Letters

Frequency Shifts in a Piezoelectric Body Due to Small Amounts of Additional Mass on Its Surface

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***Abstract*—Shifts of resonance frequencies of a three-dimensional piezoelectric body of an arbitrary shape due to the addition of a thin layer of mass to its surface are studied. A first-order perturbation integral is obtained for the frequency shifts.**

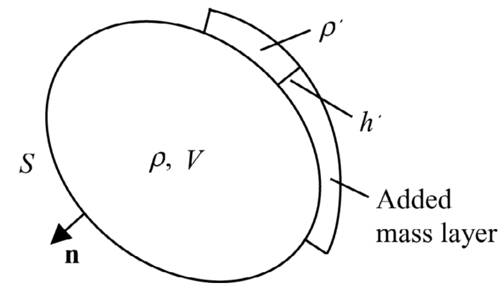

Fig. 1. A piezoelectric body with a thin layer of additional mass on part of its surface.

I. Introduction

Frequency shifts in a crystal resonator due to a thin layer of mass (e.g., an electrode) added to part of its surface have been an important issue in frequency analysis of resonators. Many chemical and biological acoustic wave sensors detect certain substances through the mass-frequency effect of a substance accumulated on the crystal surface by some chemically or biologically active films. For plate resonators, Mindlin [1] developed a procedure to analyze the mass effect on the surface of a crystal plate. Mindlin's approach was adopted by many researchers [2], [3] and is a subject of continued study [4]. However, only plate resonators were considered in [1]–[4]. The widely used perturbation integral by Tiersten [5] for frequency shifts in a piezoelectric body was for a general three-dimensional body. The perturbation integral was obtained from a first-order perturbation analysis of eigenvalue problems [6]. In [5], the source of frequency shifts was limited to be inside the body, not on the surface. In this letter, we analyze frequency shifts in a three-dimensional body due to a thin layer of surface-added mass. In Section II the eigenvalue problem is formulated, with an interesting feature that the eigenvalue (resonance frequency) appears in both the differential equations and boundary conditions. In Section III a perturbation analysis is performed and a general formula is obtained. Then some observations are made in Section IV. A simple example is given in Section V. Some conclusions are drawn in Section VI.

II. Formulation of the Problem

Consider a piezoelectric body with a thin film of thickness h' and mass density ρ' on part of its surface (see Fig. 1). Let the region occupied by the piezoelectric body be V and its boundary surface be S. The unit outward normal of S is $\mathbf{n}$. The mass layer is assumed to be very thin. For the lowest order effect of the mass layer, only the inertial effect of the layer needs to be considered; its stiffness can be neglected [7]. For free vibrations with a frequency ω, the traction boundary condition on the surface area with the added mass is [4], [7]:

$$-T_{ji}n_j = \rho' h' \ddot{u}_i = -\rho' h' \omega^2 u_i. \tag{1}$$

Let S_u be the part of S on which the mechanical displacement is prescribed and S_T the part of S with the mass layer. The S_ϕ represents the part of S which is electroded, and S_D is the unelectroded part. Then the eigenvalue problem for the resonance frequencies and modes of a crystal with surface added mass is [4], [8]:

$$\begin{aligned} &-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj} = \rho\lambda u_i, \ \text{ in } V, \\ &-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki} = 0, \ \text{ in } V, \\ &u_i = 0, \ \text{ on } S_u, \\ &T_{ji}n_j = (c_{jikl}u_{k,l} + e_{kji}\phi_{,k})\, n_j = \varepsilon\lambda\rho' h' u_i, \ \text{ on } S_T, \\ &\phi = 0, \ \text{ on } S_\phi, \\ &D_i n_i = (e_{ikl}u_{k,l} - \varepsilon_{ik}\phi_{,k})\, n_i = 0, \ \text{ on } S_D, \end{aligned} \tag{2}$$

where we have denoted

$$\lambda = \omega^2. \tag{3}$$

In $(2)_4$ we have artificially introduced a dimensionless number ε to be used in our perturbation analysis. The real physical problem is represented by $\varepsilon = 1$; (2) can be written in a more compact form as [8]:

$$\begin{aligned} &\mathbf{AU} = \lambda\mathbf{BU}, \ \text{ in } V, \\ &u_i = 0, \ \text{ on } S_u, \\ &T_{ji}(\mathbf{U})n_j = \varepsilon\lambda\rho' h' u_i, \ \text{ on } S_T, \\ &\phi = 0, \ \text{ on } S_\phi, \\ &D_i(\mathbf{U})n_i = 0, \ \text{ on } S_D, \end{aligned} \tag{4}$$

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where $\mathbf{U} = \{u_k, \phi\}$ is a 4-vector. The differential operators $\mathbf{A}$ and $\mathbf{B}$ are defined by [8]:

$$\begin{aligned} &\mathbf{AU} = \{-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj}, -e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki}\}, \\ &\mathbf{BU} = \{\rho u_i, 0\}, \end{aligned} \tag{5}$$

where $T_{ji}(\mathbf{U})$ and $D_i(\mathbf{U})$ are the stress tensor and electric displacement vector in terms of the 4-vector.

III. Perturbation Analysis

We make the following perturbation expansion:

$$\begin{aligned} &\lambda \cong \lambda^{(0)} + \varepsilon\lambda^{(1)}, \\ &\mathbf{U} = \begin{Bmatrix} u_i \\ \phi \end{Bmatrix} \cong \begin{Bmatrix} u_i^{(0)} \\ \phi^{(0)} \end{Bmatrix} + \varepsilon \begin{Bmatrix} u_i^{(1)} \\ \phi^{(1)} \end{Bmatrix} \\ &= \mathbf{U}^{(0)} + \varepsilon\mathbf{U}^{(1)}. \end{aligned} \tag{6}$$

Substituting (6) into (4), collecting terms of equal powers of ε, we can obtain a series of perturbation problems of successive orders. We are interested in the lowest order effect of the mass layer. Therefore, we collect coefficients of terms with powers of ε^0 and ε^1 only and will not study higher order problems. The zero-order problem is:

$$\begin{aligned} &-c_{jikl}u_{k,lj}^{(0)} - e_{kji}\phi_{,kj}^{(0)} = \rho\lambda^{(0)}u_i^{(0)}, \text{ in } V, \\ &-e_{ikl}u_{k,li}^{(0)} + \varepsilon_{ik}\phi_{,ki}^{(0)} = 0, \text{ in } V, \\ &u_i^{(0)} = 0, \text{ on } S_u, \\ &\left(c_{jikl}u_{k,l}^{(0)} + e_{kji}\phi_{,k}^{(0)}\right)n_j = 0, \text{ on } S_T, \\ &\phi^{(0)} = 0, \text{ on } S_\phi, \\ &\left(e_{ikl}u_{k,l}^{(0)} - \varepsilon_{ik}\phi_{,k}^{(0)}\right)n_i = 0, \text{ on } S_D. \end{aligned} \tag{7}$$

This represents free vibrations of the body without the surface mass. The solution to the zero-order problem, $\lambda^{(0)}$ and $\mathbf{U}^{(0)}$, is assumed known as usual in a perturbation analysis. The first-order problem below is to be solved:

$$\begin{aligned} &-c_{jikl}u_{k,lj}^{(1)} - e_{kji}\phi_{,kj}^{(1)} = \rho\lambda^{(1)}u_i^{(0)} + \rho\lambda^{(0)}u_i^{(1)}, \text{ in } V, \\ &-e_{ikl}u_{k,li}^{(1)} + \varepsilon_{ik}\phi_{,ki}^{(1)} = 0, \text{ in } V, \\ &u_i^{(1)} = 0, \text{ on } S_u, \\ &\left(c_{jikl}u_{k,l}^{(1)} + e_{kji}\phi_{,k}^{(1)}\right)n_j = \rho' h' \lambda^{(0)}u_i^{(0)}, \text{ on } S_T, \\ &\phi^{(1)} = 0, \text{ on } S_\phi, \\ &\left(e_{ikl}u_{k,l}^{(1)} - \varepsilon_{ik}\phi_{,k}^{(1)}\right)n_i = 0, \text{ on } S_D. \end{aligned} \tag{8}$$

Terms such as $\lambda^{(1)}\mathbf{U}^{(1)}$ belong to higher order problems. The equations for the first-order problem, $(8)_{1,2}$, can be written as:

$$\mathbf{AU}^{(1)} = \lambda^{(0)}\mathbf{BU}^{(1)} + \lambda^{(1)}\mathbf{BU}^{(0)}. \tag{9}$$

Multiply both sides of (9) by $\mathbf{U}^{(0)}$ and integrate the resulting equation over V, we have (10) (see next page), where, for simplicity, we have used $\langle \bullet \rangle$ to represent the product of two 4-vectors and the integration over V [8]. With integration by parts we get (11) (see next page).

With the boundary conditions in $(7)_{3\text{–}6}$ and $(8)_{3\text{–}6}$, (11) becomes:

$$\begin{aligned} &\left\langle \mathbf{AU}^{(0)}; \mathbf{U}^{(1)} \right\rangle \\ &= \int_{S_T} \rho' h' \lambda^{(0)} u_k^{(0)} u_k^{(0)} dS + \left\langle \mathbf{U}^{(0)}; \mathbf{AU}^{(1)} \right\rangle. \end{aligned} \tag{12}$$

Substitute (12) into (10):

$$\begin{aligned} &\left\langle \mathbf{AU}^{(0)}; \mathbf{U}^{(1)} \right\rangle - \int_{S_T} \rho' h' \lambda^{(0)} u_k^{(0)} u_k^{(0)} dS \\ &= \lambda^{(0)} \left\langle \mathbf{BU}^{(1)}; \mathbf{U}^{(0)} \right\rangle + \lambda^{(1)} \left\langle \mathbf{BU}^{(0)}; \mathbf{U}^{(0)} \right\rangle, \end{aligned} \tag{13}$$

which further can be written as:

$$\begin{aligned} &\left\langle \mathbf{AU}^{(0)} - \lambda^{(0)}\mathbf{BU}^{(0)}; \mathbf{U}^{(1)} \right\rangle - \int_{S_T} \rho' h' \lambda^{(0)} u_k^{(0)} u_k^{(0)} dS \\ &= \lambda^{(1)} \left\langle \mathbf{BU}^{(0)}; \mathbf{U}^{(0)} \right\rangle. \end{aligned} \tag{14}$$

With $(7)_{1,2}$, we obtain, from (14):

$$\begin{aligned} \lambda^{(1)} &= -\frac{\int_{S_T} \rho' h' \lambda^{(0)} u_k^{(0)} u_k^{(0)} dS}{\left\langle \mathbf{BU}^{(0)}; \mathbf{U}^{(0)} \right\rangle} \\ &= -\lambda^{(0)} \frac{\int_{S_T} \rho' h' u_k^{(0)} u_k^{(0)} dS}{\int_V \rho u_i^{(0)} u_i^{(0)} dV}. \end{aligned} \tag{15}$$

The above expression is for the eigenvalue $\lambda = \omega^2$. For ω we make the following expansion:

$$\omega \cong \omega^{(0)} + \varepsilon\omega^{(1)}. \tag{16}$$

Then:

$$\begin{aligned} \lambda = \omega^2 &\cong \left(\omega^{(0)} + \varepsilon\omega^{(1)}\right)^2 \\ &\cong \left(\omega^{(0)}\right)^2 + 2\varepsilon\omega^{(0)}\omega^{(1)} \cong \lambda^{(0)} + \varepsilon\lambda^{(1)}. \end{aligned} \tag{17}$$

Hence:

$$\begin{aligned} \frac{\varepsilon\omega^{(1)}}{\omega^{(0)}} &\cong \frac{1}{2\left(\omega^{(0)}\right)^2}\varepsilon\lambda^{(1)} \\ &= -\frac{1}{2\left(\omega^{(0)}\right)^2}\varepsilon\lambda^{(0)} \frac{\int_{S_T} \rho' h' u_k^{(0)} u_k^{(0)} dS}{\int_V \rho u_i^{(0)} u_i^{(0)} dV}. \end{aligned} \tag{18}$$

Setting $\varepsilon = 1$ in (18), we obtain:

$$\frac{\omega - \omega^{(0)}}{\omega^{(0)}} \cong -\frac{1}{2} \frac{\int_{S_T} \rho' h' u_k^{(0)} u_k^{(0)} dS}{\int_V \rho u_i^{(0)} u_i^{(0)} dV}. \tag{19}$$

$$\left\langle \mathbf{AU}^{(1)};\, \mathbf{U}^{(0)} \right\rangle = \int_V \left[\left(-c_{jikl} u_{k,lj}^{(1)} - e_{kji} \phi_{,kj}^{(1)} \right) u_i^{(0)} + \left(-e_{ikl} u_{k,li}^{(1)} + \varepsilon_{ik} \phi_{,ki}^{(1)} \right) \phi^{(0)} \right] dV$$
$$= \lambda^{(0)} \left\langle \mathbf{BU}^{(1)};\, \mathbf{U}^{(0)} \right\rangle + \lambda^{(1)} \left\langle \mathbf{BU}^{(0)};\, \mathbf{U}^{(0)} \right\rangle, \quad (10)$$

$$\left\langle \mathbf{AU}^{(0)};\, \mathbf{U}^{(1)} \right\rangle = -\int_S \left[T_{ji} \left(\mathbf{U}^{(0)} \right) n_j u_i^{(1)} + D_i \left(\mathbf{U}^{(0)} \right) n_i \phi^{(1)} \right] dS$$
$$+ \int_S \left[T_{kl} \left(\mathbf{U}^{(1)} \right) n_l u_k^{(0)} + D_k \left(\mathbf{U}^{(1)} \right) n_k \phi^{(0)} \right] dS + \left\langle \mathbf{U}^{(0)};\, \mathbf{AU}^{(1)} \right\rangle. \quad (11)$$

IV. Discussion

We make the following observations from (19):

- Clearly, we have $\omega - \omega^{(0)} \leq 0$. This shows that small amounts of mass added to the surface tends to lower the resonance frequencies, as expected. However, if a thin layer of material is removed from the surface, resonance frequencies increase.
- Larger $\rho' h'$ causes more frequency shifts.
- In an area in which the surface displacement is large, the added mass has a larger effect on resonance frequencies.
- If the additional mass is essentially a concentrated mass m at a point with Cartesian coordinates y_k on the surface (e.g., a local contamination), then (19) reduces to:

$$\frac{\omega - \omega^{(0)}}{\omega^{(0)}} \cong -\frac{1}{2} \frac{m u_k^{(0)}(\mathbf{y}) u_k^{(0)}(\mathbf{y})}{\int_V \rho u_i^{(0)} u_i^{(0)} dV}. \quad (20)$$

  Both (19) and (20) seem to suggest that the frequency shift is related to the ratio between the kinetic energy of the surface mass to the kinetic energy of the crystal.
- Obviously, S_T can be several disadjoint areas.

V. An Example

As an example, consider an unbounded plate of rotated Y-cut quartz (Fig. 2). The plate carries a thin layer on both of its surfaces. Quartz is a material with very weak piezoelectric coupling. For a frequency analysis, usually the weak piezoelectric coupling can be neglected and an elastic analysis is performed. This is sufficient for our present purpose. We are interested in the fundamental thickness-shear mode important for resonator application. When the surface mass is not present, the frequency and mode of interest are given by [3]:

$$\omega^{(0)} = \frac{\pi}{2h} \sqrt{\frac{c_{66}}{\rho}}, \quad u_1 = \sin \frac{\pi}{2h} x_2, \quad u_2 = u_3 = 0. \quad (21)$$

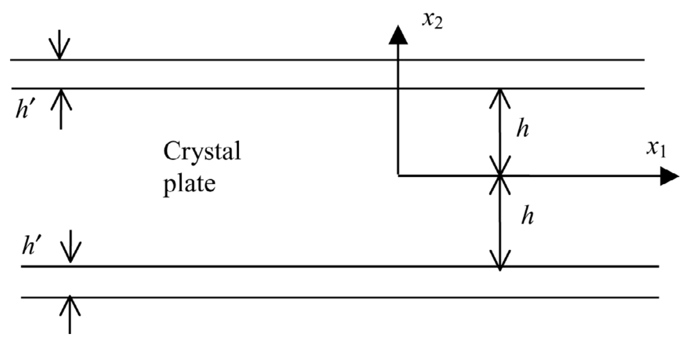

Fig. 2. A rotated Y-cut quartz plate with identical mass layers.

When the mass layers are present, from (19) we obtain:

$$\frac{\omega - \omega^{(0)}}{\omega^{(0)}} = \frac{\rho' h'}{\rho h}. \quad (22)$$

For the plate and mass layers shown in Fig. 2, the exact resonance frequency for the fundamental thickness-shear mode as directly determined from (2) is given by [3]:

$$\tan \left(\omega h \sqrt{\frac{\rho}{c_{66}}} \right) = \frac{1}{\frac{\rho' h'}{\rho h} \omega h \sqrt{\frac{\rho}{c_{66}}}}. \quad (23)$$

It also was shown in [3] that from small h', to the first order effect of h', the ω determined by (23) is approximately given by (22). Thus the perturbation integral predicts the lowest order effect of the mass layer.

VI. Conclusions

A general formula for frequency shifts in a piezoelectric body due to additional mass on its surface is obtained. The formula is useful for the analysis and design of piezoelectric resonators and acoustic wave sensors.

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