TSINGHUA SCIENCE AND TECHNOLOGY
ISSN 1007-0214 18/18 pp128-132
Volume 10, Number 1, February 2005

Improved Ternary Subdivision Interpolation Scheme*

WANG Huawei (王华维), QIN Kaihuai (秦开怀)**

Department of Computer Science and Technology, Tsinghua University, Beijing 100084, China

Abstract: An improved ternary subdivision interpolation scheme was developed for computer graphics applications that can manipulate open control polygons unlike the previous ternary scheme, with the resulting curve proved to be still C^2-continuous. Parameterizations of the limit curve near the two endpoints are given with expressions for the boundary derivatives. The split joint problem is handled with the interpolating ternary subdivision scheme. The improved scheme can be used for modeling interpolation curves in computer aided geometric design systems, and provides a method for joining two limit curves of interpolating ternary subdivisions.

Key words: curve; interpolation; subdivision; ternary subdivision scheme

Introduction

Computer aided geometric design is not only the basis of computer-aided designs of machines, ships, and buildings, but also has wide applications in many fields, such as computer animations, three-dimensional geometric measurements, and medical surgery simulations. Subdivision curves and surfaces are an important part of such design systems because they provide flexibility to computer aided geometric design.

Interpolating subdivision schemes are more attractive than approximating schemes in computer aided geometric designs because of their interpolation property. All vertices in the control mesh are located on the limit curve or surface of the interpolation subdivision, which facilitates fast multi-resolution representations and wavelet analyses and simplifies the graphics algorithms. In most engineering designs, the interpolating curve or surface must pass through all data points. In addition, the interpolation subdivisions are more intuitive to the users.

Interpolating subdivision schemes for curves have been used as the foundation of various interpolation schemes for curves and surfaces. In 1987, Dyn et al.[1] introduced a 4-point interpolation subdivision scheme for curve design, which was later generalized to non-stationary 4-point schemes[2, 3]. Cai[4] modified the 4-point scheme with a formula for computing vertices near the two endpoints to solve the split joint problem for 4-point interpolation curves. The 4-point scheme and the convexity preservation property have been used to define many other interpolation schemes [5-8]. Kobbelt and Schroder[9, 10] presented a class of interpolation subdivision schemes by solving an optimization problem such that the 4-point scheme is a special case. Kobbelt[11] also generalized the 4-point scheme to quad meshes to obtain an interpolation subdivision scheme for surfaces, but his scheme cannot deal with extraordinary faces in control meshes so it is only applicable to meshes with only quadrilaterals. In the 4-point scheme, a new vertex is inserted between every two adjacent old vertices in each subdivision step so the corresponding parameter interval is divided into two equal parts. Recently, Hassan et al.[12] proposed a novel 4-point ternary interpolation subdivision scheme, where two new vertices are inserted between every two adjacent vertices each time to divide each parameter

Received: 2003-03-21; revised: 2003-11-12
* Supported by the National Natural Science Foundation of China (No. 60273013), the Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20010003048), and Research Grants Council of Hong Kong (RGC) (No. CUHK4189/01E)
** To whom correspondence should be addressed.
E-mail: qkh-dcs@tsinghua.edu.cn;
Tel: 86-10-62783505-2

interval into three equal parts so that the resulting limit curve has higher continuity than the 4-point scheme. This paper presents an improved scheme of interpolating ternary subdivision as well as an analysis of its convergence and continuity properties. The parameterization of the limit curve near the two endpoints is given with the expression for the boundary derivatives. The improved ternary scheme can also be used for the joining of two interpolation curves.

1 Improved Ternary Scheme

Given a sequence of vertices $\{\boldsymbol{p}_i\}$ as the initial control polygon, insert new vertices recursively using the ternary scheme to obtain finer and finer control polygons that converge to a limiting smooth curve. In contrast to the 4-point scheme, two new vertices are inserted between every two adjacent vertices in each subdivision step, as shown in Fig. 1, where solid dots denote the old vertices and the crosses denote the new vertices. The subdivision masks for computing the new vertices are also shown in Fig. 1. For the sequence of any single coordinate $\{f_i\}$, the vertices of the new control polygon are computed from the following equations[12]:

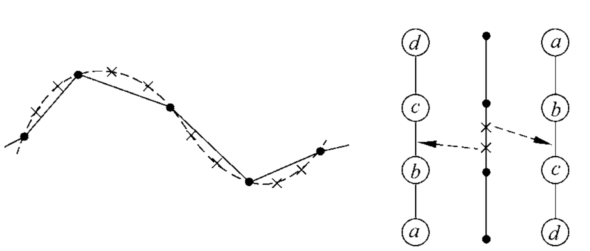
$$\begin{cases} f_{3i}^{k+1} = f_i^k, \\ f_{3i+1}^{k+1} = af_{i-1}^k + bf_i^k + cf_{i+1}^k + df_{i+2}^k, \\ f_{3i+2}^{k+1} = df_{i-1}^k + cf_i^k + bf_{i+1}^k + af_{i+2}^k \end{cases} \quad (1)$$

where $f_i^0 = f_i$ and the four coefficients *a*, *b*, *c*, and *d* are defined as:

$$a = -\frac{1}{18} - \frac{1}{6}\mu, \quad b = \frac{13}{18} + \frac{1}{2}\mu,$$

$$c = \frac{7}{18} - \frac{1}{2}\mu, \quad d = -\frac{1}{18} + \frac{1}{6}\mu.$$

The precision set for the ternary scheme is the quadratics. Hassan et al.[12] proved that the limit function of this scheme is C^2 for $1/15 < \mu < 1/9$.



(a) Subdivision scheme (b) Subdivision masks

Fig. 1 Ternary interpolation scheme

However, the control polygon is often not bi-infinite in practical applications. Therefore, consider the control polygon $\boldsymbol{p}_0\boldsymbol{p}_1\cdots\boldsymbol{p}_n$, where $\boldsymbol{p}_0 \neq \boldsymbol{p}_n$ and $n \geq 3$. In this case, new vertices near the two endpoints cannot be calculated using the original ternary scheme, so the control polygon shrinks step by step in the subdivision process until it converges to the limit curve, as shown in Fig. 2a, where the limit curve is shown together with the corresponding control polygon. To avoid this problem, an improved ternary scheme computes the new vertices using Eq. (1) in the interior of the control polygon, but using the following subdivision scheme near the left endpoint:

$$\begin{cases} f_0^{k+1} = f_0^k, \\ f_1^{k+1} = \alpha_0 f_0^k + \alpha_1 f_1^k + \alpha_2 f_2^k, \\ f_2^{k+1} = \beta_0 f_0^k + \beta_1 f_1^k + \beta_2 f_2^k \end{cases} \quad (2)$$

where

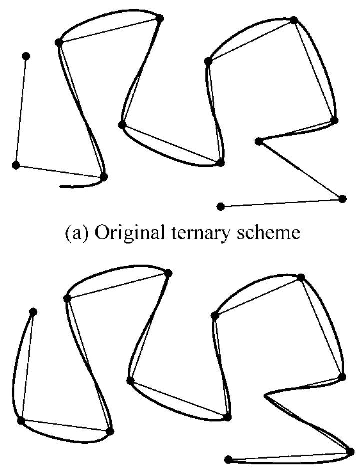
$$\alpha_0 = 3a + b, \quad \alpha_1 = c - 3a, \quad \alpha_2 = a + d,$$

$$\beta_0 = c + 3d, \quad \beta_1 = b - 3d, \quad \beta_2 = a + d.$$

The subdivision rules near the right endpoint are:

$$\begin{cases} f_{3^{k+1}n}^{k+1} = f_{3^k n}^k, \\ f_{3^{k+1}n-1}^{k+1} = \alpha_0 f_{3^k n}^k + \alpha_1 f_{3^k n-1}^k + \alpha_2 f_{3^k n-2}^k, \\ f_{3^{k+1}n-2}^{k+1} = \beta_0 f_{3^k n}^k + \beta_1 f_{3^k n-1}^k + \beta_2 f_{3^k n-2}^k \end{cases} \quad (3)$$

The improved ternary scheme preserves the precision set, which will be shown in Section 2. For the same control polygon, the limit curve generated by the improved scheme is shown in Fig. 2b. The improved interpolating subdivision scheme provides accurate endpoint interpolation.



(a) Original ternary scheme

(b) Improved ternary scheme

Fig. 2 Comparing curves generated by the original and improved ternary scheme for μ=1/11

2 Convergence and Continuity Analyses

Let f^k be the piecewise linear interpolation to the values $\{f_i^k\}_{i=0}^{3^k n}$, i.e., the control polygon with vertices f_i^k, where

$$f^k(3^{-k}i) = f_i^k, \quad 0 \le i \le 3^k n.$$

Obviously, $f^k \in C[0, n]$. Next, we investigate the limit of f^k as $k \to \infty$.

Consider any given positive integer *m*. In the interval $[2/3^m, n-2/3^m]$, the sequence of control polygons $\{f^{m+k}\}_{k=0}^{+\infty}$ can be shown to be exactly the same as the sequence of control polygons generated by the original ternary scheme, so the sequence $\{f^k\}$ has the same convergence and continuity properties the original scheme in $[2/3^m, n-2/3^m]$. Consequently, let $m \to \infty$, and then the sequence $\{f^k\}$ has the same convergence and continuity properties as the original scheme in the interval $(0, n)$. Note that $f^k(0) \equiv f_0$, $f^k(n) \equiv f_n$, so the sequence $\{f^k\}$ has the same convergence property as the original scheme in the whole interval $[0, n]$. For $1/15 < \mu < 1/9$, the sequence generated by the original scheme is convergent[12] and has C^2 limit function, so in $[0, n]$ the sequence $\{f^k\}$ converges to a function, denoted by $f(t)$, and $f(t)$ is C^2 in $(0, n)$. The remaining question is the continuity property near the two endpoints.

In the initial control polygon, the left three vertices f_0^0, f_1^0, and f_2^0 can be used to define a quadratic polynomial,

$$L(t) = \frac{(1-t)(2-t)}{2} f_0^0 + t(2-t) f_1^0 + \frac{t}{2}(t-1) f_2^0,$$

such that $L(0) = f_0^0, L(1) = f_1^0$, and $L(2) = f_2^0$. Then, $L(-1) = 3f_0^0 - 3f_1^0 + f_2^0$ and Eq. (2) can be used to show that

$$f_1^1 = aL(-1) + bL(0) + cL(1) + dL(2) = L(1/3) \quad (4)$$

In the same way, $f_2^1 = L(2/3)$, so $f_i^1 = L(i/3)$, $i = 0, 1, 2, 3$. Hence, one can recursively obtain

$$f_i^k = L(3^{-k}i), i = 0, 1, ..., T_k,$$

where $k = 0, 1, 2, ...$ as shown in Fig. 3. T_k satisfies the recursive relation $T_{k+1} = 3(T_k - 1)$ with $T_0 = 2$, that is, $T_k = (3^k + 3)/2$. Note that $3^{-k}T_k > 1/2$ for any *k*, that is to say, $\{f^k\}$ is the sequence of linear functions gradually approximating $L(t)$ in the interval $[0, 1/2]$, and accordingly $f(t) \equiv L(t), \ \forall t \in [0, 1/2]$. Similarly, $f(t) \equiv R(t-n+1), \ \forall t \in [n-1/2, n]$, where

$$R(t) = \frac{t}{2}(t-1) f_{n-2}^0 + (1-t^2) f_{n-1}^0 + \frac{t}{2}(t+1) f_n^0.$$

$L(t)$ and $R(t)$ are C^2 at least, so $f(t)$ is also C^2 near the two endpoints. Equation (4) shows that the precision set for the improved ternary interpolation scheme is the quadratics.

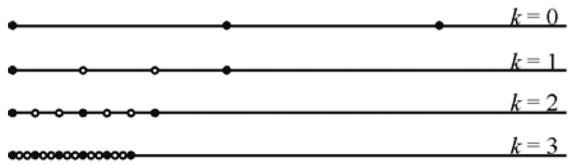


Fig. 3 Control vertices located in *L*(*t*)

Therefore, the control polygons generated recursively by the improved ternary interpolation scheme converge to a limit curve with C^2 continuity for $1/15 < \mu < 1/9$.

3 Joining Two Curves

In computer aided geometric design, two curves are usually joined to produce a more complicated curve. The improved ternary interpolation curves can also be applied to curve joining. The continuity of the boundary derivatives is the key to the continuity of the joined curve. Since $f(t) \equiv L(t), \ \forall t \in [0, 1/2]$, the derivatives of the limit curve at the left endpoint are:

$$f'(0) = -\frac{3}{2} f_0 + 2f_1 - \frac{1}{2} f_2,$$

$$f''(0) = f_0 - 2f_1 + f_2.$$

In the same way, the derivatives at the right endpoint are:

$$f'(n) = \frac{1}{2} f_{n-2} - 2f_{n-1} + \frac{3}{2} f_n,$$

$$f''(n) = f_{n-2} - 2f_{n-1} + f_n.$$

When two curves are joined, the resulting curve is smooth if the two curves have the same tangential derivatives at the joint. The two improved ternary interpolation curves, defined by the control polygons

$\boldsymbol{p}_0\boldsymbol{p}_1\cdots\boldsymbol{p}_n$ and $\boldsymbol{q}_0\boldsymbol{q}_1\cdots\boldsymbol{q}_m$, can be joined at P_n and q_0 to achieve a C^1 curve if and only if

$$\begin{cases}\boldsymbol{p}_n=\boldsymbol{q}_0,\\ 3\boldsymbol{p}_n-4\boldsymbol{p}_{n-1}+\boldsymbol{p}_{n-2}=\gamma(-3\boldsymbol{q}_0+4\boldsymbol{q}_1-\boldsymbol{q}_2),\end{cases}$$

where $\boldsymbol{p}_i$ and $\boldsymbol{q}_j$ represent the coordinate vectors of the control vertices for $0\le i\le n$ and $0\le j\le m$, and $\gamma>0$ is some constant. The resulting curve is C^2 if and only if

$$\begin{cases}\boldsymbol{p}_n=\boldsymbol{q}_0,\\ 3\boldsymbol{p}_n-4\boldsymbol{p}_{n-1}+\boldsymbol{p}_{n-2}=\gamma(-3\boldsymbol{q}_0+4\boldsymbol{q}_1-\boldsymbol{q}_2),\\ \boldsymbol{p}_n-2\boldsymbol{p}_{n-1}+\boldsymbol{p}_{n-2}=\gamma^2(\boldsymbol{q}_0-2\boldsymbol{q}_1+\boldsymbol{q}_2).\end{cases}$$

Note that the boundary derivatives of any order greater than 2 equal 0 near each endpoint, so the C^2 joint ternary curve is C^∞ at the joint. Two joint curves are shown in Fig. 4, where a grey curve and a dark curve are joined at the common endpoint indicated by an arrow for each case.

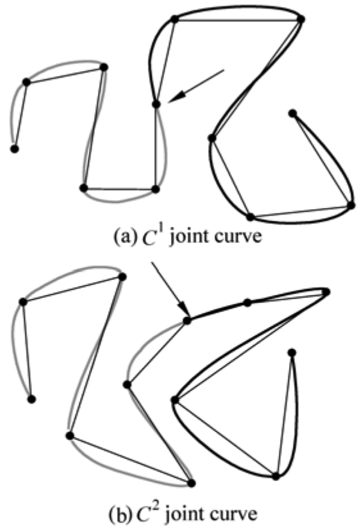


Fig. 4 Joint curves formed using the improved ternary curves with $\mu=1/11$

4 Conclusions

An improved ternary interpolation scheme was developed based on the scheme presented by Hassan et al. [12] to address the problem that the original scheme cannot deal with open control polygons. The improved ternary scheme preserves the subdivision rules in the control polygon interior and the resulting curve is still C^2. The precision set for the improved ternary interpolation scheme is also the quadratics and the limit curve near each endpoint is exactly the quadratic polynomial defined by the three adjacent control vertices, so the curve is C^∞ near an endpoint. Therefore, the derivatives can be computed to any order near each endpoint. Sufficient and necessary conditions are given for C^1 and C^2 joints between two improved ternary curves. The improved ternary scheme offers another method for modeling interpolation curves and provides methods for joining two ternary interpolation curves.

Further work is needed to generalize the improved ternary interpolation scheme to any topological surfaces to generate high-order smooth interpolation surfaces. Although Kobbelt presented an interpolation subdivision scheme for surfaces based on the 4-point scheme, his scheme can deal with control meshes with only quadrilaterals. Arbitrary control meshes of any topologies may be uniformly subdivided using an interpolating scheme generalized from the improved ternary scheme because we insert more points so there are more degrees of freedom in the ternary subdivision.

References

[1] Dyn N, Levin D, Gregory J A. A 4-point interpolatory subdivision scheme for curve design. *Computer Aided Geometric Design*, 1987, **4**: 257-268.

[2] Cai Zhijie. Parameter-varying 4-point scheme and its application. *Chinese Annals of Mathematics A*, 1995, **16**(4): 524-531. (in Chinese)

[3] Jin Jianrong, Wang Guozhao. A non-uniform 4-point interpolatory subdivision scheme to construct curve. *Appl. Math. J. Chinese Univ. Ser. A*, 2000, **15**(1): 97-100. (in Chinese)

[4] Cai Zhijie. Modified four-point scheme and its application. *Computer Aided Geometric Design*, 1998, **15**: 251-260.

[5] Cai Zhijie. Four-point scheme and convexy-preserving algorithm. *Chinese Journal of Computer Aided Design and Computer Graphics*, 1994, **6**(1): 33-36. (in Chinese)

[6] Ding Youdong. A type of nonlinear 4-point preserving-convexity interpolatory discrete subdivision scheme and its application. *Journal of Fudan University* (*Natural Science*), 2000, **39**(1): 9-14. (in Chinese)

[7] Ding Youdong, Hua Xuanji. A type of convexity-preserving interpolatory subdivision scheme for smooth curve generation and its properties. *Chinese Journal of Computer Aided Design and Computer Graphics*, 2000, **12**(7): 492-496. (in Chinese)

[8] Dyn N, Kuijt F, Levin D, Van Damme R. Convexity pres-

ervation of the four-point interpolatory subdivision scheme. *Computer Aided Geometric Design*, 1999, **16**: 789-792.

[9] Kobbelt L. A variational approach to subdivision. *Computer Aided Geometric Design*, 1996, **13**: 743-761.

[10] Kobbelt L, Schroder P. A multiresolution framework for variational subdivision. *ACM Trans. on Graphics*, 1998, **17**(4): 209-237.

[11] Kobbelt L. Interpolatory subdivision on open quadrilateral nets with arbitrary topology. *Computer Graphics Forum* (*Proceedings of Eurographics*'96), 1996, **15**(3): 409-420.

[12] Hassan M F, Ivrissimitzis I P, Dodgson N A, Sabin M A. An interpolating 4-point C^2 ternary stationary subdivision scheme. *Computer Aided Geometric Design*, 2002, **19**: 1-18.

China Displays New Nuclear Reactor

The high-temperature gas-cooled reactor, designed at prestigious Tsinghua University in Beijing, was on display at a location near the Great Wall, roughly 40 km north of downtown Beijing, China. More than 60 atomic energy experts from over 30 countries watched the safety operation, in which the reactor successfully cooled down after the control stick was pulled out. The operation had been demonstrated before. Scientists have said the major safety issue regarding nuclear reactors lies in how to cool them efficiently, as they continue produce heat even after shutdown.

Gas-cooled reactors are now widely considered the most secure. They do not need additional safety systems, as do water cooled reactor, and they discharge surplus heat, which could damage elements of the device. “It will not cause a catastrophe such as the one at Chernobyl in the Ukraine at any time,” said Qian Jihui, former deputy chief of International Atomic Energy Agency (IAEA) and a noted atomic scientist with an international reputation. IAEA official Byung-Koo Kim said that the operation of the reactor was “rather impressive.” Owing to technological improvement, Kim acknowledged, gas-cooled reactors will be introduced extensively for business purposes in the coming decades, and international cooperation will also be greatly reinforced.

China is the fifth nation in the world to master the technology, the others being the United States, Britain, Germany, and Japan, and remains in the lead in the peaceful application of nuclear energy, said Qian.

Andrew C. Kadak, former president of the American Nuclear Society and a professor at Massachusetts Institute of Technology, said after the demonstration that MIT has reached an agreement with Tsinghua University on research cooperation.

With a budget of more than 250 million yuan (approximately 30 million US dollars), the gas-cooled reactor was constructed in 1995 and incorporated into the power network in 2003. With helium refrigerant and ceramic components, the fuel temperature in the reactor can reach up to 1600℃.

Experts believe the use of gas-cooled reactors will significantly cut costs and enhance the competitive edge of nuclear power plants, which might finally trigger a new revolution in the energy field. Analysts held that China would surely run short of petroleum due to its rapid economic development and energy consumption.

Nuclear electricity accounts for 2 percent of China’s energy consumption. It is likely to reach 6 percent in 2020, still low compared with world average of 16 percent, the analysts said.

(From http://news.xinhuanet.com/english/2004-10/01/content_2044497.htm)