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Modeling and Simulation of a Flexible Inverted Pendulum System

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Abstract: This paper presents a dynamic model of a planar flexible inverted pendulum system under the frame of multi-body dynamics by floating frame of reference formulation (FFRF). By proper simplification and linearization, the state space equation of the system was established for linear analysis and control design. The simulation method for such a coupled system by multi-body dynamics program was also provided. The designed controller with a simple low-pass filter for the flexible inverted pendulum was validated by the simulation of a simple flexible pendulum sample. The result demonstrates a new method of designing and verifying a feedback controller of a flexible multi-body system.

Key words: inverted pendulum; flexible multi-body dynamics; state space; transfer function

Introduction

Inverted pendulum is a classical ideal model in the control theory as an absolute-unstable, high-order, multi-variables, and strong-coupled system. It was first predicted by Stephensen about 100 years ago that the inverted rigid pendulum system can be stabilized under the control force of a suitable high frequency, and Acheson expanded the theory to the multi-linked rigid pendulum system[1]. Chao studied the elasticity of the pendulum under the beam assumption[2] and other recent researches on the inverted pendulum concerned the system with additional appendix[3]. Due to the fast development and widely application of the control theory[4-6], the control technology derived from the inverted pendulum was also applied in many industrial and engineering products such as high-precision control of robot arm, stability control of launching rocket, and attitude control of satellite. In the recent years, the research on the human-like walking robots have also propelled the study on inverted pendulum[7-10]. Therefore, many experimental and numerical researches of the inverted pendulum system are still being carried on all over the world.

The classical inverted pendulum system was described as a rigid-body system and the elasticity of the inverted rod was neglected. However, in the modern industrial production, the flexibility of the structure and the system is more and more important due to the higher demanding of the accuracy and performance. Therefore, the control of a flexible system is greatly demanded by the modern industrial design and manufacture. Since the flexibility of the system can be easily introduced into the inverted pendulum model under the frame of multi-body dynamics[11], the study on such a model can well demonstrate the difference of the behavior between the rigid model and flexible model. Besides, due to the development of the flexible multi-body dynamics, it is feasible to verify the controller of the system by the simulation program. The related research can also help the engineer design a controller for a general flexible system.

1 Dynamic Model of a Flexible Inverted Pendulum

To describe the motion of a flexible body in multi-body dynamics by floating frame of reference formulation, a

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floating frame should be introduced and the displacement of one point on the flexible body can be decomposed into two parts, the floating frame's motion, including translation and rotation, and the vibration due to the elasticity. This decomposition is unique if the floating frame is defined.

Figure 1 illustrates the model of a typical planar inverted pendulum, where the pendulum was modeled as a flexible body. The floating frame of the flexible pendulum was defined on the undeformed pendulum and the origin of the frame was jointed to the slider. Then we can apply the Lagrange equation on the model which stands for

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = \boldsymbol{Q}_{\mathrm{e}} \tag{1}$$

The general variables of the system is selected as the translational displacement of the slider, the rotation angle of the floating frame, and the elastic coordinates of the flexible pendulum, that is

$$\boldsymbol{q} = [x \quad \varphi \quad \boldsymbol{q}_{\mathrm{f}}^{\mathrm{T}}]^{\mathrm{T}}$$

where $\boldsymbol{q}_{\mathrm{f}}$ stands for the flexible coordinates.

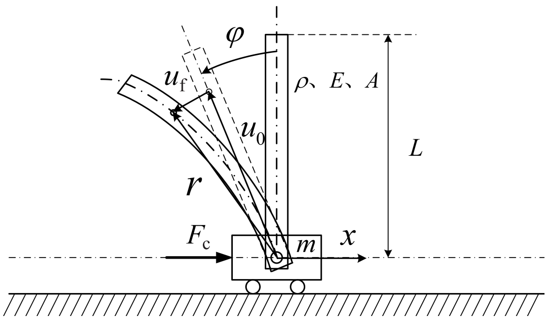

Fig. 1 Typical planar flexible inverted pendulum system

For a continuous system, the DOF of the general variables is infinite. In order to simplify the model for the control design procedure, we first introduce the modal coordinate of the flexible pendulum which can be easily got from FEM software. After that we can choose the first n modes to describe the vibration of the flexible pendulum since the high order modes influence little to the system and will decrease in a relatively short time. After the simplification, the global coordinate and velocity of point i on the pendulum can be given by the following equations:

$$\boldsymbol{r}^i = \boldsymbol{r}_0 + \boldsymbol{A}\left(\boldsymbol{u}_o^i + \sum_{j=1}^{n} \boldsymbol{N}_j^i q_{\mathrm{f},j}\right) \tag{2}$$

$$\dot{\boldsymbol{r}}^i = \dot{\boldsymbol{r}}_0 + \tilde{\boldsymbol{\omega}}\boldsymbol{A}\left(\boldsymbol{u}_o^i + \sum_{j=1}^{n} \boldsymbol{N}_j^i q_{\mathrm{f},j}\right) + \boldsymbol{A}\sum_{j=1}^{n} \boldsymbol{N}_j^i \dot{q}_{\mathrm{f},j} \tag{3}$$

where N_j stands for the shape function of the j-th mode of the flexible body.

Since the vibration of the pendulum is mainly vertical to its axis, we can assume that the elastic displacement of the pendulum is also vertical to its axis, and the Lagrangian L of the system can be described by the following equation. (The general case without the assumption can also be derived out by the following steps, although it would be a little complicated.)

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\int_0^L \rho A \dot{\boldsymbol{r}}^{\mathrm{T}}\dot{\boldsymbol{r}}\,\mathrm{d}l - \int_0^L g\rho A y\,\mathrm{d}l - \boldsymbol{K}_{\mathrm{f}}\boldsymbol{q}_{\mathrm{f}} \tag{4}$$

where y represents the vertical height of the point and $\boldsymbol{K}_{\mathrm{f}}$ represents the stiffness matrix of the flexible pendulum, which can be got from the FEM analysis. By substituting the defined general variables into the Lagrange equation, we can get the governing equation of the inverted flexible system

$$I_0\ddot{x} + (-I_1\cos\varphi + \boldsymbol{I}_3\boldsymbol{q}_{\mathrm{f}}\sin\varphi)\ddot{\varphi} - \boldsymbol{I}_3\ddot{\boldsymbol{q}}_{\mathrm{f}}\cos\varphi + (I_1\sin\varphi + \boldsymbol{I}_3\boldsymbol{q}_{\mathrm{f}}\cos\varphi)\dot{\varphi}^2 + \boldsymbol{I}_3\dot{\boldsymbol{q}}_{\mathrm{f}}\dot{\varphi}\sin\varphi = F_{\mathrm{c}} \tag{5}$$

$$(-I_1\cos\varphi + \boldsymbol{I}_3\boldsymbol{q}_{\mathrm{f}}\sin\varphi)\ddot{x} + (I_2 + \boldsymbol{q}_{\mathrm{f}}^{\mathrm{T}}\boldsymbol{I}_4\boldsymbol{q}_{\mathrm{f}})\ddot{\varphi} + \boldsymbol{I}_5\ddot{q}_{\mathrm{f}} + 2\boldsymbol{I}_4\boldsymbol{q}_{\mathrm{f}}\dot{\varphi}\dot{q}_{\mathrm{f}} - \boldsymbol{I}_3\dot{x}\dot{\boldsymbol{q}}_{\mathrm{f}}\sin\varphi - g(I_1\sin\varphi + \boldsymbol{I}_3\boldsymbol{q}_{\mathrm{f}}\cos\varphi) = \frac{1}{2}F_{\mathrm{c}}L\cos\varphi \tag{6}$$

$$-\boldsymbol{I}_3^{\mathrm{T}}\ddot{x}\cos\varphi + \boldsymbol{I}_5^{\mathrm{T}}\ddot{\varphi} + \boldsymbol{I}_4\ddot{\boldsymbol{q}}_{\mathrm{f}} - \boldsymbol{I}_4\dot{\boldsymbol{q}}_{\mathrm{f}}\dot{\varphi} - g\boldsymbol{I}_3^{\mathrm{T}}\sin\varphi + \boldsymbol{K}_{\mathrm{f}}\boldsymbol{q}_{\mathrm{f}} = \boldsymbol{N}_1^{\mathrm{T}}F_{\mathrm{c}}\cos\varphi \tag{7}$$

where

$$I_0 = \int_0^L \rho A\,\mathrm{d}l + m, \qquad I_1 = \int_0^L \rho u_0 A\,\mathrm{d}l,$$

$$I_2 = \int_0^L \rho A u_0^2\,\mathrm{d}l, \qquad \boldsymbol{I}_3 = \int_0^L \rho A\boldsymbol{N}\,\mathrm{d}l,$$

$$\boldsymbol{I}_4 = \int_0^L \rho A\boldsymbol{N}^{\mathrm{T}}\boldsymbol{N}\,\mathrm{d}l, \qquad \boldsymbol{I}_5 = \int_0^L \rho A u_0\boldsymbol{N}\,\mathrm{d}l.$$

I_0 - $\boldsymbol{I}_5$ define the six invariables of the flexible body, and $\boldsymbol{N}_1$ represents the shape function at the jointed point of the pendulum. These three equations also provide the mathematical model of the general flexible pendulum.

2 Linear State Space Equation of the Flexible Inverted Pendulum

To analyze the stability of the inverted flexible pendulum model, we should get the linear equation of the system from the governing equation. After the linearization at the balance position of the pendulum, we can get the linear equation of the system as the following form:

$$\begin{bmatrix} I_0 & -I_1 & -\boldsymbol{I}_3 \\ -I_1 & I_2 & \boldsymbol{I}_5 \\ -\boldsymbol{I}_3^{\mathrm T} & \boldsymbol{I}_5^{\mathrm T} & \boldsymbol{I}_4 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\varphi} \\ \ddot{\boldsymbol{q}}_{\mathrm f} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \mathbf{0}_{1\times n} \\ 0 & -gI_1 & -g\boldsymbol{I}_3 \\ \mathbf{0}_{n\times 1} & -g\boldsymbol{I}_3^{\mathrm T} & \boldsymbol{K}_{\mathrm f} \end{bmatrix} \begin{bmatrix} x \\ \varphi \\ \boldsymbol{q}_{\mathrm f} \end{bmatrix} - \begin{bmatrix} 1 \\ \frac{1}{2}L \\ \boldsymbol{N}_1^{\mathrm T} \end{bmatrix} F_{\mathrm c} = \mathbf{0} \tag{8}$$

To observe the status of the inverted pendulum system, sensors should be assigned on the top tip of the pendulum as well as the slider. Therefore, the orientation of the pendulum would be observed together with the vibrations.

Define the state variable of the system $\boldsymbol{X}$, output variables of the sensors $\boldsymbol{Y}$, and input variable of the system u as

$$\boldsymbol{X} = [x \quad \varphi \quad \boldsymbol{q}_{\mathrm f}^{\mathrm T} \quad \dot{x} \quad \dot{\varphi} \quad \dot{\boldsymbol{q}}_{\mathrm f}^{\mathrm T}]^{\mathrm T},$$

$$\boldsymbol{Y} = \begin{bmatrix} x_o \\ \varphi_o \\ \omega_o \end{bmatrix}, \qquad u = F_{\mathrm c},$$

then the linear state equation of the system can be easily formulated as the following standard form[12]:

$$\dot{\boldsymbol{X}} = \boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{X} + \boldsymbol{H}^{-1}\boldsymbol{B}u, \qquad \boldsymbol{Y} = \boldsymbol{C}\boldsymbol{X} \tag{9}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & \mathbf{0}_{1\times n} & 1 & 0 & \mathbf{0}_{1\times n} \\ 0 & 0 & \mathbf{0}_{1\times n} & 0 & 1 & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times 1} & \boldsymbol{I}_{n\times n} \\ 0 & 0 & \mathbf{0}_{1\times n} & 0 & 0 & \mathbf{0}_{1\times n} \\ 0 & -gI_1 & -g\boldsymbol{I}_3 & 0 & 0 & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & -g\boldsymbol{I}_3^{\mathrm T} & \boldsymbol{K}_{\mathrm f} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & \frac{1}{2}L & \boldsymbol{N}_1 & 0 & 0 & \mathbf{0}_{1\times n} \end{bmatrix}^{\mathrm T},$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \boldsymbol{N}_o' & 0 & 0 & \mathbf{0}_{1\times n} \\ 0 & 0 & \mathbf{0}_{1\times n} & 0 & 1 & \boldsymbol{N}_o' \end{bmatrix},$$

and

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 & \mathbf{0}_{1\times n} & 0 & 0 & \mathbf{0}_{1\times n} \\ 0 & 1 & 0_{1\times n} & 0 & 0 & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times 1} & \boldsymbol{I}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} \\ 0 & 0 & \mathbf{0}_{1\times n} & I_0 & -I_1 & -\boldsymbol{I}_3 \\ 0 & 0 & \mathbf{0}_{1\times n} & -I_1 & I_2 & \boldsymbol{I}_5 \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} & -\boldsymbol{I}_3^{\mathrm T} & \boldsymbol{I}_5^{\mathrm T} & \boldsymbol{I}_4 \end{bmatrix}.$$

Where N_o' stands for the gradient of shape function of the pendulum at the tip top. Many methods to design a proper controller for such a system were introduced in the textbooks of control system[12,13]. Therefore, it would be much easier to design a feedback controller if the parameters of the system are available.

Although the dimension number of the matrix is still large due to the flexibility, only the low frequency motion is concerned in the control design procedure since the high-frequency response of the system would decrease quickly due to the structural damping. So it is a practical way to discard the high frequency modes in the design procedure of the controller, and include them in the numerical simulation procedure to verify the controller.

3 Simulation of the System by Multi-body Dynamics Program

To verify the effect of the controller, we can simulate the whole system with control under the frame of multi-body dynamics, and the flowchart of the simulation was illustrated in Fig. 2. The transfer function of the controller was integrated together with the dynamic equation of the inverted pendulum system at each time step and the observed variables of the controller were got from the sensors in the dynamic system. Besides, the output of the controller is transformed to the control force applied on the dynamic system, which establish a closed-loop control simulation.

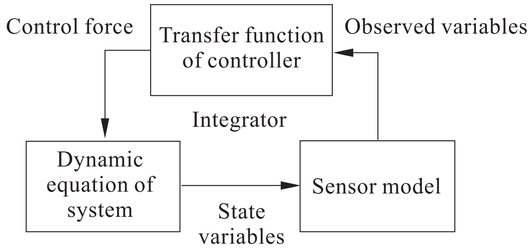

Fig. 2 Flowchart of the simulation procedure

Here we introduce a simple dimensionless inverted-pendulum sample to verify the designed controller and simulations. Set the initial orientation of the beam to be $\pi/6$ at the start of the simulation, then the parameters were listed in the Table 1.

Table 1 Property of the model (dimensionless)

m	L	ρ	EA	EI	ζ/%
1	1	7.8×10^3	4.2×10^8	8.75×10^2	0.1

The frequency of the first mode is 25.92 and we can design the controller by two steps.

First we exclude the flexible modes from the model and design a simple feedback control based on the sensor, which lead to the feedback control force as

$$F_c = -500\varphi_o - 5\omega_o.$$

The orientation of the pendulum was stabled under such control force (Fig. 3). However, the feedback control above would excite the vibration of the pendulum when the vibration modes were included into the model (Fig. 4). Figure 5 shows the vibration shape and orientation of the pendulum at different time.

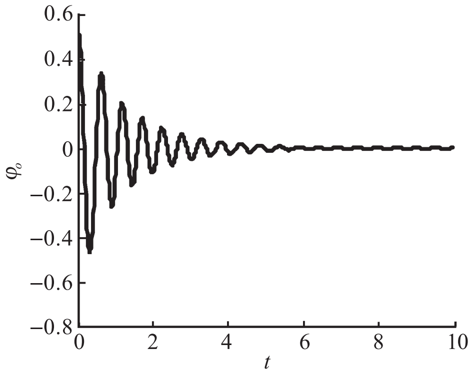

Fig. 3 Output of φ_o in rigid system

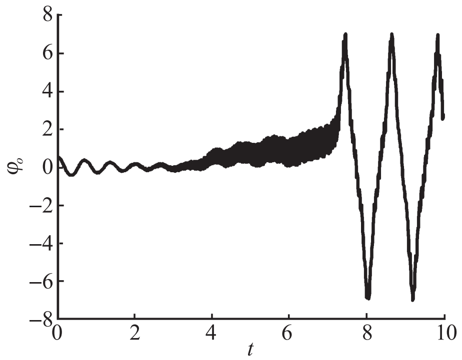

Fig. 4 Output of φ_o in flexible system

Since the existence of the structural damping, we can insert a low-pass filter after the feedback unit, which should guarantee the positive gain on the low frequency. The transfer function is presented below.

$$G(s) = \frac{62.5(s+10)}{s^2+50s+625} \tag{10}$$

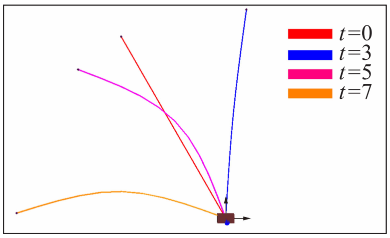

Fig. 5 Vibration shape and orientation of the pendulum at different time

By introducing the filter above, the output attitude of the pendulum is forced to be stable again (Fig. 6). Besides, the response of the pendulum's first mode (Fig. 7) depicts that the flexible vibration also decreased due to the structure damping and did not influence the stability of the pendulum. Figure 8 shows the shape and orientation of the pendulum at different seconds.

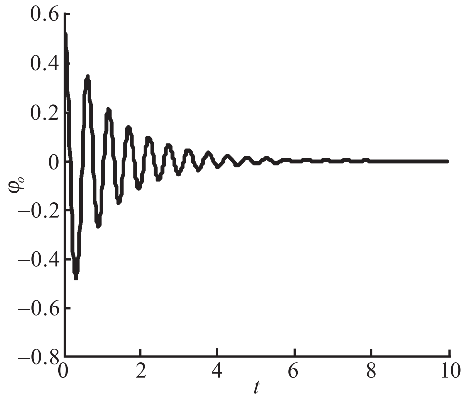

Fig. 6 Output of φ_o in flexible system

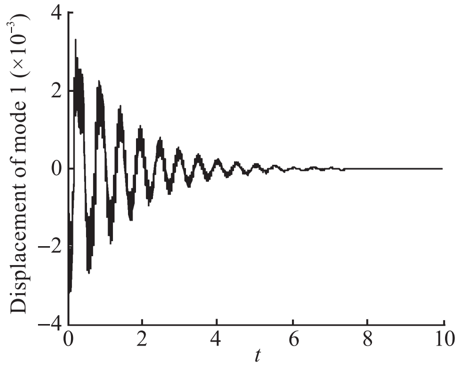

Fig. 7 Response of the first mode of the pendulum

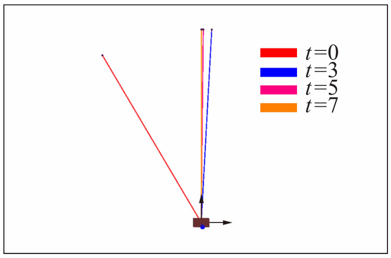

Fig. 8 Status of the pendulum at different time

4 Conclusions

By adopting the floating frame of reference formulation, a dynamic model of the general inverted flexible pendulum was derived under the frame of flexible multi-body dynamics, and the related linear governing equation was established by a few proper simplifications, which lead to a practical and feasible way to design the feedback controller of the flexible coupled system. The simulation method of the inverted pendulum by multi-body dynamics program was also provided and a simple sample was introduced to verify the related procedure. The simulation result of the flexible inverted pendulum shows that the traditional control strategy based on rigid inverted pendulum may have trouble due to the flexibility of the pendulum and the low-pass filter would take effect to keep the system stable.

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