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Vibration Analysis of Timoshenko Beams on a Nonlinear Elastic Foundation*

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Abstract: The vibrations of beams on a nonlinear elastic foundation were analyzed considering the effects of transverse shear deformation and the rotational inertia of beams. A weak form quadrature element method (QEM) is used for the vibration analysis. The fundamental frequencies of beams are presented for various slenderness ratios and nonlinear foundation parameters for both slender and short beams. The results for slender beams compare well with finite element results. The analysis shows that the transverse shear deformation and the nonlinear foundation parameter significantly affect the fundamental frequency of the beams.

Key words: weak form quadrature element method (QEM); nonlinear foundation; vibration; Timoshenko beam

Introduction

Beams resting on elastic foundations have been studied extensively over the years[1-6], with the Winkler foundation model normally used as a simple, effective method to describe the interactions between the structures and the foundation. Soils are a typical elastic foundation characterized by high nonlinearity. Although some efforts have been made to investigate the response of beams on nonlinear foundations[7-9], all these publications are based on Bernoulli-Euler beam theory which over-predicts the vibrational frequencies when transverse shear deformation and the rotational inertia of the beams are significant. Timoshenko beam theory[10], which takes into account transverse shear deformation and the rotational inertia, has been widely used to analyze short beams. While there has been considerable research on Timoshenko beams on elastic foundations[3,4,6,11,12], to the authors' knowledge, there have been no studies of Timoshenko beams on nonlinear elastic foundations.

This analysis applies the weak-form quadrature element method (QEM)[13] to a vibration analysis of Timoshenko beams on nonlinear elastic foundations. The accuracy of the weak-form quadrature element method is demonstrated by comparing the results with an available analytical solution. The effects of the shear deformation and nonlinear foundation parameter on the frequency of the Timoshenko beam vibrations are discussed in detail.

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1 Formulation

1.1 Hamilton principle for Timoshenko beams

Consider a uniform Timoshenko beam on a nonlinear elastic foundation. The length of the beam is L. Since the geometry is regular, only one quadrature element is used in the analysis. Assume that the two displacement components, the deflection and the cross-sectional rotation for the beam are $\bar{w}$ and $\bar{\varphi}$, with $\bar{\boldsymbol{d}}$ as the nodal displacement vector of the element,

$$\bar{\boldsymbol{d}}^{\mathrm{T}} = [\bar{w}_1 \quad \bar{\varphi}_1 \quad \bar{w}_2 \quad \dots \quad \bar{w}_N \quad \bar{\varphi}_N] \tag{1}$$

where N is the number of nodes in the analysis. The kinetic energy is

$$T=\frac{1}{2}\int_0^L(\rho I\dot{\bar{\varphi}}^2+\rho A\dot{\bar{w}}^2)\mathrm{d}x=\frac{1}{2}\int_{-1}^{+1}(\rho I\dot{\bar{\varphi}}^2+\rho A\dot{\bar{w}}^2)\frac{L}{2}\mathrm{d}\xi \tag{2}$$

where ρ is the beam mass density, I is the moment of inertia, A is the cross-sectional area, ξ is the non-dimensional coordinate defined as $\xi=-1+2x/L$, and the single dot represents differentiation with respect to time t. The quadrature element analysis approximates the integrand in the first term. The kinetic energy can be approximated by Lagrangian interpolations,

$$\rho I\dot{\bar{\varphi}}^2=\sum_{i=1}^N l_i(\xi)(\rho I\dot{\bar{\varphi}}^2)_i \tag{3a}$$

$$\rho A\dot{\bar{w}}^2=\sum_{i=1}^N l_i(\xi)(\rho A\dot{\bar{w}}^2)_i \tag{3b}$$

where $l_i(\xi)$ are Lagrange interpolation functions.

The kinetic energy in an element is rewritten as

$$T=\frac{1}{2}\dot{\bar{\boldsymbol{d}}}^{\mathrm{T}}\sum_{i=1}^N\boldsymbol{m}_i\int_{-1}^{+1}l_i(\xi)\mathrm{d}\xi\dot{\bar{\boldsymbol{d}}}=\frac{1}{2}\dot{\bar{\boldsymbol{d}}}^{\mathrm{T}}\boldsymbol{M}\dot{\bar{\boldsymbol{d}}} \tag{4}$$

where the element mass matrix is

$$\boldsymbol{M}=\sum_{i=1}^N\boldsymbol{m}_i\int_{-1}^{+1}l_i(\xi)\mathrm{d}\xi \tag{5}$$

and the diagonal matrices

$$\boldsymbol{m}_i=\frac{\rho_i L}{2}\begin{bmatrix}\ddots & & & \\ & A_i & & \\ & & I_i & \\ & & & \ddots\end{bmatrix},\quad i=1,\ldots,N \tag{6}$$

are zero matrices except for the two diagonal entries. The beam strain energy is

$$U_{\mathrm{s}}=\int_L\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{D}_0\boldsymbol{\varepsilon}\mathrm{d}x=\int_{-1}^{+1}V_0\mathrm{d}\xi,\quad V_0=\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{D}_0\boldsymbol{\varepsilon}\frac{L}{2} \tag{7}$$

where the strain components are

$$\boldsymbol{\varepsilon}^{\mathrm{T}}=\begin{bmatrix}\frac{\mathrm{d}\bar{\varphi}}{\mathrm{d}x} & \frac{\mathrm{d}\bar{w}}{\mathrm{d}x}-\bar{\varphi}\end{bmatrix}=\begin{bmatrix}\frac{2}{L}\frac{\mathrm{d}\bar{\varphi}}{\mathrm{d}\xi} & \frac{2}{L}\frac{\mathrm{d}\bar{w}}{\mathrm{d}\xi}-\bar{\varphi}\end{bmatrix} \tag{8}$$

The elasticity matrix in Eq. (7) for an isotropic material is

$$\boldsymbol{D}_0=\begin{bmatrix}EI & 0\\ 0 & \kappa GA\end{bmatrix} \tag{9}$$

where E is the elastic modulus, G is the shear modulus, and κ is the shear correction factor. The differential quadrature analog of the derivatives by approximating the strain energy density in Eq. (7) in terms of its values at all nodes in the beam. Again, the Lagrangian interpolation functions are chosen,

$$V_0=\sum_{i=1}^N l_i(\xi)V_{0i} \tag{10}$$

Assume that the foundation stiffness is

$$k_{\mathrm{f}}=\alpha+\beta\bar{w}^2 \tag{11}$$

where α and β are the linear and nonlinear stiffness coefficients of the foundation.

The strain energy of the foundation is

$$\begin{cases}U_{\mathrm{f}}=\int_l\left(\frac{\alpha}{2}\bar{w}^2+\frac{\beta}{2}\bar{w}^4\right)\mathrm{d}x=\int_{-1}^{+1}(V_1+V_2)\mathrm{d}\xi,\\ V_1=\frac{\alpha}{2}\bar{w}^2\frac{L}{2},\quad V_2=\frac{\beta}{2}\bar{w}^4\frac{L}{2}\end{cases} \tag{12}$$

Similarly, Lagrangian interpolations are used to approximate the integrand in Eq. (12),

$$\begin{cases}V_1=\sum_{i=1}^N l_i(\xi)V_{1i}=\sum_{i=1}^N l_i(\xi)\frac{\alpha}{2}\bar{w}_i^2\frac{L}{2}=\\ \quad\frac{1}{2}\bar{\boldsymbol{d}}^{\mathrm{T}}\frac{\alpha L}{2}\sum_{i=1}^N l_i(\xi)\boldsymbol{H}_i\bar{\boldsymbol{d}},\\ V_2=\sum_{i=1}^N l_i(\xi)V_{2i}=\sum_{i=1}^N l_i(\xi)\frac{\beta}{2}\bar{w}_i^4\frac{L}{2}=\\ \quad\frac{1}{2}\bar{\boldsymbol{d}}^{\mathrm{T}}\left(\frac{\beta L}{2}\sum_{i=1}^N l_i(\xi)\boldsymbol{H}_i^{\mathrm{T}}\bar{\boldsymbol{d}}\bar{\boldsymbol{d}}^{\mathrm{T}}\boldsymbol{H}_i\right)\bar{\boldsymbol{d}}\end{cases} \tag{13}$$

where

$$\boldsymbol{H}_i=\begin{bmatrix}\ddots & & \\ & 1 & \\ & & \ddots\end{bmatrix},\quad i=1,\ldots,N \tag{14}$$

are zero matrices except for the one unit diagonal entry.

1.2 Differential quadrature

With the differential quadrature analog of the derivative[14,15], the strain components at a node are

$$\boldsymbol{\varepsilon}_i=\begin{bmatrix}\frac{2}{L}\sum_{j=1}^N C_{ij}^{(\xi)}\bar{\varphi}_j & \frac{2}{L}\sum_{j=1}^N C_{ij}^{(\xi)}\bar{w}_j-\bar{\varphi}_i\end{bmatrix}^{\mathrm{T}}=\boldsymbol{B}_i\bar{\boldsymbol{d}},\ i=1,\ldots,N \tag{15}$$

where $\boldsymbol{B}_i$ is a nodal strain matrix of the element. The weighting coefficients $C_{ij}^{(\xi)}$ for the first order derivatives are obtained using the method of Quan and Chang[16] further elaborated by Shu and Richards[17],

$$\begin{cases}C_{ik}^{(\xi)}=\dfrac{\prod_{j=1,j\neq i}^N(\xi_i-\xi_j)}{(\xi_i-\xi_k)\prod_{j=1,j\neq k}^N(\xi_k-\xi_j)}, & k=1,\ldots,N,\ \text{and}\ i\neq k;\\ C_{ii}^{(\xi)}=-\sum_{j=1,j\neq i}^N C_{ij}^{(\xi)}\end{cases} \tag{16}$$

Since the Timoshenko beam theory is C_0 continuous, the weighting coefficients for higher order derivatives are not needed.

To preserve the high order features of the solution, non-uniform node patterns are used in the quadrilateral quadrature element. The Lobatto non-uniform node distribution in the normalized range[18] is

$$-1=\xi_1,\ \ldots,\ \xi_i,\ \ldots,\ \xi_N=1,\quad i=2,\ldots,N-1 \tag{17}$$

where ξ_i is the (i–1)-th zero of

$$\frac{\mathrm{d}P_{N-1}(\xi)}{\mathrm{d}\xi}=0 \tag{18}$$

where $P_{N-1}(\xi)$ is the $(N-1)$-th order Legendre polynomial. Then, the strain energy density at a node is expressed as

$$\begin{cases} V_{0i}=\dfrac{1}{2}\bar{\boldsymbol{d}}^{\mathrm{T}}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{D}_{0i}\boldsymbol{B}_i\bar{\boldsymbol{d}}=\dfrac{1}{2}\bar{\boldsymbol{d}}^{\mathrm{T}}\boldsymbol{D}_i\bar{\boldsymbol{d}}, \\ \boldsymbol{D}_i=\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{D}_{0i}\boldsymbol{B}_i \end{cases} \tag{19}$$

which leads to

$$\begin{cases} U_{\mathrm{s}}=\dfrac{1}{2}\bar{\boldsymbol{d}}^{\mathrm{T}}\boldsymbol{K}_{\mathrm{s}}\bar{\boldsymbol{d}}, \\ \boldsymbol{K}_{\mathrm{s}}=\displaystyle\sum_{i=1}^{N}\boldsymbol{D}_i\int_{-1}^{+1}l_i(\xi)\mathrm{d}\xi \end{cases} \tag{20}$$

Ideally, the numerical integration uses the same node pattern as used to approximate the derivatives for the highest accuracy. With numerical integration, the element stiffness matrix in Eq. (20) becomes

$$\boldsymbol{K}_{\mathrm{s}}=\sum_{i=1}^{N}\boldsymbol{D}_iW_i \tag{21}$$

where W_i are the weight coefficients for the numerical integration. For the Gauss-Lobatto rule,

$$\int_{-1}^{1}l_i(\xi)\mathrm{d}\xi=W_1l_i(-1)+W_Nl_i(1)+\sum_{k=2}^{N-1}W_kl_i(\xi_k)+R_N[l_i] \tag{22}$$

where $R_N[l_i]$ is the error of order $2N-2$ and

$$\begin{cases} W_1=W_N=\dfrac{2}{N(N-1)}, \\ W_k=\dfrac{2}{N(N-1)[P_{N-1}(\xi_k)]^2},\quad k=2,\ldots,N-1 \end{cases} \tag{23}$$

Similarly,

$$U_{\mathrm{f}}=\int_{-1}^{+1}(V_1+V_2)\mathrm{d}\xi=\frac{1}{2}\bar{\boldsymbol{d}}^{\mathrm{T}}(\boldsymbol{K}_{\mathrm{f}1}+\boldsymbol{K}_{\mathrm{f}2})\bar{\boldsymbol{d}} \tag{24}$$

with

$$\boldsymbol{K}_{\mathrm{f}1}=\frac{\alpha L}{2}\sum_{i=1}^{N}W_i\boldsymbol{H}_i\quad\text{and}\quad\boldsymbol{K}_{\mathrm{f}2}=\frac{\beta L}{2}\sum_{i=1}^{N}W_i\boldsymbol{H}_i^{\mathrm{T}}\bar{\boldsymbol{d}}\bar{\boldsymbol{d}}^{\mathrm{T}}\boldsymbol{H}_i \tag{25}$$

Likewise, the mass matrix in Eq. (5) is

$$\boldsymbol{M}=\sum_{i=1}^{N}\boldsymbol{m}_iW_i \tag{26}$$

Applying Hamilton's principle,

$$\delta\int_{t_1}^{t_2}(U_{\mathrm{s}}+U_{\mathrm{f}}-T)\,\mathrm{d}t=0 \tag{27}$$

one has

$$(\boldsymbol{K}_{\mathrm{s}}+\boldsymbol{K}_{\mathrm{f}1}+2\boldsymbol{K}_{\mathrm{f}2})\bar{\boldsymbol{d}}+\boldsymbol{M}\ddot{\bar{\boldsymbol{d}}}=\boldsymbol{0} \tag{28}$$

where the double dot represents second order differentiation with respect to time.

A simply supported beam allows separation of variables in space (x) and time (t)[19]. Assume that

$$\bar{\boldsymbol{d}}=\boldsymbol{d}\cos(\omega t) \tag{29}$$

where ω is the nonlinear circular frequency of the beam. Using the Galerkin method[20] and carrying out the integration from 0 to $2\pi/\omega$, Eq. (28) then becomes

$$(\boldsymbol{K}_{\mathrm{s}}+\boldsymbol{K}_{\mathrm{f}1}+\frac{3}{2}\boldsymbol{K}_{\mathrm{f}2}-\omega^2\boldsymbol{M})\boldsymbol{d}=\boldsymbol{0} \tag{30}$$

The boundary conditions for a simply supported beam, $w_1=w_N=0$ can be used to eliminate the corresponding rows and columns of the matrices in Eq. (30), which then becomes

$$(\tilde{\boldsymbol{K}}_{\mathrm{s}}+\tilde{\boldsymbol{K}}_{\mathrm{f}1}+\frac{3}{2}\tilde{\boldsymbol{K}}_{\mathrm{f}2}-\omega^2\tilde{\boldsymbol{M}})\tilde{\boldsymbol{d}}=\boldsymbol{0} \tag{31}$$

Since $\tilde{\boldsymbol{K}}_{\mathrm{f}2}$ contains a displacement vector, the problem is nonlinear. The frequencies are extracted from Eq. (31) using an iterative algorithm[21].

2 Numerical Results

The following non-dimensional parameters are introduced to the results:

$$\bar{\omega}=\omega\sqrt{\rho AL^4/EI},\ \ \bar{\alpha}=\alpha L^4/EI,\ \ \bar{\beta}=\beta L^6/EI,$$
$$S_{\mathrm{r}}=\sqrt{AL^2/I} \tag{32}$$

where S_{r} is the beam slenderness ratio.

2.1 Verification of convergence

A slender beam with $S_{\mathrm{r}}=1000$ and $\kappa=3/2$ was analyzed to verify the convergence of the method. A closed form solution for the nonlinear frequency for a simply supported slender beam where the Bernoulli-Euler theory holds is[22]

$$\bar{\omega}=\left[\pi^4+\bar{\alpha}+\frac{3}{2}\bar{\beta}(a/L)^2\right]^{1/2} \tag{33}$$

where a is the vibration amplitude of the beam. The present analysis needs only one element since the quadrature element method is a global approximation technique. The predicted nonlinear fundamental frequencies for three amplitudes are listed in Table 1.

Table 1 Convergence of non-dimensional fundamental frequency $\bar{\varpi}$ of a slender ratio beam on nonlinear elastic foundation

a/L	N	$\bar{\varpi}$		
		$\bar{\beta}=10$	$\bar{\beta}=100$	$\bar{\beta}=1000$
0.02	5	9.8694	9.8710	9.8874
	6	9.8727	9.8748	9.8958
	7	9.8697	9.8715	9.8897
	8	9.8697	9.8718	9.8927
	9	9.8697	9.8716	9.8907
	10	9.8697	9.8718	9.8926
	11	9.8697	9.8717	9.8913
	Analytical[22]	9.8699	9.8726	9.8999
0.06	5	9.8708	9.8856	10.0321
	6	9.8745	9.8935	10.0807
	7	9.8713	9.8877	10.0496
	8	9.8716	9.8904	10.0765
	9	9.8714	9.8886	10.0590
	10	9.8716	9.8902	10.0750
	11	9.8715	9.8891	10.0637
	Analytical[22]	9.8723	9.8969	10.1395
0.10	5	9.8738	9.9147	10.3154
	6	9.8783	9.9308	10.4397
	7	9.8745	9.9199	10.3621
	8	9.8753	9.9274	10.4337
	9	9.8748	9.9225	10.3875
	10	9.8753	9.9270	10.4302
	11	9.8749	9.9238	10.4000
	12	9.8752	9.9268	10.4283
	Analytical[22]	9.8772	9.9453	10.6023

The convergence rate as N is increased decreases somewhat with the increasing of vibration amplitude and nonlinear effect. Despite the nonlinear effect on the convergence rate, the converged frequencies show excellent agreement with the analytical results. For the largest nonlinear effect, $\bar{\beta}=1000$, and the largest amplitude, $a/L=0.1$, the relative error for the nonlinear fundamental frequency is reduced by less than 0.4% when the number of nodes N is increased to 11, highlighting the good computational efficiency of the present method. To ensure the accuracy of the results, all the remaining computations were performed using $N=15$.

2.2 Vibration analysis

The effects of shear deformation and rotational inertia on the fundamental frequency are shown in Fig. 1, for $\bar{\alpha}=0, \bar{\beta}=100$, and a/L=0.1. For beams with large slenderness ratios, the transverse shear and rotational inertia have little effect on the nonlinear frequency. Thus, the Bernoulli-Euler beam theory can accurately predict the frequencies. For beams with small slenderness ratios, however, the nonlinear frequency is significantly smaller than that predicted by the Bernoulli-Euler theory. Smaller slenderness ratios result in more significant effects of shear deformation and rotational inertia on the frequency. For example, the nonlinear frequency is more than 4% less than that given by the Bernoulli-Euler theory for the slenderness ratios less than 15.

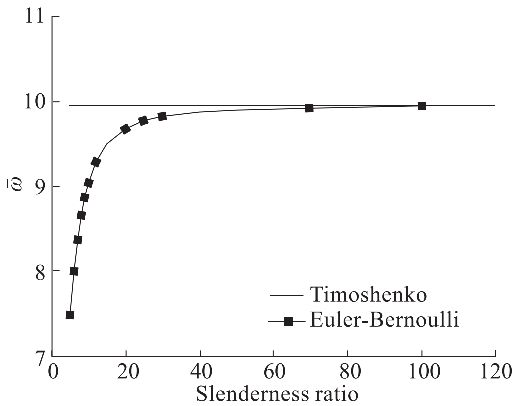

Fig. 1 Frequency comparison of two beam theories

The effect of nonlinear foundation parameters on the fundamental frequency is shown by the ratio of the predicted nonlinear fundamental frequency to the linear fundamental frequency as shown in Fig. 2 for $S_r=10$ and $\bar{\alpha}=10$. The results show that the nonlinear foundation parameter has a significant effect on the beam frequency, with larger deviations from the linear foundation ($\bar{\beta}=0$) resulting in much larger deviations of ratio of the nonlinear fundamental frequency to linear fundamental frequency $\bar{\varpi}/\bar{\varpi}_L$. Positive nonlinear foundation parameters have a "hardening" effect on the frequency while negative foundation parameters have a "softening" effect on the frequency. Nevertheless, the linear foundation parameter $\bar{\alpha}$ is still dominant in comparison with the nonlinear foundation parameter $\bar{\beta}$ since changes of $\bar{\alpha}$ are shown to have more significant effects on the fundamental frequency than changes of $\bar{\beta}$.

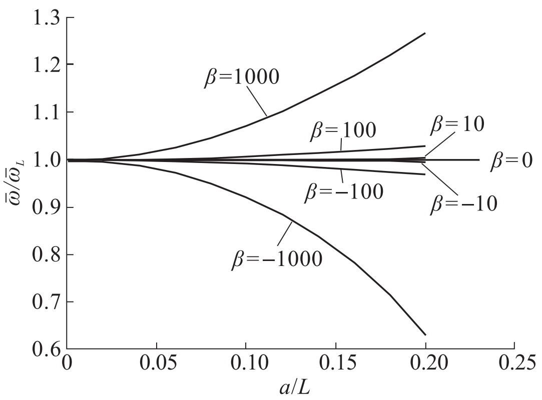

Fig. 2 Nonlinear effects on fundamental frequency

3 Concluding Remarks

The vibrations of beams on nonlinear foundations were studied on the basis of Timoshenko beam theory using the weak form quadrature element formulation for vibration analysis of beams on nonlinear foundations. The results show the excellent computational efficiency of the weak form quadrature element method. Excellent agreement with available solutions is achieved with fewer degrees of freedom. The effects of transverse shear deformation and the nonlinear foundation parameter on the vibration of beams are illustrated.

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