Letters

Effects of Piezoelectric Coupling on Energy Trapping of Thickness-Shear Modes

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***Abstract*—Energy trapping of thickness-shear vibration modes in a partially electroded piezoelectric crystal plate of monoclinic symmetry is analyzed. Effects of piezoelectric coupling on energy trapping are examined. Results show that the effect of piezoelectric coupling is comparable to the effect of electrode mass and needs to be included in the analysis of energy trapping.**

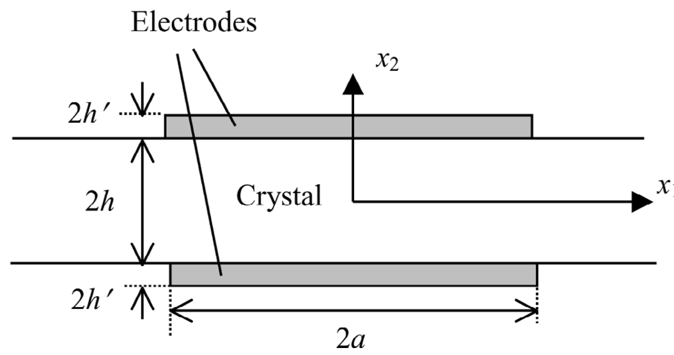

Fig. 1. A partially electroded piezoelectric plate.

I. Introduction

Thickness-shear vibration in a partially electroded quartz plate is confined to the area under and close to the electroded region of the plate. This phenomenon is called energy trapping of thickness-shear modes [1]. Energy trapping has been known and used for a long time in quartz thickness-shear resonators. Quartz is a material with very weak piezoelectric coupling. Therefore, in the calculation of resonance frequencies and in the analyses of energy trapping in quartz resonators, the small piezoelectric coupling often was neglected, and a pure elastic analysis was performed [2]. In this type of analysis, the energy-trapping effect is due to the electrode mass only.

Recently, new crystals of the langasite family with electromechanical coupling stronger than quartz have been developed [3] and used to make thickness-shear resonators [4]. In addition, other crystals with relatively strong piezoelectric coupling (e.g., lithium niobate and lithium tantalate) also have been used to make thickness-shear devices [5], [6]. For thickness-shear devices made from materials with stronger piezoelectric coupling, the understanding of energy trapping based on an elastic analysis is questionable and may be inadequate. An analysis with piezoelectric coupling is necessary to quantify the effect of piezoelectric coupling on energy trapping in materials with relatively strong coupling.

In this paper we analyze energy trapping of thickness-shear modes in a partially electroded crystal plate of monoclinic symmetry. This includes the widely used, rotated Y-cut quartz and langasite plates. Piezoelectric coupling is included in the analysis. The governing equations are summarized in Section II. The eigenvalue problem for free, thickness-shear vibrations of the plate is formulated in Section III. The frequency equation that determines resonance frequencies is obtained in Section IV, along with the modes. An approximate solution is obtained for the first resonance frequency and mode, from which some important observations are made, and some conclusions are drawn.

Manuscript received March 12, 2004; accepted April 26, 2004. This work was supported by the U.S. Army Research Office under DAAD19-01-1-0443.

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II. Governing Equations

Consider an unbounded, partially electroded piezoelectric plate of monoclinic symmetry with thickness $2h$ (Fig. 1). The mass densities of the plate and the electrodes are ρ and ρ', respectively. We consider the case of thin electrodes in which the mass effect needs to be considered, but the stiffness of the electrodes can be neglected. When the structure is in thickness-shear vibrations, there may be accompanying flexural deformations. The two-dimensional equations for coupled thickness-shear and flexural vibrations of piezoelectric plates [7] can describe such motions. When $a \gg h$ (i.e., very thin plates), the flexural deformation is small and can be eliminated through a thickness-shear approximation [8]. For the purpose of the present analysis, we assume $a \gg h$ and use the equations based on the thickness-shear approximation given in [8], which has the most basic effects of piezoelectric coupling. Consider plane-strain motions with $u_3 = 0$ and $\partial/\partial x_3 = 0$. We look for free vibration modes at a frequency ω.

To conduct a free vibration frequency analysis, the electrodes are taken as shorted, i.e., the voltage across the electrodes is zero. The equation governing the thickness-shear displacement $u_1^{(1)}(x_1)$ is given by [8]:

$$\overline{\gamma} u_{1,11}^{(1)} + \overline{\rho}(\omega^2 - \overline{\omega}_0^2) u_1^{(1)} = 0, \qquad (1)$$

where

$$\begin{aligned}
\overline{\gamma} &= \gamma_{11} + \frac{c_{66}\overline{\kappa}_6^2}{r}, \\
\gamma_{11} &= c_{11} - \frac{c_{12}^2}{c_{22}} - \frac{(c_{14} - c_{12}c_{24}/c_{22})^2}{c_{44} - c_{24}^2/c_{22}}, \\
\overline{\kappa}_6^2 &= \frac{\pi^2}{12}\left(1 + R - \frac{8k_{26}^2}{\pi^2}\right)\frac{\overline{c}_{66}}{c_{66}}, \\
\overline{\rho} &= \rho(1 + 3R), \\
R &= \frac{2\rho' h'}{\rho h}, \\
k_{26}^2 &= \frac{e_{26}^2}{\varepsilon_{22}\overline{c}_{66}}, \\
\overline{c}_{66} &= c_{66} + \frac{e_{26}^2}{\varepsilon_{22}}, \\
r &= \frac{1 + R}{1 + 3R}, \\
\overline{\omega}_0^2 &= \frac{\pi^2}{4h^2}\left(1 - R - \frac{4k_{26}^2}{\pi^2}\right)^2 \frac{\overline{c}_{66}}{\rho}.
\end{aligned} \tag{2}$$

Eq. (1) is the result of a series of subtle approximations. The most basic effects of mass loading and piezoelectric coupling are represented by the presence of R and k_{26}^2.

Similarly, in an unelectroded region of the plate we have [8]:

$$\gamma u_{1,11}^{(1)} + \rho(\omega^2 - \omega_0^2)u_1^{(1)} = 0, \tag{3}$$

where

$$\begin{aligned}
\gamma &= \gamma_{11} + c_{66}\kappa_6^2, \\
\kappa_6^2 &= \frac{\pi^2}{12}(1 + k_{26}^2), \\
\omega_0^2 &= \frac{\pi^2}{4h^2}\frac{\overline{c}_{66}}{\rho}.
\end{aligned} \tag{4}$$

At the junctions between the electroded and unelectroded regions, the continuity of $u_1^{(1)}$ and $u_{1,1}^{(1)}$ need to be imposed [8].

III. Free Vibration Analysis

We look for trapped thickness-shear modes in the following form:

$$u_1^{(1)} = \begin{cases} A \exp \xi(a + x_1), & x_1 \le -a, \\ \overline{A}\cos\overline{\xi}x_1, & |x_1| \le a, \\ A\exp\xi(a - x_1), & x_1 \ge a, \end{cases} \tag{5}$$

where A, $\overline{A}$, ξ, and $\overline{\xi}$ are undetermined constants. Modes represented by (5) are symmetric about x_1. The antisymmetric modes are not electrically excitable by the electrode configuration in Fig. 1 and will not be considered. Substituting (5) into (1) and (3), we obtain:

$$\begin{aligned}
\overline{\xi}^2 &= \frac{\overline{\rho}}{\overline{\gamma}}(\omega^2 - \overline{\omega}_0^2) > 0, \\
\xi^2 &= \frac{\rho}{\gamma}(\omega_0^2 - \omega^2) > 0.
\end{aligned} \tag{6}$$

The inequalities hold because $\overline{\omega}_0^2 < \omega_0^2$. The frequencies we are looking for are in the interval $\overline{\omega}_0^2 < \omega^2 < \omega_0^2$. The continuity of $u_1^{(1)}$ and $u_{1,1}^{(1)}$ at $x_1 = \pm a$ implies:

$$\begin{aligned}
\overline{A}\cos\overline{\xi}a &= A. \\
\overline{A}\,\overline{\xi}\sin\overline{\xi}a &= A\xi.
\end{aligned} \tag{7}$$

For nontrivial solutions of A and $\overline{A}$, the determinant of coefficient matrix of (7) has to vanish, which yields:

$$\tan\overline{\xi}a = \frac{\xi}{\overline{\xi}}. \tag{8}$$

With (6), (8) can be written as an equation for ω. From $(7)_1$, the modes can be written as:

$$u_1^{(1)} = \begin{cases} \overline{A}\cos\overline{\xi}a\exp\xi(a + x_1), & x_1 \le -a, \\ \overline{A}\cos\overline{\xi}x_1, & |x_1| \le a, \\ \overline{A}\cos\overline{\xi}a\exp\xi(a - x_1), & x_1 \ge a. \end{cases} \tag{9}$$

Consider the first resonance frequency above $\overline{\omega}_0$, which is the one most important in applications. When $a \gg h$, the first mode has an essentially uniform thickness-shear deformation (small $\overline{\xi}$) in the long, electroded region, with an exponential decay right outside the electroded region. A limit solution to (8) for this case of small $\overline{\xi}$ and large a is:

$$\overline{\xi} \cong 0, \quad \overline{\xi}a \cong \frac{\pi}{2}. \tag{10}$$

From $(6)_1$ and $(10)_1$ we have the resonance frequency for this mode approximately as:

$$\omega \cong \overline{\omega}_0. \tag{11}$$

Then from $(6)_2$:

$$\xi^2 \cong \frac{\rho}{\gamma}(\omega_0^2 - \overline{\omega}_0^2). \tag{12}$$

The decay coefficient of the mode, ξa, characterizes how the mode is trapped. From (12), (2), and (4), the decay coefficient for this mode is approximately:

$$\begin{aligned}
\xi^2 a^2 &\cong \frac{\rho}{\gamma}(\omega_0^2 - \overline{\omega}_0^2)a^2 \\
&= \frac{\rho}{\gamma}\left[\frac{\pi^2}{4h^2}\frac{\overline{c}_{66}}{\rho} - \frac{\pi^2}{4h^2}\left(1 - R - \frac{4k_{26}^2}{\pi^2}\right)^2\frac{\overline{c}_{66}}{\rho}\right]a^2 \\
&= \frac{\rho}{\gamma}\frac{\pi^2}{4h^2}\frac{\overline{c}_{66}}{\rho}a^2\left[1 - \left(1 - R - \frac{4k_{26}^2}{\pi^2}\right)^2\right] \\
&\cong \frac{\pi^2 a^2}{2h^2}\frac{\overline{c}_{66}}{\gamma}\left(R + \frac{4k_{26}^2}{\pi^2}\right).
\end{aligned} \tag{13}$$

TABLE I
Effects of Piezoelectric Coupling.

Material	Y-cut quartz	AT-cut quartz	Y-cut langasite [9]
k_{26}	13.7%	8.8%	17.2%
$\frac{4k_{26}^2}{\pi^2}$	0.76%	0.32%	1.20%

We make the following observations from (13):

- Both mass ratio and piezoelectric coupling contribute to energy trapping.
- The contribution from mass ratio is linear. The contribution from piezoelectric coupling is quadratic in the electromechanical coupling coefficient k_{26} or the piezoelectric constant e_{26}.
- Energy trapping is stronger with increasing a/h ratio.
- To get some quantitative estimates, we calculate as shown in Table I.

We observe that, even for a relatively large mass loading on the order of $R = 1\%$, the piezoelectric coupling contribution to energy trapping is comparable to the mass effect and has to be considered.

The above analysis and the result in (13) are within the approximations made in [8] for small mass ratio and small piezoelectric coupling. More refined analyses are needed and can be performed. For example, in an unelectroded region of the plate, an equation governing the small voltage across the plate thickness can be included in the manner of [10]. These are left as future work. The goal of the present paper is to point out this situation and the need for further study.

IV. Conclusions

The effect of piezoelectric coupling on energy trapping is more pronounced in langasite than in quartz. It may be comparable to the mass effect of the electrodes. Therefore, for materials with relatively strong piezoelectric coupling, a pure elastic analysis is insufficient for energy trapping. Some of the previous understanding on energy trapping needs to be reexamined and more analyses, including full piezoelectric coupling, are needed.

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