TSINGHUA SCIENCE AND TECHNOLOGY
ISSN 1007-0214 19/49 pp105-109
Volume 12, Number S1, July 2007

Diagnosability of the Incomplete Star Graphs*

ZHENG Shuxia (郑淑霞), ZHOU Shuming (周书明)**

Key Laboratory of Network Security and Cryptology, Fujian Normal University, Fuzhou 350007, China

Abstract: The growing size of the multiprocessor systems increases their vulnerability to component failures. It is crucial to local and to replace the fault processors to maintain system's high reliability. The fault diagnosis is the process of identifying faulty processors in a system through testing. This paper establishes the diagnosabilities of the incomplete star graph S_n ($n\geqslant 4$) with missing links under the PMC model and its variant, the BGM model, and shows that the diagnosabilities of incomplete star graph S_n under these two diagnostic models can be determined by the minimum degree of its topology structure. This method can also be applied to the other existing multiprocessor systems.

Key words: diagnosability; incomplete star graph; PMC model; BGM model

Introduction

In recent years, large-scale distributed and loosely coupled multiprocessor systems have been developed for many critical applications in military, commercial, and scientific computing. The growing size of the systems increases their vulnerability to component failures. To maintain systems high availability, it is crucial to locate the faulty processors therein efficiently and then replace them with spare ones. The process of identifying fault processors in a system by analyzing the outcomes of available inter-processor tests is system-level diagnosis. The diagnosability of the system is defined as the maximum number t such that the system is diagnosable as long as the number of the faulty processors is not greater than t. The foundation of system diagnosis and an original diagnostic model, namely the PMC model were established in a classic paper by Preparata et al.[1] Under the PMC model, all tests are performed between two adjacent processors, and it was assumed that the test is reliable (respectively, unreliable) if the processor that initiates the test is fault-free (respectively, unreliable).

Inspired by the work of Preparata et al.[1], Baris et al.[2] proposed another diagnostic model, referred to as the BGM model. This model uses the same testing strategy as the PMC model, but it assumes that a faulty unit is always tested as faulty regardless of the state of the tester. The rational is that tests consist of long sequences of stimuli and testing a faulty unit is very likely to result in at least one mismatch, even if the tester is also faulty. Thus, a test that produces an outcome 0 implies that the tested unit is fault-free, and a test that produces an outcome 1 implies that at least one of the involved unit is faulty.

Many topologies have been proposed to interconnect processors in a multiprocessor system. Among them, the hypercube has drawn the greatest attention, since it possesses many attractive properties such as low degree, small diameter, symmetry, high fault tolerance, and efficient routing algorithms. In the past half century, the hypercube and its various important variants, such as the crossed cubes, the mobius cubes, the twisted cubes[3,4], and the enhanced cubes[5] have

Received: 2007-02-01
* Supported by the Foundation of Fujian Provincial Department of Science & Technology (No. 2006F5035) and the National Natural Science Foundation of China (No. 60502047)
** To whom correspondence should be addressed.
E-mail: zhoushuming@sina.com.cn; Tel: 86-1489137190

received extensive investigation while they are regular structures. To study the diagnosability of hypercube in the presence of arbitrarily distributed missing links, Wang[6] established the diagnosability of incomplete hypercube under the PMC model and the BGM model. In search of viable or even better alternative for hypercube, another family of regular structure, called the star graphs[7, 8] seem to enjoy most of the desirable properties of the hypercubes at considerably less cost, they accommodates more nodes with less interconnection hardware and less communication delay. It has been shown that many parallel algorithms can be efficiently mapped on star graphs. The diagnosabilities of complete star graph under three strategies based on the PMC model were studied by Kavianpour[9]. Zheng et al.[10] investigated the diagnosability of complete star graph S_n under comparison diagnostic model. In this paper, we establish the diagnosability of incomplete star graph S_n ($n \geqslant 4$) under the PMC diagnostic model and its variant, the BGM model. This idea was attributed to Wang[6] who is the first to discuss the diagnosability of non-regular structure.

1 Preliminaries

In the study of multiprocessor systems, the topology of a system is often adequately represented by a graph $G=(V,E)$, where each node $u \in V$ denotes a processor and each edge $(u,v) \in E$ denotes a link between nodes u and v. Throughout this paper we use a graph $G=(V,E)$ to present a self-diagnosable system. For any subset $S \subset V$, the notation G-S represents the graph obtained by removing the vertices in S from G and deleting those edges with at least one end vertex in S simultaneously. If G-S is disconnect, then S is called a vertex cut or a separating set. For a node u of G, denoted by $N(u)$ the set of all its neighboring nodes, i.e., $N(u)=\{u \in V \mid (u,v) \in E\}$. For any subset $S \subset V$, let $N(S)=\bigcup_{u \in V} N(u)-S$. The connectivity $\kappa(G)$ of a graph $G=(V,E)$ is the minimum number of nodes whose removal results in a disconnected or a trivial (one node) graph. A graph G is k-connectivity if $\kappa(G) \geqslant k$.

A star graph S_n of n dimensions is defined to be a symmetric graph $G=(V,E)$ with v being the set of all permutations of $\{1,2,\cdots,n\}$ and E consisting of the symmetric edges (u,v) such that two permutations u and v are connected by an edge iff one can be reached from the other by interchanging its first symbol with the i-th symbol, i.e., $u=a_1a_2\cdots a_{i-1}a_ia_{i+1}\cdots a_n$, $v=a_ia_2\cdots a_{i-1}a_1a_{i+1}\cdots a_n$, $2 \leqslant i \leqslant n$. So the regular degree of the star graph S_n is $n-1$. Each star graph S_n can be decomposed into n sub-star graphs, which are isomorphic to S_{n-1}. The crossing edges between sub-star graphs constitute a perfect matching.

Fault diagnosis is an important step in the design of multiprocessor systems. In the classic PMC multiprocessor diagnostic model, the self-diagnosable system is represented by a directed graph $G=(V,A)$, or digraph for short, in which a node u can test all nodes v if arrow $(u \to v) \in A$, in which u is called the tester and v is called the tested vertex. The outcome of a test (u,v) is 1 (respectively, 0) if u evaluates v as faulty (respectively, fault-free). An undirected graph $G=(V,E)$ is a special digraph in which $(u \to v) \in A$ if and only if $(v \to u) \in A$. This means that two linked nodes can perform tests on each other. In order to diagnose faults, a number of tests are performed among adjacent processors and the collection of all test results is referred to as a syndrome. For a given syndrome σ, a subset of vertices $F \subset V$ is said to be consistent with σ if syndrome σ can be produced from the situation that for any $(u \to v) \in A$, such that $u \in V-F$, $\sigma(u,v)=1$ iff $v \in F$. It is worthy pointing out that for a given syndrome σ, there may be more than one subset F of V that are consistent with σ. If this happens, the system cannot diagnose for syndrome σ, because a faulty tester can lead to unreliable result, so a given set F of fault vertices may produce different syndromes. It is clear that under the PMC model, there must be some (at least one) good processors to correctly perform diagnosis.

Definition 1[3] A system $G=G(V,E)$ is t-diagnosable (or has diagnosability t) if for any syndrome σ, there is one and only one faulty subset $F \subset V$ that consists with σ given that the number of faulty vertices does not exceed t.

The following two results related to t-diagnosable systems are due to Preparata et al.[1] and Hakimi and Amin[11], respectively.

Lemma 1[1] Let $G=G(V,E)$ be the graph representing of a system G, with V representing the processors and E the interconnections, the following two conditions are necessary for G to be t-diagnosable:

(1) $|G| \geqslant 2t+1$; and

(2) Each processor is tested by at least t other processors.

Lemma 2[11] The following two conditions are sufficient for a system $G = G(V, E)$ to be t-diagnosable:

(1) $|G| \geqslant 2t+1$; and

(2) $\kappa(G) \geqslant t$.

Since star graph S_n has connectivity $n-1$, its diagnosability under the PMC model is $n-1$ by Lemma 2.

2 Diagnosability of Incomplete Star Graphs Under the PMC Model

For a general graph, establishing its diagnosability is not easier than finding the faults. The distribution of missing links in the star graph S_n greatly affects its diagnosability. One missing link will decrease the diagnosability by 1, since its connectivity is decreased by 1. But if we delete the perfect matching of $n!/2$ edges between the sub-star graphs which are isomorphic to S_{n-1}, the diagnosability will not decrease any more, and remains $n-2$, $n \geqslant 4$ (See Fig. 1, the matching consists of the crossing edges between any two of the 6-cycles).

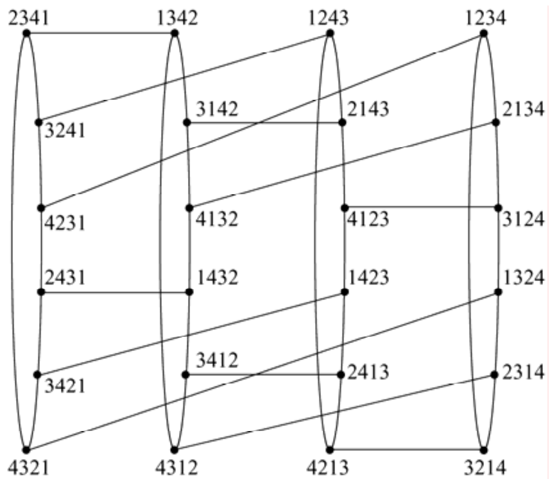

Fig. 1 Star graph S_4

The following characterization of a t-diagnosable system is the foundation of our discussing.

Lemma 3[12] A system $G = G(V, E)$ is t-diagnosable if and only if for $\forall S \subseteq V, S \neq \varnothing$, $\frac{|S|}{2} + |N(S)| > t$.

For the complete star graph S_n, the following three properties on neighbor are necessary.

Lemma 4[9,10] For any subset $S \subset V$ with $S = \{x_1, x_2\}$ in star graphs S_n, $|N(S)| \geqslant 2n-4$.

Lemma 5 For any subset $S \subset V$ with $S = \{x_1, x_2, x_3\}$ in star graph S_n, $|N(S)| \geqslant 3n-7$.

Proof We prove this lemma by discussing the distribution of the three nodes x_1, x_2, and x_3.

Case 1 These three nodes are all adjacent. Since star graph is bipartite graph, and every cycle has length of even number l, $6 \leqslant l \leqslant n!$[8], so these three nodes can only be connected as in a line, without loss of generality, suppose that x_2 is adjacent to x_1 and x_3. Thus we have

$$|N(S)| = |N(x_1) - \{x_2\}| + |N(x_2) - \{x_1, x_3\}| + |N(x_3) - \{x_2\}| = n-2+n-3+n-2 = 3n-7.$$

Case 2 Two nodes are adjacent to each other, and the other is not adjacent to the former two nodes. Without loss of generality, we consider the case that x_1 and x_2 are adjacent to each other, and x_3 is not adjacent to x_1 and x_2. Since star graph S_n has no 3-cycle, and 4-cycle, 5-cycle, $|N(x_3) \cap N(\{x_1, x_2\})| \leqslant 1$. We have

$$|N(S)| = |N(x_1) - \{x_2\}| + |N(x_2) - \{x_1\}| + |N(x_3)| + |N(x_3)| + |N(x_3) \cap N(\{x_1, x_2\})| \geqslant n-2+n-2+n-1-1 = 3n-6.$$

Case 3 These three nodes constitute an independent set. Since star graph S_n has no 3-cycle, 4-cycle, and 5-cycle, $|N(x_1) \cap N(x_2)| + |N(x_2) \cap N(x_3)| + |N(x_1) \cap N(x_3)| \leqslant 3$.

Hence we have

$$|N(S)| = |N(x_1)| + |N(x_2)| + |N(x_3)| - (|N(x_1) \cap N(x_2)| + |N(x_2) \cap N(x_3)| + |N(x_1) \cap N(x_3)|) \geqslant 3(n-1) - 3 \geqslant 3n-6.$$

Lemma 6[10] For any subset $S \subset V$ with $S = \{x_1, x_2, x_3, x_4\}$ in star graph S_n, $|N(S)| \geqslant 4n-10$.

Theorem 1 Given the star graph $S_n = G(V, E)$ with missing links. If $\delta(G) = \min\{d(u) \mid u \in V\} = t$ such that $t \geqslant 2$, then the system's diagnosability is t under the PMC model.

Proof We prove the theorem by showing that for any no-empty subset S of S_n, $\frac{|S|}{2} + |N(S)| > t$ will be satisfied, then by Lemma 3, the system is t-diagnosable. For the sake of convenience, we denote $\frac{|S|}{2}+|N(S)|$ as $\varphi(S)$.

Case 1 $|S|=1$.

For any one vertex set $S=\{x_1\}$, since $d(x_1)\geqslant t$, we have $\varphi(S)=\frac{1}{2}+t>t$.

Case 2 $|S|=2$.

For any subset $S\subset V$ with $S=\{x_1,x_2\}$, $|N(S)|\geqslant 2n-4$ for a complete star graph by Lemma 4.

Let k_{x_1} (respectively, k_{x_2}) be the number of missing links from x_1 (respectively, x_2).

Since $d(x_i)>t$, $k_{x_i}\leqslant n-1-t$, $i=1, 2$. Therefore, $|N(S)|\geqslant 2n-4-2(n-1-t)\geqslant 2t-2$, and if $t>1$, then $\varphi(S)\geqslant \frac{2}{2}+2t-2=2t-1>t$.

Case 3 $|S|=3$.

For any subset $S\subseteq V$ with $S=\{x_1,x_2,x_3\}$, $|N(S)|\geqslant 3n-7$ for a complete star graph by Lemma 5. Let k_{x_i} be the number of missing links from x_i, $i=1, 2, 3$. Since $d(x_i)>t$, $k_{x_i}\leqslant n-1-t$, $i=1, 2, 3$, we have $|N(S)|\geqslant 3n-7-3(n-t-1)\geqslant 3t-4$, then

$$\varphi(S)\geqslant \frac{3}{2}+3t-4\geqslant 3t-\frac{5}{2}>t,\text{ if } t>1.$$

Case 4 $|S|=4$.

For any subset $S\subseteq V$ with $S=\{x_1,x_2,x_3,x_4\}$, $|N(S)|\geqslant 4n-10$ for a complete star graph by Lemma 6. Similarly, let k_{x_i} be the number of missing links from x_i, $i=1,2,3,4$. Since $d(x_i)>t$, $k_{x_i}\leqslant n-1-t$ for $i=1,2,3,4$, we have $|N(S)|\geqslant 4n-10-4(n-1-t)\geqslant 4t-6$, then $\varphi(S)\geqslant \frac{4}{2}+4t-6\geqslant 4t-\ \ 4>t$, if $t>1$.

Case 5 $5\leqslant|S|\leqslant 2t$.

Observe that the addition of one vertex into S will increase $\frac{|S|}{2}$ by $\frac{1}{2}$, but if this new added vertex has been chosen from the old set $N(S)$, then the worst case is that $|N(S)|$ will be decreased by 1, resulting an overall decrease of $\varphi(S)$ by $\frac{1}{2}$. So for $S\subseteq V$ such that $5\leqslant|S|\leqslant 2t$,

$$\varphi(S)\geqslant 4t-4-(2t-4)\times\frac{1}{2}=3t-2>t,\text{ if } t\geqslant 2.$$

Case 6 $|S|\geqslant 2t+1$.

Since $\frac{|S|}{2}+|N(S)|>t$ always holds, it is not due to discuss.

Summarizing the preceding six cases, we have $\frac{|S|}{2}+|N(S)|>t$ for any nonempty $S\subseteq V$ if $t\geqslant 2$.

From Theorem 1, we can immediately have the following result:

Corollary 1[3] The complete star graph S_n $(n\geqslant 4)$ is $(n-1)$-diagnosable under the PMC model.

3 Diagnosability of Incomplete Star Graphs Under the BGM Model

Before our discussing, we mention the following three lemmas:

Lemma 7[10] Let u, v be any two vertices of star graph S_n. If $(u,v)\in E$, then $|N(u)\cap N(v)|=0$; if $(u,v)\notin E$, then $|N(u)\cap N(v)|\leqslant 1$.

Lemma 8[13] If $\delta(G)=t$ holds for the system $G=G(V,E)$, then the diagnosability under the BGM model is either t or $t-1$.

Let $V_t=\{v\in V\,\|\,N(v)|=t\}$. We define a binary relation "$\equiv$" over V_t such that $u\equiv v$ if and only if $N(u)=N(v)$. It is obvious that $\equiv$ is an equivalence relation so that it induces the nodes of V_t into several equivalence classes.

Lemma 9[12] The system's diagnosability under the BGM model is $t-1$ if and only if there exist distinct equivalence classes A and B over V_t such that $N(A)-B=N(B)-A$.

Theorem 2 Let S_n be the star graph $G=G(V,E)$ with incomplete links. If $\delta(G)=t$ holds such that $t\geqslant 2$, then the system's diagnosability under the BGM model is t.

Proof We prove the theorem by showing that in an incomplete star graph S_n, if $\delta(G)=t\geqslant 2$, there will be no distinct equivalence classes A and B over V_t such that $N(A)-B=N(B)-A$, then by Lemmas 8 and 9, the system is t-diagnosable. First, we show that every equivalence class contains only one vertex:

If $(u,v)\in E$, $|N(u)\cap N(v)|=0$ induces that $N(u)\cap N(v)=\varnothing$ by Lemma 7, i.e., $N(u)\neq N(v)$, which means that u and v do not belong to the same equivalence class also.

If $(u,v)\notin E$, the fact $|N(u)\cap N(v)|\leqslant 1$ means that $N(u)\neq N(v)$, which concludes that u and v do not belong to the same equivalence class also. We now show that there will be no distinct equivalence classes A and B over V_t such that $N(A)-B=N(B)-A$.

Let $A=\{u\}$ and $B=\{v\}$. If $(u,v)\in E$, then $N(u)\cap N(v)=\varnothing$, so we have $|N(A)-B|=|N(B)-A|=t-1\geqslant 1$, and that $N(A)-B\neq N(B)-A$ since the incomplete star graph has no 3-cycle.

If $(u,v)\notin E$, we also have $N(A)-B\neq N(B)-A$, since $N(A)-B=N(A)$, $N(B)-A=N(B)$, and $|N(A)\cap N(B)|\leqslant 1$, $|N(A)|=|N(B)|=t\geqslant 2$. But $N(A)-B\neq N(B)-A$ since the incomplete star graph has no 4-cycle.

This completes the proof of the theorem.

4 Conclusions

The fault tolerances of star graph were restricted to only complete star graph in recent studies[3,7-9]. This paper centres upon incomplete star graph S_n ($n\geqslant 4$) with arbitrarily distributed missing links. The diagnosabilities of incomplete star graph under the PMC model and the BGM model are decided by its minimum degree.

The algorithms and the time complexities of incomplete star graph based on the PMC model and the BGM model are clear from the discussing of incomplete hypercubes[6].

References

[1] Preparata F P, Metze G, Chien R T. On the connection assignment problem of diagnosable systems. *IEEE Transactions on Computers*, 1967, **16**: 848-854.

[2] Barsi F, Grandoni F, Maestrini P. A theory of diagnosablity of digital systems. *IEEE Transactions on Computers*, 1976, **6**: 585-593.

[3] Chang G Y, Chang G, Chen G H. Diagnosabilities of regular networks. *IEEE Transactions on Parallel and Distributed Systems*, 2005, **4**: 314-323.

[4] Fan Jianxi, Lin Xiaola. The *t*/*k*-diagnosable of BC graphs. *IEEE Transactions on Computers*, 2005, **2**: 176-184.

[5] Wang Dajin. Diagnosability of enhanced hypercubes. *IEEE Transactions on Computer*s, 1994, **9**: 1054-1061.

[6] Wang Dajin. The diagnosability of hypercube with arbitrarily missing links. *Journal of Systems Architecture*, 2000, **46**: 519-527.

[7] Akers S B, Harel D, Krishnamurthy B. The star graph: An attractive alternative to the *n*-cube. In: Int. Conf. Parallel Processing, 1987: 393-400.

[8] Qui K, Meijer H, Akl S G. Decomposing a star graph into disjoint cycles. *Information Processing Letters*, 1991, **39**: 125-129.

[9] Kavianpout A. Sequential diagnosability of star graphs. *Computers and Electrical Engineering*, 1996, **1**: 37-44.

[10] Zheng Jun, Latifi S, Regentova E, Luo Kai, Wu Xiaolong. Diagnosability of star graphs under the comparison diagnosis model. *Information Processing Letters*, 2005, **93**: 29-36.

[11] Hakimi S L, Amin A T. Characterization of the connection assignment of diagnosable systems. *IEEE Transactions on Computers*, 1974, C-**23**(1): 86-88.

[12] Raghavan V, Tripathi A R. Improved diagnosability algorithms. *IEEE Transactions on Computers*, 1991, **40**(2): 143-153.

[13] Narasimhan J, Nakajima K. An algorithm for determining the fault diagnosability system. *IEEE Transactions on Computers*, 1986, C-**35**(11): 1004-1008.