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Three-Dimensional Analysis of Simply Supported Functionally Graded Plate with Arbitrary Distributed Elastic Modulus*

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Abstract: Three-dimensional bending analysis is presented for a simply supported orthotropic functionally graded rectangular plate in this paper. Assuming that material properties have arbitrary variations along the plate-thickness direction, Peano-Baker series solution is obtained for the elastic fields of the functionally graded plate subjected to mechanical loads on its upper and lower surfaces by means of state space method. The correctness of the obtained series solution is validated through numerical examples. The influence of the structural response of the plate is also studied when material properties have different dependence on the thickness-coordinate. The results show that the solution is valid for the material properties of arbitrary dependence on the thickness-coordinate of the plate.

Key words: functionally graded plate; Peano-Baker series; three-dimensional bending analysis; state space method

Introduction

Functionally graded materials (FGMs) are heterogeneous composite materials with gradient compositional variations of the constituents (e.g., metallic and ceramic) from one surface of material to the other which results in continuously varying material properties. This continuously varying composition eliminates interfacial discontinuity, thus, the stress distributions are smooth.

In the past few years, a number of methods have been proposed to analyze the elastic behaviour of FGMs. Praveen and Reddy[1] analyzed the nonlinear static and dynamic response of functionally graded ceramic-metal plates using the first-order deformation theory. Della and Venini[2] developed a hierarchic family of finite elements according to the Reissner-Mindlin theory. Khoma[3] developed a method of constructing a general solution of the equilibrium equations for inhomogeneous transversely isotropic plates with elastic moduli that depend linearly on the transverse coordinate. Sankar[4] obtained an elasticity solution for a simply supported functionally graded plate under cylindrical bending. Zhong and Shang[5] presented a three-dimensional analysis for a rectangular plate made of orthotropic functionally graded piezoelectric material when the plate is simply supported and grounded along its four edges and mechanical and electric properties of the material are assumed to have the same exponent-law dependence on the thickness coordinate. Vel and Batra[6,7] obtained a closed form solutions for three-dimensional deformations of a simply supported functionally graded rectangular plate subjected to mechanical and thermal loads, and for free and forced vibrations of simply functionally graded rectangular plates. Bian et al.[8] developed a plate theory using the concept of shape function of the transverse coordinate to determine the stress distribution in an orthotropic functionally graded plate subjected to cylindrical bending. Martin et al.[9] solved the problem of an

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unbounded, three-dimensional, elastic exponentially graded solid. Pan[10] presented an exact solution for three-dimensional, anisotropic, linearly elastic, and functionally graded rectangular composite laminates under simply supported edge conditions. The solution was expressed in terms of the pseudo-Stroh formalism, and the composite laminates can be made of multilayered FGMs with their properties varying exponentially in the thickness direction. Woo and Meguid[11] developed series solutions for large deflections of functionally graded plate under transverse loading and a temperature field using von Karman theory. Cheng and Batra[12] used an asymptotic expansion method to analyse three-dimensional thermoelastic deformations of functionally graded elliptic plates, rigidly clamped at the edges. Almajid et al.[13] applied a modified classical lamination theory to predict the stress and out-of-displacement of a newly proposed piezoelectric functionally graded bimorph.

In the above studies, most scholars used specific distribution of material properties. However, for arbitrary graded distribution, effective methods are very few. The objective of this work is to present a Peano-Baker series solution of a simply supported functionally graded rectangular plate of arbitrary graded distribution of material properties based on three-dimensional elasticity theory.

1 Formulation of the Problem

Consider an FGM rectangular plate of uniform thickness h, as shown in Fig. 1. Introduce a Cartesian coordinate system $\{x_i\}$ $(i=1,2,3)$ such that the bottom and top surfaces of the undeformed plate lie in the plane $x_3=0$ and $x_3=h$. The functionally graded plate is assumed to have length a and width b in x_1-x_2 plane. In this paper, the Einsteinian summation convention over repeated indices of tensor components is used, with Latin indices ranging from 1 to 3 while Greek indices over 1 and 2.

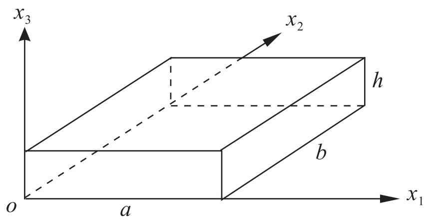

Fig. 1 Sketch of rectangular plate

In the absence of body forces, the field equations of elastic equilibrium is

$$\sigma_{ij,j}=0 \tag{1}$$

where σ_{ij} is the stress tensor, a comma denotes partial differentiation with respect to the coordinate x_i. The strain ε_{ij} is related to the elastic displacements u_i through the following formula:

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \tag{2}$$

The constitutive relationship of FGMs is

$$\sigma_{ij}=c_{ijkl}\varepsilon_{kl} \tag{3}$$

where c_{ijkl} is the elastic stiffness tensor, with the interchanging symmetries $c_{ijkl}=c_{jikl}=c_{ijlk}=c_{klij}$. Unlike in a homogeneous material, c_{ijkl} is now functions of the coordinates x_i ($i=1,2,3$). In most real cases, the material property parameter is varied continuously in one direction, which is assumed to be in x_3 direction for the present analysis.

Next, an orthotropic functionally graded material was considered, for which the nonzero components of the elastic stiffness tensor are c_{1111}, c_{2222}, c_{3333}, c_{1122}, c_{1133}, c_{2233}, c_{2323}, c_{1313}, and c_{1212}. Using state space method, the following relationships in matrix form can be obtained from Eqs. (1)-(3):

$$\partial_3\begin{bmatrix}\boldsymbol{\Pi}\\ \boldsymbol{\Gamma}\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \boldsymbol{A}\\ \boldsymbol{B} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Pi}\\ \boldsymbol{\Gamma}\end{bmatrix} \tag{4}$$

$$\boldsymbol{P}=\boldsymbol{\Phi}\boldsymbol{\Lambda} \tag{5}$$

where $\partial_i=\partial/\partial x_i$, and

$$\boldsymbol{\Pi}=[u_1\quad u_2\quad \sigma_{33}]^{\mathrm{T}},\quad \boldsymbol{\Gamma}=[\sigma_{13}\quad \sigma_{23}\quad u_3]^{\mathrm{T}} \tag{6}$$

$$\boldsymbol{\Lambda}=[\boldsymbol{\Pi}\quad \boldsymbol{\Gamma}]^{\mathrm{T}},\quad \boldsymbol{P}=[\sigma_{11}\quad \sigma_{22}\quad \sigma_{12}]^{\mathrm{T}} \tag{7}$$

The operator matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{\Phi}$ contain the in-plane differential operators ∂_1 and ∂_2, and depend on x_3 only through the material moduli:

$$\boldsymbol{A}=\begin{bmatrix}c_{1313}^{-1} & 0 & -\partial_1\\ 0 & c_{2323}^{-1} & -\partial_2\\ -\partial_1 & -\partial_2 & 0\end{bmatrix} \tag{8}$$

$$\boldsymbol{B}=\begin{bmatrix}-k_1\partial_1^2-c_{1212}\partial_2^2 & k_2\partial_1\partial_2 & -k_3\partial_1\\ k_2\partial_1\partial_2 & -c_{1212}\partial_1^2-k_5\partial_2^2 & -k_4\partial_2\\ -k_3\partial_1 & -k_4\partial_2 & c_{3333}^{-1}\end{bmatrix} \tag{9}$$

$$\boldsymbol{\Phi}=\begin{bmatrix}k_1\partial_1 & k_6\partial_2 & k_3 & 0 & 0 & 0\\ k_6\partial_1 & k_5\partial_2 & k_4 & 0 & 0 & 0\\ c_{1212}\partial_2 & c_{1212}\partial_1 & 0 & 0 & 0 & 0\end{bmatrix} \tag{10}$$

where

$k_1=c_{1111}-c_{1133}^2/c_{3333}$, $k_2=c_{2233}c_{1133}/c_{3333}-c_{1122}-c_{1212}$, $k_3=c_{1133}/c_{3333}$, $k_4=c_{2233}/c_{3333}$, $k_5=c_{2222}-c_{2233}^2/c_{3333}$, and $k_6 = c_{2222} - c_{1133}c_{2233} / c_{3333}$.

For a rectangular plate that is simply supported on all its four lateral edges, the boundary conditions are given by

$$\begin{aligned} &\sigma_{11} = u_2 = u_3 = 0 \quad \text{at} \quad x_1 = 0 \text{ and } a, \\ &\sigma_{22} = u_1 = u_3 = 0 \quad \text{at} \quad x_2 = 0 \text{ and } b \end{aligned} \tag{11}$$

Boundary conditions at the top and bottom surfaces are:

at $x_3 = 0$ (bottom surface),

$$\sigma_{33}=Z^0(x_1,x_2), \sigma_{23}=Y^0(x_1,x_2), \sigma_{13}=X^0(x_1,x_2) \tag{12}$$

at $x_3 = h$ (top surface),

$$\sigma_{33}=Z^h(x_1,x_2), \sigma_{13}=X^h(x_1,x_2), \sigma_{23}=Y^h(x_1,x_2) \tag{13}$$

2 Solutions

The state variables that satisfy the boundary condition Eq. (11) can be assumed as

$$\begin{bmatrix} u_1 \\ u_2 \\ \sigma_{33} \end{bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{bmatrix} U_{mn}(x_3)\cos(m\pi x_1/a)\sin(n\pi x_2/b) \\ V_{mn}(x_3)\sin(m\pi x_1/a)\cos(n\pi x_2/b) \\ Z_{mn}(x_3)\sin(m\pi x_1/a)\sin(n\pi x_2/b) \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} \sigma_{13} \\ \sigma_{23} \\ u_3 \end{bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{bmatrix} X_{mn}(x_3)\cos(m\pi x_1/a)\sin(n\pi x_2/b) \\ Y_{mn}(x_3)\sin(m\pi x_1/a)\cos(n\pi x_2/b) \\ W_{mn}(x_3)\sin(m\pi x_1/a)\sin(n\pi x_2/b) \end{bmatrix} \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (4), the following equation is obtained,

$$\frac{\partial \boldsymbol{M}_{mn}}{\partial x_3} = \boldsymbol{K}_{mn}\boldsymbol{M}_{mn} \tag{16}$$

where

$$\boldsymbol{M}_{mn} = [U_{mn} \quad V_{mn} \quad Z_{mn} \quad X_{mn} \quad Y_{mn} \quad W_{mn}]^{\mathrm{T}} \tag{17}$$

and

$$K_{mn} = \begin{bmatrix} 0 & K_{mn}^1 \\ K_{mn}^2 & 0 \end{bmatrix} \tag{18}$$

with

$$\boldsymbol{K}_{mn}^1 = \begin{bmatrix} \dfrac{1}{c_{1313}} & 0 & -\dfrac{m\pi}{a} \\ 0 & \dfrac{1}{c_{2323}} & -\dfrac{n\pi}{b} \\ \dfrac{m\pi}{a} & \dfrac{n\pi}{b} & 0 \end{bmatrix},$$

$$\boldsymbol{K}_{mn}^2 = \begin{bmatrix} \left(c_{1111} - \dfrac{c_{1133}^2}{c_{3333}}\right)\left(\dfrac{m\pi}{a}\right)^2 + c_{1212}\left(\dfrac{n\pi}{b}\right)^2 & -\left(c_{2233}\dfrac{c_{1133}}{c_{3333}} - c_{1122} - c_{1212}\right)\dfrac{m\pi}{a}\dfrac{n\pi}{b} & -\dfrac{c_{1133}}{c_{3333}}\dfrac{m\pi}{a} \\ -\left(c_{2233}\dfrac{c_{1133}}{c_{3333}} - c_{1122} - c_{1212}\right)\dfrac{m\pi}{a}\dfrac{n\pi}{b} & c_{1212}\left(\dfrac{m\pi}{a}\right)^2 + \left(c_{2222} - \dfrac{c_{2233}^2}{c_{3333}}\right)\left(\dfrac{n\pi}{b}\right)^2 & -\dfrac{c_{2233}}{c_{3333}}\dfrac{n\pi}{b} \\ \dfrac{c_{1133}}{c_{3333}}\dfrac{m\pi}{a} & \dfrac{c_{2233}}{c_{3333}}\dfrac{n\pi}{b} & \dfrac{1}{c_{3333}} \end{bmatrix}.$$

The solution to Eq.(16) can be written as[14]

$$\boldsymbol{M}_{mn}(x_3) = \boldsymbol{T}(x_3)\boldsymbol{M}_{mn}(0) \tag{19}$$

where $\boldsymbol{T}(x_3)$ is called a transfer matrix, which can be expanded with respect to K_{mn} into a Peano-Baker series as follows[15]:

$$\boldsymbol{T}(x_3)=\boldsymbol{I}+\int_0^{x_3}\boldsymbol{K}_{mn}(\tau)\mathrm{d}\tau+\int_0^{x_3}\boldsymbol{K}_{mn}(\tau_1)\mathrm{d}\tau_1\int_0^{\tau_1}\boldsymbol{K}_{mn}(\tau_2)\mathrm{d}\tau_2+ \int_0^{x_3}\boldsymbol{K}_{mn}(\tau_1)\mathrm{d}\tau_1\int_0^{\tau_1}\boldsymbol{K}_{mn}(\tau_2)\mathrm{d}\tau_2\int_0^{\tau_2}\boldsymbol{K}_{mn}(\tau_3)\mathrm{d}\tau_3+\cdots \tag{20}$$

From Eq. (19), we can get

$$\boldsymbol{M}_{mn}(h) = \boldsymbol{T}(h)\boldsymbol{M}_{mn}(0) \tag{21}$$

If the mechanical loads on the top and bottom surfaces of the plate (see boundary conditions Eqs. (12) and (13)) can be further expanded as

$$\begin{bmatrix} X^0(x_1,x_2) \\ Y^0(x_1,x_2) \\ Z^0(x_1,x_2) \end{bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{bmatrix} X_{mn}^0\cos(m\pi x_1/a)\sin(n\pi x_1/b) \\ Y_{mn}^0\sin(m\pi x_1/a)\cos(n\pi x_1/b) \\ Z_{mn}^0\sin(m\pi x_1/a)\sin(n\pi x_1/b) \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} X^h(x_1,x_2) \\ Y^h(x_1,x_2) \\ Z^h(x_1,x_2) \end{bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{bmatrix} X_{mn}^h\cos(m\pi x_1/a)\sin(n\pi x_1/b) \\ Y_{mn}^h\sin(m\pi x_1/a)\cos(n\pi x_1/b) \\ Z_{mn}^h\sin(m\pi x_1/a)\sin(n\pi x_1/b) \end{bmatrix} \tag{23}$$

where

$$\begin{bmatrix} X_{mn}^0 \\ Y_{mn}^0 \\ Z_{mn}^0 \end{bmatrix} = \frac{4}{ab}\begin{bmatrix} \int_0^a\int_0^b X^0(x_1,x_2)\cos(m\pi x_1/a)\sin(n\pi x_2/b)\mathrm{d}x_1\mathrm{d}x_2 \\ \int_0^a\int_0^b X^0(x_1,x_2)\sin(m\pi x_1/a)\cos(n\pi x_2/b)\mathrm{d}x_1\mathrm{d}x_2 \\ \int_0^a\int_0^b X^0(x_1,x_2)\sin(m\pi x_1/a)\sin(n\pi x_2/b)\mathrm{d}x_1\mathrm{d}x_2 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} X_{mn}^h \\ Y_{mn}^h \\ Z_{mn}^h \end{bmatrix} =$$

$$\frac{4}{ab}\begin{bmatrix}\int_0^a\int_0^b X^h(x_1,x_2)\cos(m\pi x_1/a)\sin(n\pi x_2/b)\mathrm{d}x_1\mathrm{d}x_2\\ \int_0^a\int_0^b Y^h(x_1,x_2)\sin(m\pi x_1/a)\cos(n\pi x_2/b)\mathrm{d}x_1\mathrm{d}x_2\\ \int_0^a\int_0^b Z^h(x_1,x_2)\sin(m\pi x_1/a)\sin(n\pi x_2/b)\mathrm{d}x_1\mathrm{d}x_2\end{bmatrix} \quad (25)$$

Then, substituting Eqs. (14) and (15) into boundary conditions Eqs. (12) and (13), with Eqs. (22) and (23) being considered, the following equations are obtained.

$$X_{mn}(0)=X_{mn}^0, Y_{mn}(0)=Y_{mn}^0, Z_{mn}(0)=Z_{mn}^0 \quad (26)$$

$$X_{mn}(h)=X_{mn}^h, Y_{mn}(h)=Y_{mn}^h, Z_{mn}(h)=Z_{mn}^h \quad (27)$$

Substituting Eqs. (26) and (27) into Eq. (21), these six algebraic equations are solved for six unknowns, $U_{mn}(0)$, $V_{mn}(0)$, $W_{mn}(0)$, $U_{mn}(h)$, $V_{mn}(h)$, and $W_{mn}(h)$. Hence, all the components of $\boldsymbol{M}_{mn}(0)$ are obtained.

3 Numerical Examples

Example 1

In this section, FGM square plate is numerically studied ($a=b=1$, $h/a=0.1$), which is simply supported on its four lateral edges, and is made based on the above series solutions. The material chosen for the study has the material properties at $x_3=0$, as follows[16]:

$c_{1111}^0=c_{3333}^0=7.38$ GPa, $c_{2222}^0=173.41$ GPa, $c_{1122}^0=2.31$ GPa, $c_{1133}^0=1.87$ GPa, $c_{2233}^0=2.31$ GPa, $c_{2323}^0=c_{1212}^0=3.45$ GPa, and $c_{1313}^0=1.38$ GPa.

Numerical results are presented for cases of sinusoidal loading for which only one term solution is needed ($m=n=1$). The cases considered here are

$$Z^h(x_1,x_2)=Z_0\sin(\pi x_1/a)\sin(\pi x_2/b),\ Z_0=-1\text{ Pa},$$

$$Z^0(x_1,x_2)=X^h(x_1,x_2)=X^0(x_1,x_2)=$$

$$Y^h(x_1,x_2)=Y^0(x_1,x_2)=0.$$

In order to validate the correctness of the Peano-Baker series solutions, assuming that the material elastic modulus c_{ijkl} is an exponential distribution, i.e.,

$$c_{ijkl}=c_{ijkl}^0\exp(\alpha x_3/h) \quad (28)$$

where α is the material property gradient index, h is the plate thickness, c_{ijkl}^0 is the value at the plane $x_3=0$. When $\alpha=1$, in Table 1, the result is listed at a location ($x_1/a=0.25$, $x_2/b=0.25$, $x_3/h=0.25$), at the same time, listed in Table 1 also the exact solution according to Ref. [10].

Table 1 Comparison between the present solutions and the exact solutions

	u_1/(10^{-11}m)	u_2/(10^{-11}m)	u_3/(10^{-10}m)	σ_{11}/Pa	σ_{22}/(10Pa)	σ_{33}/(10^{-1}Pa)	σ_{12}/Pa	σ_{13}/(10^{-1}Pa)	σ_{23}/Pa
Exact solutions	−4.4124	−2.9648	−4.5135	1.4105	2.0928	−1.0066	−1.0252	−2.3428	−2.1684
N=5	−2.3743	−1.8596	−2.4233	7.7622	1.3100	−5.8788	−5.8838	−1.2936	−1.2836
N=7	−3.9893	−2.7344	−4.0818	1.2787	1.9297	−9.1941	−9.3438	−2.1253	−1.9851
N=9	−4.3792	−2.9468	−4.4797	1.4001	2.0800	−9.9978	−1.0181	−2.3258	−2.1541
N=11	−4.4108	−2.9639	−4.5119	1.4100	2.0922	−1.0063	−1.0248	−2.3420	−2.1678
N=13	−4.4123	−2.9647	−4.5134	1.4104	2.0927	−1.0066	−1.0252	−2.3428	−2.1684

Table 1 shows that the present solution is rapid convergence to the exact solution with increasing the truncated terms N of the Peano-Baker series in Eq. (20). When the truncated terms N of the series is 9, the relative error between the present solutions and the exact solutions is less than 1%. In the next numerical example, the truncated terms N of the series is taken as 9.

Example 2

Consider an FGM square plate ($a=b=1$ m, $h/a=0.1$), which is simply supported on its four lateral edges. The material properties at $x_3=0$ and applied loading are the same as those in Example 1. Assuming that material properties have three power distributions as Eq. (29).

$$c_{ijkl}(x_3)/c_{ijkl}^0=1+(\alpha-1)\left(\frac{x_3}{h}\right)^n \quad (29)$$

where n is taken as 1/2, 1, and 2.

The variation of displacements and stresses, as a function of the plate thickness coordinate x_3 for different material properties distribution, at a chosen location ($x_1=a/4$, $x_2=b/4$), are shown in Fig. 2. In Fig. 2, the width-to-thickness ratio is $a/h=0.1$ and the material property gradient index is $\alpha=3$. The displacement u_2, stresses σ_{22}, and σ_{23} are not depicted since their distributions along the plate thickness direction are similar to those of u_1, σ_{11}, and σ_{13}, respectively, due to the symmetry of the problem. Figure 2 shows that the transverse displacement u_3

demonstrates essentially uniform distribution along the plate thickness direction, while in-plane displacements u_1 and u_2 show the linear variations across the thickness of the plate, which means that the classical Kichhoff assumptions for homogeneous thin plate are still valid for functionally graded thin plate.

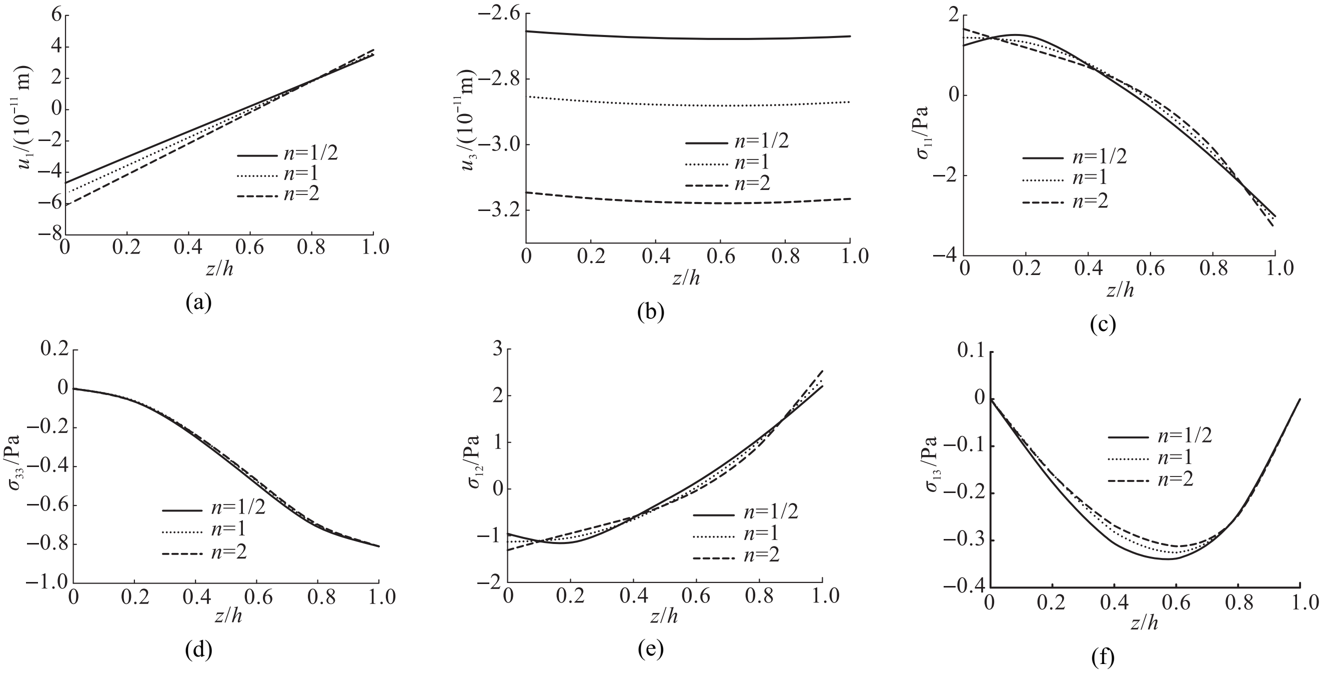

Fig. 2 Variation of physical quantities with coordinate x_3 at a location (x_1=a/4 and x_2=b/4) for different material distributions: (a) in-plane displacement u_1, (b) transverse displacement u_3, (c) in-plane normal stress σ_{11}, (d) out-of-plane normal stress σ_{33}, (e) in-plane shear stress σ_{12}, and (f) out-of-plane shear stress σ_{13}.

4 Conclusions

A three-dimensional solution based on Peano-Baker series is obtained for an FGM rectangular plate simply supported along its four edges by means of the state space method. The solution is valid for the material properties of arbitrary dependence on the thickness-coordinate of the plate.

References

[1] Praveen G N, Reddy J N. Nonlinear transient thermoelastic analysis of functionally graded ceramic-metal plates. *Int. J. Solids Struct.*, 1998, **35**(33): 4457-4476.

[2] Della C L, Venini P. Finite elements for functionally graded Ressner-Mindlin plates. *Comput. Meth. Appl. Mech. Eng.*, 2004, **193**(9-11): 705-725.

[3] Khoma I Y. Integration of the equilibrium equqtions for inhomogeneous, transversely isotropic plates. *Int. Appl. Mech.*, 2002, **38**(11): 1371-1380.

[4] Sankar B V. An elasticity solution for functionally graded beams. *Comp. Sci. Technol.*, 2001, **61**(5): 689-696.

[5] Zhong Z, Shang E T. Three-dimensional exact analysis of a simply supported functionally gradient piezoelectric plate. *Int. J. Solids Struct.*, 2003, **40**(20): 5335-5352.

[6] Vel S S, Batra R C. Three-dimensional analysis of transient thermal stress in functionally graded plates. *Int. J. Solids Struct.*, 2003, **40**(25): 7181-7196.

[7] Vel S S, Batra R C. Three-dimensional exact solution for the vibration of functionally graded rectangular plates. *J. Sound Vib.*, 2004, **272**(3-5): 703-730.

[8] Bian Z G, Chen W Q, Lim C W, et al. Analytical solution for single- and multi-span functionally graded plates in cylindrical bending. *Int. J. Solids Struct.*, 2005, **42**(24): 6433-6456.

[9] Martin P A, Richardson J D, Gray L J, et al. On Green's function for a three-dimensional exponentially graded elastic solid. *Proc. R. Soc. London Ser. A-Math. Phy. Eng. Sci.*, 2002, **458**(20-24): 1931-1947.

[10] Pan E. Exact solution for functionally graded anisotropic elastic composite laminates. *J. Compos. Mater.*, 2003, **37**(21): 1903-1919.

[11] Woo J, Meguid S A. Nonlinear analysis of functionally graded plates and shallow shell. *Int. J. Solids Struct.*, 2001, **38**(42-43): 7409-7421.

[12] Cheng Z Q, Batra R C. Three-dimensional thermoelastic

deformations of a functionally graded elliptic plate. *Composites Part B*, 2000, **31**(1): 97-106.

[13] Almajid A A, Taya M, Hudnut S. Analysis of out-of-plane displacement and stress field in a piezocomposite plate with functionally graded microstructure. *Int. J. Solids Struct.*, 2001, **38**(19): 3377-3391.

[14] Gantmacher F R. The Theory of Matrxi, vol. 2. Chelsea, New York, 1980.

[15] Dacunha J J. Transition matrix and generalized matrix exponential via the Peano-Baker series. *J. Difference Equ. Appl.*, 2005, **11**(15): 1245-1264.

[16] Ramirez F, Heyliger P R, Pan E. Static analysis of functionally graded elastic anisotropic plates using a discrete layer approach. *Composites Part B*, 2006, **37**(1): 10-20.