}+\alpha_2^{-1})^{-1}I\end{bmatrix}<0 \tag{27}$$

$$\begin{bmatrix}Y & I\\ I & X\end{bmatrix}>0 \tag{28}$$

If there is no such controller, then stop and the algorithm fails to obtain a solution.

(2) Let $K=K^0_{\mathrm{opt}}$, minimize $[\mathrm{tr}(\sum_{i=1}^{N}P_i)]$ subject to $P_i>0$ and the optimization problems described in Theorem 2, then we get the initial Lyapunov function variables P^0_{i0}.

(3) At the k-th (k>0) iteration, let $P^k_{i0}=P^{k-1}_{i,\mathrm{opt}}$ and $K^k_0=K^{k-1}_{\mathrm{opt}}$, minimize $[\mathrm{tr}(\sum_{i=1}^{N}P_i)]$ subject to $P_i>0$ and the optimization problems described in Theorem 3, then we get the k-th Lyapunov function variables $P^k_{i,\mathrm{opt}}$ and controller gain K^k_{opt}.

(4) If $\left|\mathrm{tr}[\sum_{i=1}^{N}(P^k_{i0}-P^{k-1}_{i0})]\right|<\delta$ where δ is a given error tolerance, the calculated $K=K^k_{\mathrm{opt}}$ is the optimal fault-tolerant H_∞ controller gain, stop. Otherwise, let k=k+1 and return to Step 3.

It should be noted that the convex optimization problem for K and P_i in Theorem 3 can be solved by using the LMI Toolbox in the MATLAB environment for given initial gains K^0 and P^0_i.

2.2 Mixed H_2/H_∞ fault-tolerant controller synthesis

It is well known that H_∞ control guarantees robust stability of a system only in the face of uncertainties and disturbances. In practice, transient performance of a system also needs to be dealt with. To manage the trade-off between the system performance and robustness, many works of the mixed H_2/H_∞ control[6,13-16] have been carried out. So we extend our results to the mixed H_2/H_∞ FTC in this subsection.

Introduce the different controlled outputs $z_\infty(t)$ and $z_2(t)$ as measurement of robustness and system performances, respectively. Then the closed-loop system (9) is represented as

$$\begin{cases}\dot{x}(t)=[A(\theta)+B(\theta)F_0(I+L)K]x(t)+G(\theta)v(t),\\ y(t)=C(\theta)x(t),\\ z_\infty(t)=[C_\infty+D_\infty F_0(I+L)K]x(t),\\ z_2(t)=[C_2+D_2F_0(I+L)K]x(t)\end{cases} \tag{29}$$

Definition 2[15] Mixed H_2/H_∞ problem. Give an H_∞ level γ, find an admissible SOF controller gain K which stabilizes the closed-loop system satisfying

$$\min\|T_{z_2v}\|_2=\min\left\{\mathrm{Trace}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}T(\mathrm{j}\omega)T^{\mathrm{T}}(\mathrm{j}\omega)\,\mathrm{d}\omega\right)^{1/2}\right\}$$

$$\text{s.t.} \quad \|\boldsymbol{T}_{z_\infty v}\|_\infty = \left(\sup_{\omega}(\sigma_{\max}(\boldsymbol{T}(\mathrm{j}\omega)))\right) < \gamma \tag{30}$$

where $\boldsymbol{T}_{z_\infty v}$ and $\boldsymbol{T}_{z_2 v}$ denote the transfer functions from $\boldsymbol{v}(t)$ to $z_\infty(t)$ and $z_2(t)$, respectively. The H_∞ norm of $\boldsymbol{T}_{z_\infty v}$ is defined as supermum of its largest singular value over all frequencies. The H_2 norm of $\boldsymbol{T}_{z_2 v}$ is defined as the output energy of impulse response of $z_2(t)$.

Theorem 4 Consider the uncertain closed-loop system (29). For given positive scalars γ and λ, if there exist affine parameter-dependent Lyapunov variable $\boldsymbol{P}(\boldsymbol{\theta}) = \sum_{i=1}^{N} \theta_i \boldsymbol{P}_i$ and symmetric positive definite matrix $\boldsymbol{Q}$ satisfying the optimization problem,

$$\min[\mathrm{tr}(\boldsymbol{Q})] \quad \text{s. t.}$$

$$\begin{bmatrix} [\boldsymbol{A}(\boldsymbol{\theta})+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\theta})+ \\ \boldsymbol{P}(\boldsymbol{\theta})[\boldsymbol{A}(\boldsymbol{\theta})+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}] & * & * \\ \boldsymbol{G}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{P}(\boldsymbol{\theta}) & -\gamma\boldsymbol{I} & * \\ \boldsymbol{C}_\infty+\boldsymbol{D}_\infty\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K} & 0 & -\gamma\boldsymbol{I} \end{bmatrix} < 0 \tag{31}$$

$$[\boldsymbol{A}(\boldsymbol{\theta})+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\theta})+\boldsymbol{P}(\boldsymbol{\theta})[\boldsymbol{A}(\boldsymbol{\theta})+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]+[\boldsymbol{C}_2+\boldsymbol{D}_2\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]^{\mathrm{T}}\cdot[\boldsymbol{C}_2+\boldsymbol{D}_2\boldsymbol{F}_0(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{K}]<0 \tag{32}$$

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{G}^{\mathrm{T}}(\boldsymbol{\theta})\boldsymbol{P}(\boldsymbol{\theta}) \\ \boldsymbol{P}(\boldsymbol{\theta})\boldsymbol{G}(\boldsymbol{\theta}) & \boldsymbol{P}(\boldsymbol{\theta}) \end{bmatrix} > 0 \tag{33}$$

$$\mathrm{Trace}(\boldsymbol{Q}) < \lambda^2 \tag{34}$$

the obtained controller guarantees the robust stability and mixed H_2/H_∞ performance of the uncertain closed-loop system in both normal and fault cases.

Although Theorem 4 is not a direct consequence of Theorem 1, we can derive it by following Definition 2 and similar arguments to the proof of Theorem 1. So it is omitted here.

Theorem 5 For given positive scalars γ_i and λ_i, the robust stability and mixed performance of Eq. (29) guarantee if there exist symmetric positive definite matrices $\boldsymbol{P}_i$ and $\boldsymbol{Q}_i$ satisfying

$$\min[\mathrm{tr}(\sum_{i=1}^{N}\boldsymbol{Q}_i)] \quad \text{s. t.}$$

$$\begin{bmatrix} (\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P}_{\infty i}+\boldsymbol{P}_{\infty i}(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K}) & * & * & * & * \\ \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i} & -\gamma_i\boldsymbol{I} & * & * & * \\ \boldsymbol{C}_\infty+\boldsymbol{D}_\infty\boldsymbol{F}_0\boldsymbol{K} & 0 & -\gamma_i\boldsymbol{I}+\alpha_2\boldsymbol{D}_\infty\boldsymbol{F}_0\boldsymbol{W}\boldsymbol{F}_0\boldsymbol{D}_\infty^{\mathrm{T}} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i} & 0 & 0 & -\alpha_1^{-1}\boldsymbol{I} & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & 0 & -(\alpha_1^{-1}+\alpha_2^{-1})^{-1}\boldsymbol{I} \end{bmatrix} < 0 \tag{35}$$

$$\begin{bmatrix} (\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P}_{2i}+\boldsymbol{P}_{2i}(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K}) & * & * & * \\ \boldsymbol{C}_2+\boldsymbol{D}_2\boldsymbol{F}_0\boldsymbol{K} & -\boldsymbol{I}+\alpha_4\boldsymbol{D}_2\boldsymbol{F}_0\boldsymbol{W}\boldsymbol{F}_0\boldsymbol{D}_2^{\mathrm{T}} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{2i} & 0 & -\alpha_3^{-1}\boldsymbol{I} & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & -(\alpha_3^{-1}+\alpha_4^{-1})^{-1}\boldsymbol{I} \end{bmatrix} < 0 \tag{36}$$

$$\begin{bmatrix} \boldsymbol{Q}_i & \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_{2i} \\ \boldsymbol{P}_{2i}\boldsymbol{G}_i & \boldsymbol{P}_{2i} \end{bmatrix} > 0 \tag{37}$$

$$\mathrm{Trace}(\boldsymbol{Q}_i) < \lambda_i^2 \tag{38}$$

$$\begin{bmatrix} (\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P}_{\infty i}+ \\ \boldsymbol{P}_{\infty i}(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})+\mu_{1i}\boldsymbol{I} & * & * & * \\ \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i} & \mu_{1i}\boldsymbol{I} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i} & 0 & -\alpha_5^{-1}\boldsymbol{I} & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & -\alpha_5\boldsymbol{I} \end{bmatrix} > 0 \tag{39}$$

$$\begin{bmatrix} (\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P}_{2i}+ \\ \boldsymbol{P}_{2i}(\boldsymbol{A}_i+\boldsymbol{B}_i\boldsymbol{F}_0\boldsymbol{K})+\mu_{2i}\boldsymbol{I} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{2i} & -\alpha_6^{-1}\boldsymbol{I} & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & -\alpha_6\boldsymbol{I} \end{bmatrix} > 0 \tag{40}$$

Adopting the concept of Definition 2, it is clear that the expressions (35)-(38) in Theorem 5 are sufficient conditions for mixed H_2/H_∞ performance at the corners of the polytope. The proof of formulas (39) and (40) in Theorem 5 is again an application of Lemmas 1 and 2. Thus, the proof is omitted for brevity.

By following a similar discussion as in Remark 2, we see that there is still a coupling between the Lyapunov function variables and the controller gain in Theorem 5. Hence, we eliminate the coupling with the aid of the following theorem.

Theorem 6 For given positive scalars γ_i and λ_i, the robust stability and mixed performance of Eq. (29) guarantee if there exist symmetric positive definite matrices $\boldsymbol{P}_i$ and $\boldsymbol{Q}_i$ satisfying

$$\min[\mathrm{tr}(\sum_{i=1}^{N}\boldsymbol{Q}_i)] \quad \text{s. t.}$$

$$\begin{bmatrix} \boldsymbol{\Phi}_\infty(\boldsymbol{K},\boldsymbol{P}_{\infty i}) & * & * & * & * & * \\ \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i} & -\gamma_i \boldsymbol{I} & * & * & * & * \\ \boldsymbol{C}_\infty+\boldsymbol{D}_\infty\boldsymbol{F}_0\boldsymbol{K} & 0 & -\gamma_i\boldsymbol{I}+\alpha_2\boldsymbol{D}_z\boldsymbol{F}_0\boldsymbol{W}\boldsymbol{F}_0\boldsymbol{D}_z^{\mathrm{T}} & * & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i} & 0 & 0 & -\alpha_1^{-1}\boldsymbol{I} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & 0 & -(\alpha_1^{-1}+\alpha_2^{-1})^{-1}\boldsymbol{I} & * \\ \boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i}+F_0\boldsymbol{K} & 0 & 0 & 0 & 0 & -\boldsymbol{I} \end{bmatrix} < 0 \quad (41)$$

$$\begin{bmatrix} \boldsymbol{\Phi}_2(\boldsymbol{K},\boldsymbol{P}_{2i}) & * & * & * & * \\ \boldsymbol{C}_2+\boldsymbol{D}_2\boldsymbol{F}_0\boldsymbol{K} & -\boldsymbol{I}+\alpha_4\boldsymbol{D}_2\boldsymbol{F}_0\boldsymbol{W}\boldsymbol{F}_0\boldsymbol{D}_2^{\mathrm{T}} & * & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{2i} & 0 & -\alpha_3^{-1}\boldsymbol{I} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & -(\alpha_3^{-1}+\alpha_4^{-1})^{-1}\boldsymbol{I} & * \\ \boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{2i}+\boldsymbol{F}_0\boldsymbol{K} & 0 & 0 & 0 & -\boldsymbol{I} \end{bmatrix} < 0 \quad (42)$$

$$\begin{bmatrix} \boldsymbol{Q}_i & \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_{2i} \\ \boldsymbol{P}_{2i}\boldsymbol{G}_i & \boldsymbol{P}_{2i} \end{bmatrix} > 0 \quad (43)$$

$$\mathrm{Trace}(\boldsymbol{Q}_i) < \lambda_i^2 \quad (44)$$

$$\begin{bmatrix} \boldsymbol{\Phi}_\infty(\boldsymbol{K},\boldsymbol{P}_{\infty i})+\mu_{1i}\boldsymbol{I} & * & * & * & * \\ \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i} & \mu_{1i}\boldsymbol{I} & * & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i} & 0 & -\alpha_5^{-1}\boldsymbol{I} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & 0 & -\alpha_5\boldsymbol{I} & * \\ \boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i}+\boldsymbol{F}_0\boldsymbol{K} & 0 & 0 & 0 & -\boldsymbol{I} \end{bmatrix} > 0 \quad (45)$$

$$\begin{bmatrix} \boldsymbol{\Phi}_2(\boldsymbol{K},\boldsymbol{P}_{2i})+\mu_{2i}I & * & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{F}_0\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{2i} & -\alpha_6^{-1}\boldsymbol{I} & * & * \\ \boldsymbol{W}^{1/2}\boldsymbol{K} & 0 & -\alpha_6\boldsymbol{I} & * \\ \boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{2i}+\boldsymbol{F}_0\boldsymbol{K} & 0 & 0 & -\boldsymbol{I} \end{bmatrix} > 0 \quad (46)$$

where

$$\boldsymbol{\Phi}_\infty(\boldsymbol{K},\boldsymbol{P}_{\infty i}) = \boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i}+\boldsymbol{P}_{\infty i}\boldsymbol{A}_i-\boldsymbol{P}_{\infty i}\boldsymbol{B}_i\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i0}- \boldsymbol{P}_{\infty i0}\boldsymbol{B}_i\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i}+\boldsymbol{P}_{\infty i0}\boldsymbol{B}_i\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{\infty i0}-\boldsymbol{K}^{\mathrm{T}}\boldsymbol{F}_0\boldsymbol{F}_0\boldsymbol{K}_0- \boldsymbol{K}_0^{\mathrm{T}}\boldsymbol{F}_0\boldsymbol{F}\boldsymbol{K}_0+\boldsymbol{K}_0^{\mathrm{T}}\boldsymbol{F}_0\boldsymbol{F}_0\boldsymbol{K}_0,$$

$$\boldsymbol{\Phi}_2(\boldsymbol{K},\boldsymbol{P}_{2i}) = \boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{P}_{2i}+\boldsymbol{P}_{2i}\boldsymbol{A}_i-\boldsymbol{P}_{2i}\boldsymbol{B}_i\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{2i0}- \boldsymbol{P}_{2i0}\boldsymbol{B}_i\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{2i}+\boldsymbol{P}_{2i0}\boldsymbol{B}_i\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{2i0}-\boldsymbol{K}^{\mathrm{T}}\boldsymbol{F}_0\boldsymbol{F}_0\boldsymbol{K}_0- \boldsymbol{K}_0^{\mathrm{T}}\boldsymbol{F}_0\boldsymbol{F}\boldsymbol{K}_0+\boldsymbol{K}_0^{T}\boldsymbol{F}_0\boldsymbol{F}_0\boldsymbol{K}_0.$$

By following similar lines to the proof of Theorem 3, the proof of Theorem 6 can be readily obtained.

The discussions of the iterative algorithm for mixed H_2/H_∞ fault-tolerant controller design follow those in Subsection 2.1 and so are also omitted here.

3 Simulation Results and Analysis

A numerical example of flight tracking control for the F-18 aircraft is presented to demonstrate the merits of the proposed method.

The decoupled linearized longitudinal dynamical motion equations of the F-18 aircraft[17] are given as

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \boldsymbol{A}_{\mathrm{long}}\begin{bmatrix} \alpha \\ q \end{bmatrix} + \boldsymbol{B}_{\mathrm{long}}\begin{bmatrix} \delta_{\mathrm{E}} \\ \delta_{\mathrm{PTV}} \end{bmatrix},$$

where α is the angle of attack, q is the pitch rate, δ_{E} is the symmetric elevator position, and δ_{PTV} is the symmetric pitch thrust velocity nozzle position. For the considered flight case,

$$\boldsymbol{A}_{\mathrm{long}}^{\mathrm{m7h14}} = \begin{bmatrix} -1.175 & 0.9871 \\ -8.458 & -0.8776 \end{bmatrix},\ \boldsymbol{B}_{\mathrm{long}}^{\mathrm{m7h14}} = \begin{bmatrix} -0.194 & -0.0359 \\ -19.29 & -3.803 \end{bmatrix};$$

$$\boldsymbol{A}_{\mathrm{long}}^{\mathrm{m85h5}} = \begin{bmatrix} -2.328 & 0.9831 \\ -30.44 & -1.493 \end{bmatrix},\ \boldsymbol{B}_{\mathrm{long}}^{\mathrm{m85h5}} = \begin{bmatrix} -0.3012 & -0.0587 \\ -38.43 & -7.815 \end{bmatrix};$$

$$\boldsymbol{A}_{\mathrm{long}}^{\mathrm{m9h10}} = \begin{bmatrix} -2.452 & 0.9856 \\ -38.61 & -1.34 \end{bmatrix},\ \boldsymbol{B}_{\mathrm{long}}^{\mathrm{m9h10}} = \begin{bmatrix} -0.2757 & -0.0523 \\ -37.36 & -7.247 \end{bmatrix}.$$

Following the nomenclature in Ref. [17], $\boldsymbol{A}_{\mathrm{long}}^{\mathrm{m7h14}}$ denotes the longitudinal state matrix at Mach 0.7 and 14-kft altitude.

For the F-18 aircraft model, consider the actuator fault matrix $\boldsymbol{F}=\mathrm{diag}[f_1, f_2]$, $0.4 \leqslant f_1 \leqslant 1,\ 0 \leqslant f_2 \leqslant 1$, weight parameters $\alpha_1=\cdots=\alpha_6=1,\ \mu_{1i}=\mu_{2i}=0.01$ and select the designed output as

$$z(t) = \begin{bmatrix} \boldsymbol{I}_{2\times 2} \\ \boldsymbol{0}_{2\times 2} \end{bmatrix}\begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} \boldsymbol{0}_{2\times 2} \\ 0.1*\boldsymbol{I}_{2\times 2} \end{bmatrix}\begin{bmatrix} \delta_{\mathrm{E}} \\ \delta_{\mathrm{PTV}} \end{bmatrix}.$$

Then, we have taken the LMI toolbox function mincx applied it to the optimization problem in Section 2 for obtaining the H_∞ and mixed H_2/H_∞ fault-tolerant controller gains,

$$\boldsymbol{K}_{H_\infty} = \begin{bmatrix} -0.2862 & 0.6379 \\ -0.0178 & 0.0386 \end{bmatrix}$$

and

$$\boldsymbol{K}_{H_2/H_\infty} = \begin{bmatrix} -0.3617 & 1.4367 \\ -0.0739 & 0.1640 \end{bmatrix}.$$

For comparison purposes, the standard H_∞ controller[12] without considering actuator faults are also carried out.

Table 1 shows the H_∞ performance and iteration counter of the three control methods.

Table 1 Comparison of the three control methods

Methods	H_∞ performance			Iteration counter
	m7h14	m85h5	m9h10	
Standard H_∞ control	0.5006	0.5006	0.5006	–
H_∞ FTC	0.2469	0.3095	0.4236	17
Mixed H_2/H_∞ FTC	0.2586	0.3319	0.4701	42

Subsequently, we consider the following actuator faults at 0.5 s in simulations.

Fault-free: $\boldsymbol{F} = \mathrm{diag}[1,\ 1]$;

Fault 1: $\boldsymbol{F} = \mathrm{diag}[0.8,\ 0.4]$;

Fault 2: $\boldsymbol{F} = \mathrm{diag}[0.4,\ 0]$.

The fault 1 means that the effectiveness of δ_{E} and δ_{PTV} lose 20% and 60%, respectively. The fault 2 means that the effectiveness of δ_{E} loses 60% and δ_{PTV} is the total outage. Obviously, the fault 2 is more severe than fault 1.

The response curves of the closed-loop system with fault free, fault 1 and fault 2 are given in Figs. 1-9, respectively. The simulation results show that the system performance of the standard H_∞ controller is similar to that of the fault-tolerant controllers when the actuators are healthy. However, as the magnitude of faults increases, the standard H_∞ controller cannot stabilize the closed-loop system and fault-tolerant controllers just suffer from slight performance degradation.

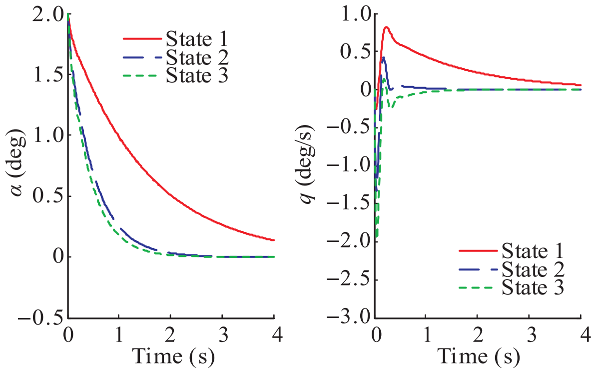

Fig. 1 Responses of standard H_∞ controller with fault free

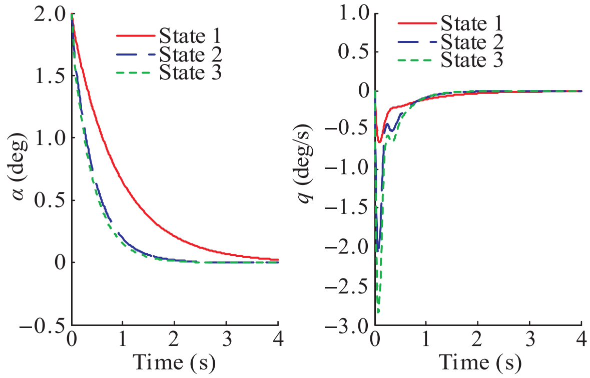

Fig. 2 Responses of H_∞ fault-tolerant controller with fault free

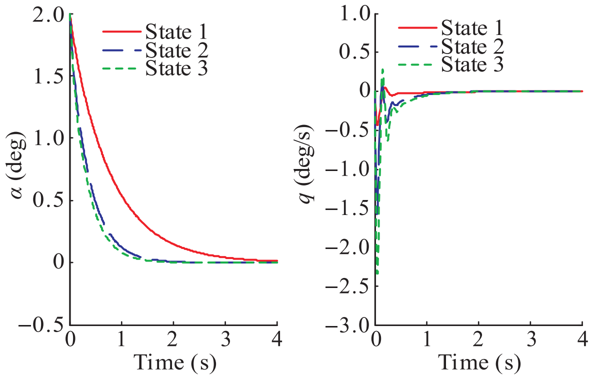

Fig. 3 Responses of mixed H_2/H_∞ fault-tolerant controller with fault free

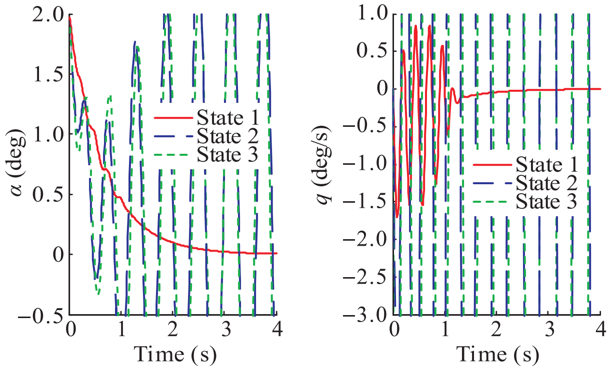

Fig. 4 Responses of standard H_∞ controller with fault 1

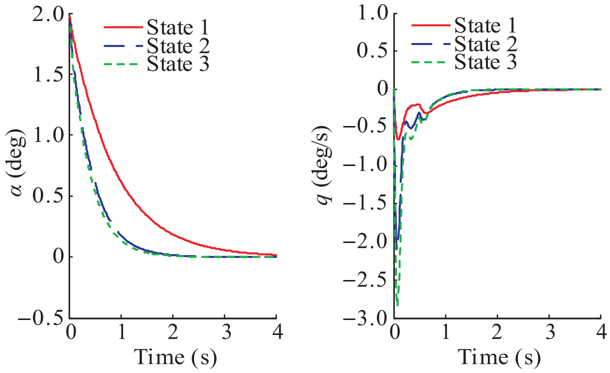

Fig. 5 Responses of H_∞ fault-tolerant controller with fault 1

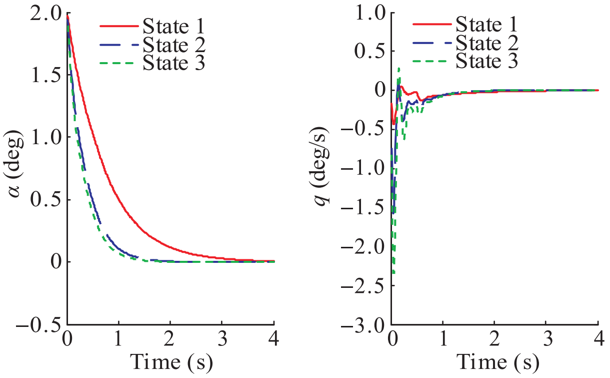

Fig. 6 Responses of mixed H_2/H_∞ fault-tolerant controller with fault 1

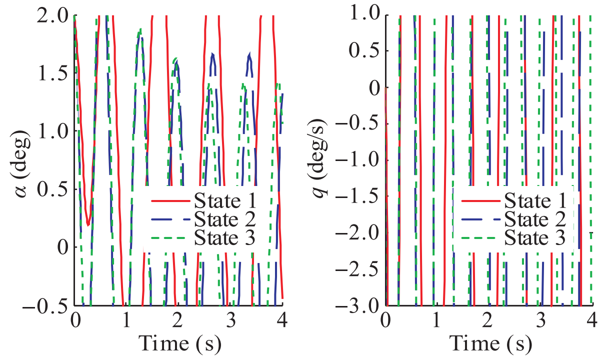

Fig. 7 Responses of standard H_∞ controller with fault 2

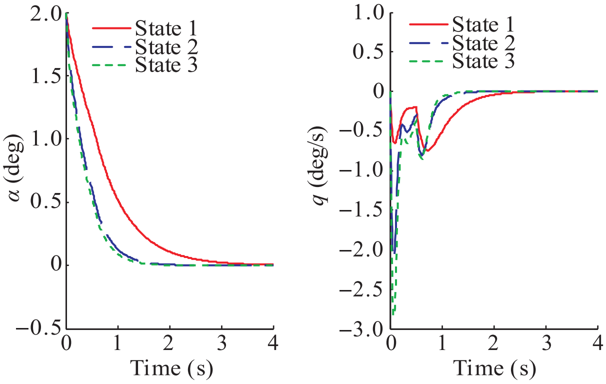

Fig. 8 Responses of H_∞ fault-tolerant controller with fault 2

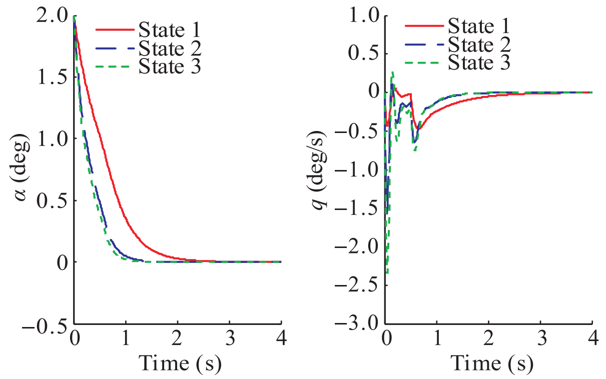

Fig. 9 Responses of mixed H_2/H_∞ fault-tolerant controller with fault 2

The simulation results also show that the mixed H_2/H_∞ fault-tolerant controller yields transient behaviors superior to the H_∞ fault-tolerant controller in all cases.

Summarizing the simulation results, our approaches can improve the system performance in the event of fault cases as compared to the standard design method.

4 Conclusions and Future Work

A fault-tolerant controller design approach has been developed for polytopic uncertain systems with actuator faults, and with extensions to H_∞ and mixed H_2/H_∞ problems. Moreover, an iterative algorithm is obtained to reduce the conservativeness of controller design by employing the additional variables. The resulting controllers are reliable in that it provides guaranteed robust stability and system performance in both normal and fault cases. Simulation results of F-18 aircraft illustrate that our methods result in performance superior to previous methods.

The convenience of controller design has been achieved by using the iterative algorithm. Unfortunately, the algorithm yields a suboptimal solution since additional variables were introduced. Therefore, future work will investigate how to develop a new LMI relaxation method to get an optimal solution of the non-convex optimization problem in Theorem 3 and 5 without additional variables, namely, how to decouple the Lyapunov variables and system matrices.

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