Letters

Propagation of Thickness-Twist Waves in a Quartz Plate with Asymmetric Mass Layers

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***Abstract*—An exact solution is obtained for thickness-twist waves in a quartz plate with asymmetric mass layers. A simple, approximate expression for cutoff frequencies is given.**

I. Introduction

Thickness-twist vibration modes of quartz plates are widely used as the operating modes of devices. and they have been under long and sustained study. The literature on the subject is too numerous to enumerate. The classical result on the subject was given by Mindlin [1] in which the inertial effect of symmetric electrode mass was considered. In resonator manufacturing, one electrode is deposited first with a predetermined thickness. Then the electrode on the other side of the crystal plate has a thickness that is determined by the fact that the electroded plate has a desired frequency. This usually results in a crystal plate with two electrodes of different thickness. The effect of asymmetric electrodes is a problem of current interest [2]–[4]. The results in [2]–[4] are for pure thickness-shear modes without in-plane variation, which are modes of unbounded plates. In device applications, due to the finite size of devices, modes with slow in-plane variations are used. These are plate waves whose wavelength is much larger than the plate thickness. The understanding of these waves requires the study of waves with in-plane variations. In this letter, we analyze thickness-twist waves in a quartz plate with asymmetric electrode mass and in-plane variation. This problem is a more realistic representation of the operating modes in real devices.

II. Thickness-Twist Waves

Consider thickness-twist waves $u_1(x_2, x_3, t)$ propagating in the x_3 direction in a rotated Y-cut quartz plate with asymmetric mass layers (see Fig. 1). In addition to the x_2 dependence for pure thickness modes, we also allow the dependence of the in-plane coordinate x_3. The governing equation and the stress component relevant to boundary conditions are [1]:

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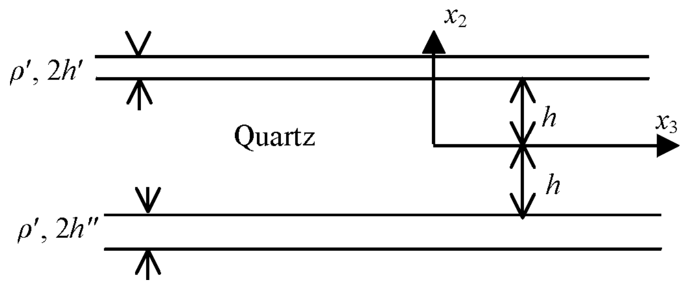

Fig. 1. A quartz plate with asymmetric mass layers.

$$c_{66}u_{1,22} + 2c_{56}u_{1,23} + c_{55}u_{1,33} = \rho\ddot{u}_1, \quad (1)$$

$$T_{21} = c_{56}u_{1,3} + c_{66}u_{1,2}. \quad (2)$$

Let:

$$u_1 = A\exp(i\eta x_2)\exp[i(\varsigma x_3 - \omega t)]. \quad (3)$$

Substitution of (3) into (1) implies that:

$$c_{66}\eta^2 + 2c_{56}\eta\varsigma + c_{55}\varsigma^2 - \rho\omega^2 = 0. \quad (4)$$

The two roots of (4) are:

$$\eta_1 = \frac{-c_{56}\varsigma + \sqrt{c_{56}^2\varsigma^2 + c_{66}(\rho\omega^2 - c_{55}\varsigma^2)}}{c_{66}}, \quad \eta_2 = \frac{-c_{56}\varsigma - \sqrt{c_{56}^2\varsigma^2 + c_{66}(\rho\omega^2 - c_{55}\varsigma^2)}}{c_{66}}. \quad (5)$$

We make the following observation from (5),which is important to the solution procedure:

$$c_{66}\eta_2 + c_{56}\varsigma = -(c_{66}\eta_1 + c_{56}\varsigma). \quad (6)$$

The general solution to (1) in the form of (3), along with T_{21}, can be written as:

$$u_1 = [A_1\exp(i\eta_1 x_2) + A_2\exp(i\eta_2 x_2)]\exp[i(\varsigma x_3 - \omega t)],$$
$$T_{21} = \Big[i(c_{56}\varsigma + c_{66}\eta_1)A_1\exp(i\eta_1 x_2) + i(c_{56}\varsigma + c_{66}\eta_2)A_2\exp(i\eta_2 x_2)\Big]\exp[i(\varsigma x_3 - \omega t)], \quad (7)$$

where A_1 and A_2 are undetermined constants. The boundary conditions to be satisfied are:

$$-T_{21} = \rho' 2h'\ddot{u}_1 = -\rho' 2h'\omega^2 u, \quad x_2 = h,$$
$$T_{21} = \rho' 2h''\ddot{u}_1 = -\rho' 2h''\omega^2 u, \quad x_2 = -h. \quad (8)$$

Substitution of (7) into (8) gives:

$$\begin{aligned} & i\left(c_{56}\varsigma+c_{66}\eta_1\right)A_1\exp\left(i\eta_1 h\right) \\ & +i\left(c_{56}\varsigma+c_{66}\eta_2\right)A_2\exp\left(i\eta_2 h\right) \\ & \quad =\rho' 2h'\omega^2\left[A_1\exp\left(i\eta_1 h\right)+A_2\exp\left(i\eta_2 h\right)\right], \\ & i\left(c_{56}\varsigma+c_{66}\eta_1\right)A_1\exp\left(-i\eta_1 h\right) \\ & +i\left(c_{56}\varsigma+c_{66}\eta_2\right)A_2\exp\left(-i\eta_2 h\right) \\ & \quad =-\rho' 2h''\omega^2\left[A_1\exp\left(-i\eta_1 h\right)+A_2\exp\left(-i\eta_2 h\right)\right]. \end{aligned} \tag{9}$$

For nontrivial solutions:

$$\begin{vmatrix} i\left(c_{56}\varsigma+c_{66}\eta_1\right)\exp\left(i\eta_1 h\right)-\rho' 2h'\omega^2\exp\left(i\eta_1 h\right) & i\left(c_{56}\varsigma+c_{66}\eta_2\right)\exp\left(i\eta_2 h\right)-\rho' 2h'\omega^2\exp\left(i\eta_2 h\right) \\ i\left(c_{56}\varsigma+c_{66}\eta_1\right)\exp\left(-i\eta_1 h\right)+\rho' 2h''\omega^2\exp\left(-i\eta_1 h\right), & i\left(c_{56}\varsigma+c_{66}\eta_2\right)\exp\left(-i\eta_2 h\right)+\rho' 2h''\omega^2\exp\left(-i\eta_2 h\right) \end{vmatrix}=0. \tag{10}$$

With (6) we can write (10) as:

$$\exp\left[i2\left(\eta_2-\eta_1\right)h\right]=\frac{1-k'k''+i\left(k'+k''\right)}{1-k'k''-i\left(k'+k''\right)}, \tag{11}$$

where:

$$k'=\frac{c_{66}\eta_1+c_{56}\varsigma}{\rho' 2h'\omega^2}, \quad k''=\frac{c_{66}\eta_1+c_{56}\varsigma}{\rho' 2h''\omega^2}. \tag{12}$$

Let θ be such that:

$$\exp(i\theta)=\frac{1-k'k''+i\left(k'+k''\right)}{\sqrt{\left(1-k'k''\right)^2+\left(k'+k''\right)^2}}, \quad \tan\theta =\frac{k'+k''}{1-k'k''}. \tag{13}$$

Then (11) can be written as:

$$\exp\left[i2\left(\eta_2-\eta_1\right)h\right]=\exp(i2\theta). \tag{14}$$

Hence:

$$\tan\left(\eta_1-\eta_2\right)h=\frac{k'+k''}{1-k'k''}. \tag{15}$$

Together (5), (12), and (15) determine the dispersion relations.

III. Cutoff Frequencies

At cutoff ($\varsigma=0$), the above equations reduce to:

$$\eta_1=-\eta_2=\eta, \quad \frac{\omega}{\omega_0}=\frac{2\eta h}{\pi}, \quad \omega_0^2=\frac{\pi^2 c_{66}}{4\rho h^2}, \tag{16}$$

$$\tan 2\eta h=\frac{\left(R'+R''\right)\eta h}{R'R''\eta^2h^2-1}, \tag{17}$$

$$R'=\frac{2\rho' h'}{\rho h}, \quad R''=\frac{2\rho' h''}{\rho h}. \tag{18}$$

The roots of technological interest are those when m is a small, odd integer, which are given by $2\eta h=m\pi$ when the mass layers are not present. With the mass layers, we let:

$$2\eta h=m\pi-\varepsilon, \quad 0<\varepsilon\ll 1. \tag{19}$$

Substituting (19) into (17), we obtain:

$$2\eta h\cong m\pi\left(1-\bar{R}\right), \tag{20}$$

$$\bar{R}=\left(R'+R''\right)/2. \tag{21}$$

Hence, the cutoff frequencies are approximately:

$$\frac{\omega}{\omega_0}=m\left(1-\bar{R}\right). \tag{22}$$

IV. Conclusions

The mass layers reduce the cutoff frequencies. To the lowest order effect, the cutoff frequencies are affected by the average thickness of the mass layers. The difference of the mass layer thickness is a higher-order effect. The above results reduce to those in [1] in which the mass layers are symmetric ($h''=h'$).

References

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