INVITED PAPER

Electromagnetic Field Theory in $(N+1)$-Space-Time: A Modern Time-Domain Tensor/Array Introduction

A consistent tensor/array notation is used in this paper to present electromagnetic theory in $(N+1)$-space-time; this leads to considerable simplifications for spacial dimensions greater than 3.

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ABSTRACT | In this paper, a modern time-domain introduction is presented for electromagnetic field theory in $(N+1)$-space-time. It uses a consistent tensor/array notation that accommodates the description of electromagnetic phenomena in N-dimensional space (plus time), a requirement that turns up in present-day theoretical cosmology, where a unified theory of electromagnetic and gravitational phenomena is aimed at. The standard vectorial approach, adequate for describing electromagnetic phenomena in $(3+1)$-space-time, turns out to be not generalizable to $(N+1)$-space-time for $N > 3$ and the tensor/array approach that, in fact, has been introduced in Einstein's theory of relativity, proves, together with its accompanying notation, to furnish the appropriate tools. Furthermore, such an approach turns out to lead to considerable simplifications, such as the complete superfluousness of standard vector calculus and the standard condition on the right-handedness of the reference frames employed. Since the field equations do no more than interrelate (in a particular manner) changes of the field quantities in time to their changes in space, only elementary properties of (spatial and temporal) derivatives are needed to formulate the theory. The tensor/array notation furthermore furnishes indications about the structure of the field equations in any of the space-time discretization procedures for time-domain field computation. After discussing the field equations, the field/source compatibility relations and the constitutive relations, the field radiated by sources in an unbounded, homogeneous, isotropic, lossless medium is determined. All components of the radiated field are shown to be expressible as elementary operations acting on the scalar Green's function of the scalar wave equation in $(N+1)$-space-time. Time-convolution and time-correlation reciprocity relations conclude the general theory. Finally, two items on field computation are touched upon: the space-time-integrated field equations method of computation and the time-domain Cartesian coordinate stretching method for constructing perfectly matched computational embeddings. The performance of these items is illustrated in a demonstrator showing the 1-D pulsed electric-current and magnetic-current sources excited wave propagation in a layered medium.

KEYWORDS | Electromagnetic theory; field computation; tensor/array notation

I. INTRODUCTION

All numerical procedures for handling the computation of electromagnetic fields somehow or other employ the concept of "array" for manipulating the data. In analytical tools for field evaluation, on the contrary, the notations of 3-D vector calculus (boldface symbols for the field and source quantities, dot product, cross product) standardly occur. As a consequence, in combining the two, some kind of conversion has to be carried out. One source of errors in this conversion is the condition that the reference frame for denoting the position of an observer in space has to be

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right-handed in its orientation, a condition stemming from the integral form of the field equations (circulation integrals of electric and magnetic field strengths) one traditionally starts with. The elimination of this step seems certainly recommendable.

Quite another demand on electromagnetic field theory is put by the recent developments in theoretical cosmology [1]. Here, it is conjectured that a unification of the theory of the electromagnetic and gravitational fields requires the description of the electromagnetic field constituent in $(N+1)$-space-time, with $N > 3$. Now, such a generalization is not feasible via the standard 3-D vector calculus. In fact, the cosmological aspect (including its relativistic consequences) inspired the search for the formulation of electromagnetic field theory as it is presented in the sections below.

Einstein's view on the tensorial/array structure of field quantities and field equations proved to furnish the key to the requested generalization. Surprisingly, considerable simplifications and economizations manifested themselves. The whole machinery of standard vector calculus proves to be superfluous, while also the orientation of the reference frame used to specify the position of an observer in space turns out to be irrelevant. As is shown in the sections to follow, a more or less complete account of the basic notions of the time-domain physics of electromagnetic wave propagation can be covered in a very limited number of pages. The implications for the teaching of the theory are evident. Once one is familiar with the notation (the standard one in tensor calculus) and the manipulation of the expressions via the Einstein summation convention, the rest is—to speak with Albert Einstein—"details."

The basic material covered is as follows.

- The observer in $(N+1)$-space-time, tensor quantities (arrays), and Einstein subscript notation and summation convention in N-dimensional Euclidean space (Section II).
- The structure of the electromagnetic field equations, intensive and extensive field quantities, source quantities, and field/source compatibility relations (Section III).
- Constitutive relations (Section IV).
- Interface boundary conditions (Section V).
- Radiation from sources in an unbounded, homogeneous, isotropic, lossless medium (Section VI).
- Field/source reciprocity of the time-convolution type (Section VII).
- Field/source reciprocity of the time-correlation type (Section VIII).

Applications discussed are as follows.

- Green's tensors and the direct source problem (Section IX).
- Field representations in a subdomain of $\mathbb{R}^N$, equivalent surface sources, Huygens' principle, and the Oseen–Ewald extinction theorem (Section X).
- The Calderón identities (Section XI).

Finally, two items on field computation are touched upon.

- The space-time-integrated field equations method of computation (Section XII).
- The time-domain, causality preserving, Cartesian coordinate stretching method for constructing perfectly matched embeddings (Section XIII).

A (tentative) IEEE Xplore website demonstrator illustrates an application of Sections XII and XIII in a Matlab driven example.

- One-dimensional pulsed electric-current and magnetic-current sources excited wave propagation in a layered medium (Section XIV).

At the end, the principal formulas are collected in tabular form.

II. THE OBSERVER IN $(N+1)$-SPACE-TIME, TENSOR/ARRAY QUANTITIES, AND EINSTEIN SUBSCRIPT NOTATION AND SUMMATION CONVENTION IN N-DIMENSIONAL SPACE

The electromagnetic phenomena that we consider manifest themselves in $(N+1)$-space-time. An observer delineates them into a *spatial aspect* and a separate *temporal aspect*. To locate position in space, the observer employs the ordered sequence of Cartesian coordinates $\{x_1,\ldots,x_N\} \in \mathbb{R}^N$, or $\boldsymbol{x} \in \mathbb{R}^N$ for short, with $N = 1,2,3,\ldots$, with respect to a given *origin* $\mathcal{O}$, while distances are measured through the Euclidean norm $|\boldsymbol{x}| = (x_1^2 + \ldots + x_N^2)^{1/2} \geq 0$. The time coordinate used by the observer is $t \in \mathbb{R}$. Differentiation with respect to x_m is denoted by ∂_m; ∂_t is a reserved symbol to denote differentiation with respect to t.

In accordance with Einstein's postulate (in the theory of relativity), the quantitative representation of any physical quantity in N-dimensional space consists of N^p numbers, arranged as p-dimensional arrays of size N (also denoted as tensors of rank p in N-dimensional space), where $p = 0,1,2,\ldots$. The notation for such a quantity is a (usually internationally normalized) symbol supplied with an ordered sequence of p subscripts, each of which runs through the values $\{1,\ldots,N\}$ (*subscript notation*). An example of a tensor of rank two is the symmetrical unit tensor (Kronecker tensor)

$$\delta_{m,n} = 1 \text{ for } m = n, \quad \delta_{m,n} = 0 \text{ for } m \neq n. \tag{1}$$

Products of tensors are defined in the same way as products of arrays. Notationally, tensor products are handled via the *Einstein summation convention*, i.e., in any term of an expression, a product of two tensors is evaluated by summing the contributions that are indicated by common subscripts (see also Table 1).

Table 1 Observer in $(N+1)$-Space-Time, Subscript Notation, and Summation Convention

Observer in $(N+1)$-spacetime	
Position	$\boldsymbol{x} = \{x_1, \ldots, x_N\} \in \mathbb{R}^N$
Distance	$\lvert\boldsymbol{x}\rvert = (x_1^2 + \cdots + x_N^2)^{1/2} \geq 0$
Time	$t \in \mathbb{R}$
Subscript notation, summation convention	
$\lvert\boldsymbol{x}\rvert = (x_m x_m)^{1/2} = (\sum_{m=1}^{N} x_m x_m)^{1/2} \geq 0$	
$\delta_{m,n} x_n = x_m,\ \delta_{m,n}\delta_{n,p} = \delta_{m,p},\ \delta_{m,m} = N$	

III. THE STRUCTURE OF THE ELECTROMAGNETIC FIELD EQUATIONS

Any wavefield theory describes the pertaining wave phenomena through the occurrence of two *intensive field quantities* in conjunction with two *extensive field quantities*. The array/tensor product of the two intensive field quantities yields a tensor of rank one that represents the *transfer of field energy* via the wave's *area density of power flow*; the array/tensor product of the two extensive field quantities yields a tensor of rank one that represents the *volume density of field momentum* that exercises the wave's *Maxwell radiation pressure*. In the electromagnetic field, the *electric* field and source quantities are tensors of rank one, while the *magnetic* field and source quantities are antisymmetric tensors of rank two. The intensive field quantities are denoted as *field strengths*, the extensive field quantities as *flux densities*, and the source quantities as *volume source densities of current*. The corresponding symbols are (see also Table 2) as follows:

E_r	:	electric field strength
D_k	:	electric flux density
J_k	:	volume source density of electric current
$[H_{p,q}]^- = -[H_{q,p}]^-$	:	magnetic field strength
$[B_{i,j}]^- = -[B_{j,i}]^-$	:	magnetic flux density
$[K_{i,j}]^- = -[K_{j,i}]^-$	:	volume source density of magnetic current
$S_m = [H_{m,k}]^- E_k$	:	area density of power flow
$G_i = [B_{i,j}]^- D_j$	:	volume density of electromagnetic momentum

The *wavefield equations* relate the time rate of change of an extensive field quantity to the spatial rate of change of its "dual" intensive field quantity, thus enabling the existence of solutions with a wavelike character. The excitation of such solutions is accommodated in accordance with the (Einstein) requirement that in a field equation all terms should be tensors of equal ranks/arrays of equal sizes. For the electromagnetic field equations, this results in the (Maxwell) field equations (see also Table 3)

$$\partial_m [H_{m,k}]^- + \partial_t D_k = -J_k \tag{2}$$
$$[\partial_i E_j]^- + \partial_t [B_{i,j}]^- = -[K_{i,j}]^- \tag{3}$$

in which

$$[\partial_i E_j]^- = (\partial_i E_j - \partial_j E_i)/2. \tag{4}$$

Operating on (2) with ∂_k and noting that $\partial_k \partial_m [H_{m,k}]^- = 0$, we obtain the *electric field/source compatibility relation*

$$\partial_t \partial_k D_k = -\partial_k J_k. \tag{5}$$

Operating on (3) with ∂_k where $k \neq i \neq j$, cyclically permuting the subscripts and adding the results, we obtain the *magnetic field/source compatibility relation*

$$\partial_t \left[\partial_k [B_{i,j}]^-\right]^{\circlearrowright} = -\left[\partial_k [K_{i,j}]^-\right]^{\circlearrowright} \tag{6}$$

where

$$\left[\partial_k [B_{i,j}]^-\right]^{\circlearrowright} = \partial_k [B_{i,j}]^- + \partial_i [B_{j,k}]^- + \partial_j [B_{k,i}]^- (i \neq j \neq k). \tag{7}$$

Evidently, the condition $i \neq j \neq k$ can only be met if $N \geq 3$, which implies that $N = 3$ *is the minimum number of spatial dimensions for which a field structure of the electromagnetic type can exist.*

In adherence to the physical concept that the volume densities of current are associated with the (collective) motion of charged particles in a flow in which the conservation of particles holds [2, Sec. 19.4], the *volume density of electric charge* is introduced as (see also Table 4)

$$\rho \stackrel{\text{def}}{=} -\partial_t^{-1} \partial_k J_k \tag{8}$$

where ∂_t^{-1} denotes integration with respect to time from the instant of onset of the sources onward. Equation (8) entails the *continuity equation of electric charge*

$$\partial_k J_k + \partial_t \rho = 0. \tag{9}$$

Similarly, the *volume density of magnetic charge* is introduced as (see also Table 4)

$$\sigma_{i,j,k} \stackrel{\text{def}}{=} -\partial_t^{-1} \left[\partial_i [K_{j,k}]^-\right]^{\circlearrowright} \tag{10}$$

which entails the *continuity equation of magnetic charge*

$$\left[\partial_i [K_{j,k}]^-\right]^{\circlearrowright} + \partial_t \sigma_{i,j,k} = 0. \tag{11}$$

Table 2 EM Field and Source Quantities

EM field and source quantities			
	Intensive field quantity	Extensive field quantity	Source quantity
	'field strength'	'flux density'	'volume source density of current'
Electric	E_r	D_k	J_k
Magnetic	$[H_{p,q}]^- = -[H_{q,p}]^-$	$[B_{i,j}]^- = -[B_{j,i}]^-$	$[K_{i,j}]^- = -[K_{j,i}]^-$
	Product $\Downarrow$	Product $\Downarrow$	
	Area density of EM power flow	Volume density of EM momentum	
	$S_m = [H_{m,r}]^- E_r$	$G_i = [B_{i,j}]^- D_j$	

Table 3 Maxwell Field Equations, Field/Source Compatibility Relations

Maxwell field equations	Operation
$\partial_m [H_{m,k}]^- + \partial_t D_k = -J_k$	$\partial_k \ \Downarrow^{[1]}$
$[\partial_i E_j]^- + \partial_t [B_{i,j}]^- = -[K_{i,j}]^-$	$\partial_k +$ cyclic $\Downarrow^{[2]}$
$[\partial_i E_j]^- = (\partial_i E_j - \partial_j E_i)/2$	
Field/source compatibility relations	
$\partial_t \partial_k D_k = -\partial_k J_k$	[1]
$\partial_t [\partial_k [B_{i,j}]^-]^{\circlearrowleft} = -[\partial_k [K_{i,j}]^-]^{\circlearrowleft}$	[2]
$[\partial_k [B_{i,j}]^-]^{\circlearrowleft} = \partial_k [B_{i,j}]^- + \partial_i [B_{j,k}]^- + \partial_j [B_{k,i}]^- \ (i \neq j \neq k)$	

Table 4 Volume Densities of Charge, Conservation Laws

	Volume density of charge	Conservation law
Electric	$\rho \stackrel{\text{def}}{=} -\partial_t^{-1} \partial_k J_k$	$\partial_k J_k + \partial_t \rho = 0$
Magnetic	$\sigma_{i,j,k} \stackrel{\text{def}}{=} -\partial_t^{-1} [\partial_i [K_{j,k}]^-]^{\circlearrowleft}$	$[\partial_i [K_{j,k}]^-]^{\circlearrowleft} + \partial_t \sigma_{i,j,k} = 0$
	$\partial_t^{-1} =$ time integration	

From the procedure it follows that the volume density of electric charge is a *scalar quantity* (*tensor of rank zero*), while the volume density of magnetic charge is a *cyclic symmetrical tensor of rank three*. (Since for $N = 3$ only a single number is involved, this tensor is commonly mistaken to be a scalar.) The tensorial character of the magnetic charge has implications for the Dirac theory of the magnetic "monopole" [3].

Evidently, the number of unknowns in the field equations is, so far, twice the number of equations. As a consequence, the fundamental physical condition of the *uniqueness* of the solution to the *initial-value problems* is not yet met. This condition requires that, given the physical state of a system at some instant t_0, its *time evolution* into $t > t_0$ should in a unique manner follow from the pertaining field equations. To meet this condition, the field equations developed thus far have to be supplemented with the *constitutive relations* that characterize the medium in which the field is present. Standardly, these constitutive relations express the values of the extensive field quantities in terms of the values of the intensive field quantities. For the electromagnetic field, the relevant general necessary and sufficient conditions are, for the most general case, not known. Only sufficient conditions (for a large class of media met in practice) are well established. These are discussed in the next section.

IV. THE ELECTROMAGNETIC CONSTITUTIVE RELATIONS

In this section, the electromagnetic constitutive relations for the class of *linear*, *time-invariant*, *passive*, *causally*, and *locally reacting media* are presented (see also Table 5). For this class of media, the uniqueness of the initial-value problem can be proved [4], [5].[1] Full inhomogeneity, anisotropy, and (Boltzmann) relaxation losses [6] are included.

In general, the medium's response consists of an instantaneous part and a time-delayed part (relaxation). In the Lorentz theory of electrons [7], the instantaneous part of the response is associated with vacuum, while the relaxation is representative for the presence of matter in the

[1] Freely downloadable from http://ursi-test.intec.ugent.be/files/rsb_june_2003.pdf

Table 5 Electromagnetic Constitutive Relations

Electromagnetic constitutive relations	
Linear | Time-invariant | Locally reacting media	
$D_k = \epsilon_{k,r} \overset{(t)}{*} E_r$	$\epsilon_{k,r}(\boldsymbol{x},t) =$ electric permittivity
$[B_{i,j}]^- = \mu^-_{i,j,p,q} \overset{(t)}{*} [H_{p,q}]^-$	$\mu^-_{i,j,p,q}(\boldsymbol{x},t) =$ magnetic permeability
$\overset{(t)}{*} =$ time convolution	
Causality	
$\{\epsilon_{k,r}, \mu^-_{i,j,p,q}\}(\boldsymbol{x},t) = 0$ for $t < 0$	
Special media	
Homogeneous	
$\{\epsilon_{k,r}, \mu^-_{i,j,p,q}\}(\boldsymbol{x},t) = \{\epsilon_{k,r}, \mu^-_{i,j,p,q}\}(t)$	
Instantaneously reacting	
$\{\epsilon_{k,r}, \mu^-_{i,j,p,q}\}(\boldsymbol{x},t) = \{\epsilon_{k,r}, \mu^-_{i,j,p,q}\}(\boldsymbol{x})\delta(t)$	
Isotropic	
$\epsilon_{k,r}(\boldsymbol{x},t) = \epsilon(\boldsymbol{x},t)\delta_{k,r}$	$D_k = \epsilon \overset{(t)}{*} E_k$
$\mu^-_{i,j,p,q}(\boldsymbol{x},t) = \mu^-(\boldsymbol{x},t)\delta_{i,p}\delta_{j,q}$	$[B_{i,j}]^- = \mu^- \overset{(t)}{*} [H_{i,j}]^-$
Vacuum: $c_0 = 299792458$ m/s	
$\epsilon(\boldsymbol{x},t) = \epsilon_0\, \delta(t)$	$\epsilon_0 = (1/c_0^2\, \mu_0)$ F/m
$\mu^-(\boldsymbol{x},t) = 2\, \mu_0\, \delta(t)$	$\mu_0 = 4\pi{\cdot}10^{-7}$ H/m

background vacuum. Classic atomic models for the relaxation functions, based on the Lorentz theory of electrons, can be found in [2, Ch. 19]. With $\overset{(t)}{*}$ denoting time convolution, the pertaining relations are (see also Table 5)

$$D_k(\mathbf{x},t) = \epsilon_{k,r}(\mathbf{x},t) \overset{(t)}{*} E_r(\mathbf{x},t) \tag{12}$$

where

$$\epsilon_{k,r}(\mathbf{x},t) = \text{electric permittivity} \tag{13}$$

and

$$[B_{i,j}]^-(\mathbf{x},t) = \mu^-_{i,j,p,q}(\mathbf{x},t) \overset{(t)}{*} [H_{p,q}]^-(\mathbf{x},t) \tag{14}$$

where

$$\mu^-_{i,j,p,q}(\mathbf{x},t) = \text{magnetic permeability.} \tag{15}$$

For *homogeneous media*, we have $\{\epsilon_{k,r}, \mu^-_{i,j,p,q}\}(\mathbf{x},t) = \{\epsilon_{k,r}, \mu^-_{i,j,p,q}\}(t)$. For *instantaneously reacting media*, we have $\{\epsilon_{k,r}, \mu^-_{i,j,p,q}\}(\mathbf{x},t) = \{\epsilon_{k,r}, \mu^-_{i,j,p,q}\}(\mathbf{x})\delta(t)$. For *isotropic media*, we have $\epsilon_{k,r}(\mathbf{x},t) = \epsilon(\mathbf{x},t)\delta_{k,r}$ and $\mu^-_{i,j,p,q}(\mathbf{x},t) = \mu^-(\mathbf{x},t)\delta_{i,p}\delta_{j,q}$, which entails $D_k = \epsilon \overset{(t)}{*} E_k$ and $[B_{i,j}]^- = \mu^- \overset{(t)}{*} [H_{i,j}]^-$, respectively. The vacuum values are $\mu^-(\mathbf{x},t) = 2\,\mu_0\,\delta(t)$, with $\mu_0 = 4\pi\cdot 10^{-7}$ H/m and $\epsilon(\mathbf{x},t) = \epsilon_0\,\delta(t)$ with $\epsilon_0 = (1/c_0^2\,\mu_0)$ F/m and $c_0 = 299\,792\,458$ m/s.

Causality and the Time Laplace Transformation: The properties associated with the causality of the medium's response are most adequately handled via the *time Laplace transformation*

$$\left\{\hat{\epsilon}_{k,r}, \hat{\mu}^-_{i,j,p,q}\right\}(\mathbf{x},s) = \int_{t=0}^{\infty} \exp(-st)\left\{\epsilon_{k,r}, \mu^-_{i,j,p,q}\right\}(\mathbf{x},t)\,dt \quad \text{for } s \in \mathbb{C}, \quad \mathrm{Re}(s) > 0. \tag{16}$$

The transforms in the left-hand side are *analytic* in the right half $\{s \in \mathbb{C}, \mathrm{Re}(s) > 0\}$ of the complex s-plane (Fig. 1).

Their limiting values on the imaginary axis of the s-plane yield the *spectral behavior* of the medium's response. The diagram, in which $20\log_{10}[|\{\hat{\epsilon}_{k,r}, \hat{\mu}^-_{i,j,p,q}\}(\mathbf{x}, 2\pi \mathrm{j}f)|]$, where j is the imaginary unit and f is the *frequency*, is plotted against $\log_{10}(f)$, is denoted as the *spectral diagram* or *Bode diagram* [8]. The *Debije* relaxation function is a standard tool in the modeling of relaxation in electrical conduction properties; the *Lorentz* relaxation function is a standard tool in the modeling of relaxation in dielectric properties (Fig. 2).

Uniqueness of the Initial-Value Problem: There seems not to be a time-domain uniqueness proof of the initial-value (time-evolution) problem for media that show an arbitrary relaxation behavior. The known proof goes via the time Laplace transformed field equations and constitutive relations [5] through their properties at the sequence of equidistant values of the transform parameter s (Fig. 1)

$$\mathcal{L} = \{s \in \mathbb{R}; s = s_0 + n\,h, s_0 > 0, h > 0, n = 0,1,2,\ldots\} \quad \text{(Lerch sequence)} \tag{17}$$

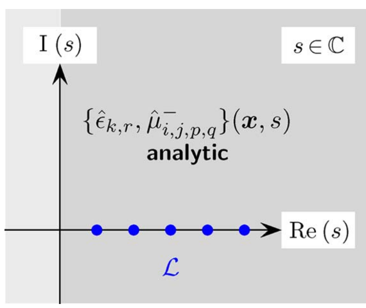

Fig. 1. ***Domain of analyticity and Lerch sequence in the complex time Laplace transform plane.***

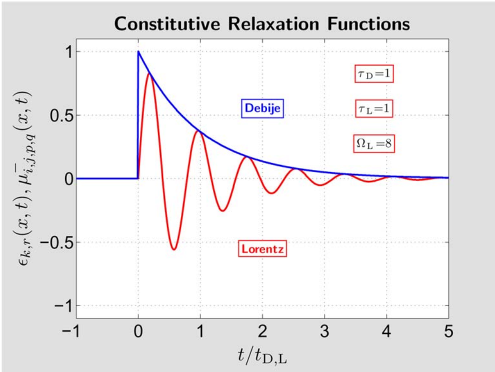

Fig. 2. ***Debije and Lorentz relaxation functions.***

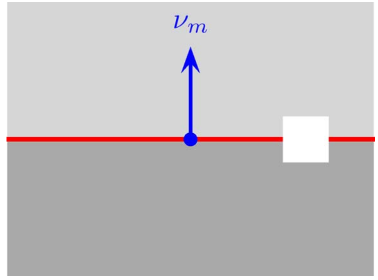

Fig. 3. ***Passive interface between two media with different constitutive parameters.***

on the positive real s-axis. The corresponding uniqueness in the time domain then follows from Lerch's theorem [9, p. 63].

Sufficient conditions for the uniqueness of the initial-value (time-evolution) problem are (the proof runs parallel to the one presented for $N = 3$ in [5])

$$\hat{E}_k \hat{\epsilon}_{k,r} \hat{E}_r > 0, \qquad \text{for } s \in \mathcal{L}$$

and

$$[\hat{H}_{i,j}]^- \hat{\mu}^-_{i,j,p,q} [\hat{H}_{p,q}]^- > 0, \qquad \text{for } s \in \mathcal{L} \tag{18}$$

and any nonzero field values. For radiation problems in unbounded domains, a restriction occurs in that outside some sphere of finite radius the medium should be homogeneous and isotropic. For media that are, in addition, instantaneously reacting (lossless) the radiation from sources is discussed in Section VI.

V. INTERFACE BOUNDARY CONDITIONS

At the passive interface between two media across which the constitutive parameters jump by finite amounts, also the field components show jump discontinuities. Certain components, however, remain continuous. The pertaining continuity conditions follow from the field equations upon decomposing the spatial differentiation ∂_m into a component normal to the interface $(\partial_m)_\perp$ and a component parallel to it $(\partial_m)_\parallel$. Let ν_m denote the unit vector along the normal to the interface, then (Fig. 3)

$$(\partial_m)_\perp = \nu_m(\nu_n \partial_n) \quad \text{and} \quad (\partial_m)_\parallel = \partial_m - (\partial_m)_\perp. \tag{19}$$

If, now, the operation of differentiation perpendicular to the interface would act on a field component that jumps across the interface, this would lead to a Dirac delta distribution operative at the interface and this would violate the assumed passivity of the interface. Hence, $(\partial_m)_\perp$ can only act at field components that are continuous across the interface. This consideration leads to the interface boundary conditions (see also Table 6)

$$\nu_m [H_{m,k}]^- |_-^+ = 0 \tag{20}$$

$$[\nu_i E_j]^- |_-^+ = 0. \tag{21}$$

Note that (21) implies $\nu_i[\nu_i E_j]^- |_-^+ = 0$ and, hence, $[E_j - (\nu_i E_i)\nu_j]|_-^+ = 0$ or $(E_j)_\parallel|_-^+ = 0$.

VI. RADIATION FROM SOURCES IN UNBOUNDED, HOMOGENEOUS, ISOTROPIC, LOSSLESS MEDIA

In this section, the radiation from sources in unbounded $\mathbb{R}^N$, filled with a homogeneous, isotropic, and lossless medium is discussed. It will be shown that only elementary mathematical operations such as spatial differentiation, temporal differentiation, spatial convolution, and temporal convolutions are needed in this case to arrive at explicit expressions for the electric and magnetic field components. All of these operations are commutable. Another feature is that the orientation of the spatial reference frame employed will turn out to be irrelevant. The source quantities J_k and $[K_{i,j}]^-$ that excite the field will be assumed to have the bounded spatial supports $\mathcal{D}^J \subset \mathbb{R}^N$ and $\mathcal{D}^K \subset \mathbb{R}^N$, respectively. The constitutive coefficients of the medium are $\epsilon > 0$ and $\mu^- > 0$. The electric field

Table 6 (Passive) Interface Boundary Conditions

(Passive) interface boundary conditions	
$\nu_m[H_{m,k}]^-\|_-^+ = 0$	$[\nu_i E_j]^-\|_-^+ = 0$
Note: $[\nu_i E_j]^-\|_-^+ = 0 \Rightarrow \nu_i[\nu_i E_j]^-\|_-^+ = 0 \Rightarrow$ $[E_j - (\nu_i E_i)\nu_j]\|_-^+ = 0 \Rightarrow (E_j)_\parallel\|_-^+ = 0$	

strength E_r and the magnetic field strength $[H_{p,q}]^-$ then satisfy the Maxwell equations

$$\partial_m[H_{m,k}]^- + \epsilon\partial_t E_k = -J_k \tag{22}$$
$$[\partial_i E_j]^- + \mu^-\partial_t[H_{i,j}]^- = -[K_{i,j}]^- \tag{23}$$

with the corresponding source/field compatibility relations

$$\epsilon\partial_t\partial_k E_k = -\partial_k J_k \tag{24}$$
$$\mu^-\partial_t\left[\partial_k[H_{i,j}]^-\right]^{\circlearrowleft} = -\left[\partial_k[K_{i,j}]^-\right]^{\circlearrowleft}, \qquad i \neq j \neq k. \tag{25}$$

Note that (25) implies that $N \geq 3$.

Elimination of $[H_{i,j}]^-$ from (22) and (23) and use of the compatibility relation (24) lead to the *electric-field vector wave equation*

$$(\partial_m\partial_m)E_k - c^{-2}\partial_t^2 E_k = -Q_k \tag{26}$$

in which

$$c = 1/(2\mu^-\epsilon)^{1/2} \tag{27}$$

and

$$Q_k = -2\mu^-\partial_t J_k + (1/\epsilon)\partial_t^{-1}\partial_k(\partial_m J_m) + 2\partial_m[K_{m,k}]^- \tag{28}$$

where ∂_t^{-1} denotes time integration from the instant of onset of the sources onward. Introducing the *vector potential* A_k as the solution of the wave equation

$$(\partial_m\partial_m)A_k - c^{-2}\partial_t^2 A_k = -J_k \tag{29}$$

and the antisymmetric *tensor potential* $[\Psi_{i,j}]^-$ as the solution of the wave equation

$$(\partial_m\partial_m)[\Psi_{i,j}]^- - c^{-2}\partial_t^2[\Psi_{i,j}]^- = -[K_{i,j}]^- \tag{30}$$

and using the property that, for constant ϵ and μ^-, the wave operator $(\partial_m\partial_m) - c^{-2}\partial_t^2$ and the operations ∂_m and ∂_t commute, (26)–(30) lead to

$$E_k = -2\mu^-\partial_t A_k + (1/\epsilon)\partial_t^{-1}\partial_k\partial_m A_m + 2\partial_m[\Psi_{m,k}]^-. \tag{31}$$

Substituting this result in (23), we arrive at

$$[H_{i,j}]^- = -2\epsilon\partial_t[\Psi_{i,j}]^- + (1/\mu^-)\partial_t^{-1}\partial_m\left[\partial_m[\Psi_{i,j}]^-\right]^{\circlearrowleft} + 2[\partial_i A_j]^-. \tag{32}$$

Finally, upon introducing the *Green's function* $G(\boldsymbol{x}, t)$ of the scalar wave equation as the solution of

$$(\partial_m\partial_m)G - c^{-2}\partial_t^2 G = -\delta(\boldsymbol{x}, t) \tag{33}$$

where $\delta(\boldsymbol{x}, t)$ is the $(N+1)$-space-time Dirac distribution operative at $\boldsymbol{x} = \boldsymbol{0}$ and $t = 0$, and using the property

$$\left\{J_k, [K_{i,j}]^-\right\}(\boldsymbol{x}, t) = \delta(\boldsymbol{x}, t) \overset{\boldsymbol{x}}{*}\overset{t}{*} \left\{J_k, [K_{i,j}]^-\right\}(\boldsymbol{x}, t) \tag{34}$$

where $\overset{(\boldsymbol{x})}{*}$ denotes spatial convolution and $\overset{(t)}{*}$ denotes temporal convolution, (29) and (30) lead to the representations

$$\left\{A_k, [\Psi_{i,j}]^-\right\}(\boldsymbol{x}, t) = G(\boldsymbol{x}, t) \overset{\boldsymbol{x}}{*}\overset{t}{*} \left\{J_k, [K_{i,j}]^-\right\}(\boldsymbol{x}, t) \tag{35}$$

where the convolutions are extended over the spatio–temporal supports of the pertaining sources. For $N > 3$, the Green's function is of a complicated nature that fundamentally differs for even and odd values of N. The simple case for $N = 3$ is further discussed below.

Radiation in $(3+1)$-Space-Time: In $(3+1)$-space-time, $G(\boldsymbol{x}, t)$ is given by

$$G(\boldsymbol{x}, t) = \frac{\delta(t - |\boldsymbol{x}|/c)}{4\pi|\boldsymbol{x}|}, \qquad \text{for } \boldsymbol{x} \neq \boldsymbol{0}. \tag{36}$$

For this case, (35) leads to the well-known *retarded potentials*

$$\left\{A_k, [\Psi_{i,j}]^-\right\}(\boldsymbol{x}, t) = \int_{\mathcal{D}^{J,K}} \frac{\left\{J_k, [K_{i,j}]^-\right\}(\boldsymbol{x}', t - |\boldsymbol{x} - \boldsymbol{x}'/c)}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|}\, dV(\boldsymbol{x}'). \tag{37}$$

The Far-Field Approximation $[(3+1)$-Space-Time$]$: The far-field approximation with respect to the *reference center*

$\mathcal{O}$ is the leading term in the expansion of the field expressions as $|\mathbf{x}| \to \infty$. With

$$|\mathbf{x} - \mathbf{x}'| = |\mathbf{x}| - \xi_m x'_m + O\left(|\mathbf{x}|^{-1}\right), \qquad \text{as } |\mathbf{x}| \to \infty \quad (38)$$

where

$$\xi_m = x_m/|\mathbf{x}| \quad (39)$$

is the unit vector in the direction of observation, we obtain

$$\left\{A_k, [\Psi_{i,j}]^-\right\}(\mathbf{x}, t) = \frac{\left\{A_k^\infty, \left[\Psi_{i,j}^\infty\right]^-\right\}(\boldsymbol{\xi}, t - |\mathbf{x}|/c)}{4\pi|\mathbf{x}|} \times \left[1 + O\left(|\mathbf{x}|^{-1}\right)\right], \qquad \text{as } |\mathbf{x}| \to \infty \quad (40)$$

with

$$\left\{A_k^\infty, \left[\Psi_{i,j}^\infty\right]^-\right\}(\xi, t) = \int_{\mathcal{D}^{JK}} \left\{J_k, [K_{i,j}]^-\right\}\left(\mathbf{x}', t + \xi_m x'_m/c\right) dV(\mathbf{x}'). \quad (41)$$

Observing that

$$\partial_m(\cdot) = -(\xi_m/c)\partial_t(\cdot)\left[1 + O\left(|\mathbf{x}|^{-1}\right)\right], \qquad \text{as } |\mathbf{x}| \to \infty \quad (42)$$

the far-field approximations for the field strengths are obtained as

$$\left\{E_r, [H_{p,q}]^-\right\}(\mathbf{x}, t) = \frac{\left\{E_r^\infty, \left[H_{p,q}^\infty\right]^-\right\}(\boldsymbol{\xi}, t - |\mathbf{x}|/c)}{4\pi|\mathbf{x}|} \times \left[1 + O\left(|\mathbf{x}|^{-1}\right)\right], \qquad \text{as } |\mathbf{x}| \to \infty \quad (43)$$

in which

$$E_r^\infty = -2\mu^-(\delta_{r,k} - \xi_r\xi_k)\partial_t A_k^\infty - 2(\xi_m/c)\partial_t\left[\Psi_{m,r}^\infty\right]^- \quad (44)$$

$$\left[H_{i,j}^\infty\right]^- = -2\epsilon\left(\partial_t\left[\Psi_{i,j}^\infty\right]^- - \xi_m\left[\xi_m\partial_t\left[\Psi_{i,j}^\infty\right]^-\right]^\cup\right) - 2\left[(\xi_i/c)\partial_t A_j^\infty\right]^-. \quad (45)$$

Note that the far-field spherical wave amplitudes satisfy the local *plane-wave relations*

$$(-\xi_m/c)\left[H_{m,k}^\infty\right]^- + \epsilon E_k^\infty = 0 \quad (46)$$

$$\left[(-\xi_i/c)E_j^\infty\right]^- + \mu^-\left[H_{i,j}^\infty\right]^- = 0 \quad (47)$$

for a wave traveling in the direction of $\boldsymbol{\xi}$.

VII. TIME-CONVOLUTION RECIPROCITY

Reciprocity theorems belong to the category of most fundamental theorems in wave physics. As has been discussed in [10] and [11], various particular cases can be considered as the basis for such computational techniques as the domain integral equations method, the boundary integral equations method, and the method of moments, while the concept of introducing the different point-source solutions (Green's functions) leads to such results as Huygens' principle and the Oseen–Ewald extinction theorem (related to the null-field method) and the source-to-receiver data transfer in imaging and constitutive parameter inversion procedures.

Reciprocity deals with the interaction of two states, both of which can exist in a certain domain $\mathcal{D} \subset \mathbb{R}^N$ in space. The two states are associated with, in general, different excitations and are present in, in general, media with different constitutive properties, and, hence, exhibit different field values. The category of configurations for which reciprocity will be discussed is the same as the one for which uniqueness of the time evolution can be proved, i.e., for time-invariant configurations with piecewise continuous, linear, passive, locally, and causally reacting media (Section IV). For such configurations, two types of reciprocity relation can be distinguished [2, Sec. 28.2 and 28.3], viz., the one of the *time-convolution type*, where the interaction between the two states involves their time convolution, and the one of the *time-correlation type*, where the interaction between the two states involves their time correlation. In this respect, it is of importance to observe that the time-convolution one *preserves causality*, whereas the time-correlation one has no such property. This distinction plays an important role in case the theorems are applied to unbounded domains. The time-correlation one leads, for zero correlation time and applied to two identical states, to the energy theorem.

The two states are indicated by the superscripts A and B. The field equations applying to state A are

$$\partial_m\left[H_{m.k}^A\right]^- + \partial_t\left(\epsilon_{k,r}^A \overset{(t)}{*} E_r^A\right) = -J_k^A \quad (48)$$

$$\left[\partial_i E_j^A\right]^- + \partial_t\left(\mu_{i,j,p,q}^{-;A} \overset{(t)}{*} \left[H_{p,q}^A\right]^-\right) = -\left[K_{i,j}^A\right]^-. \quad (49)$$

The field equations applying to state B are

$$\partial_m[H^B_{m.r}]^- + \partial_t\left(\epsilon^B_{r,k} \overset{(t)}{*} E^B_k\right) = -J^B_r \tag{50}$$

$$\left[\partial_p E^B_q\right]^- + \partial_t\left(\mu^{-;B}_{p,q,i,j} \overset{(t)}{*} \left[H^B_{i,j}\right]^-\right) = -\left[K^B_{p,q}\right]^-. \tag{51}$$

Upon carrying out the operation

$$(48) \overset{(t)}{*} E^B_k - (49) \overset{(t)}{*} \left[H^B_{i,j}\right]^- - (50) \overset{(t)}{*} E^A_r + (51) \overset{(t)}{*} \left[H^A_{p,q}\right]^-$$

we arrive at the *local form* of the *time-convolution reciprocity relation*

$$\partial_m S^{AB}_m + \partial_t U^{AB} = W^{AB} \tag{52}$$

in which

$$S^{AB}_m = \left[H^A_{m,k}\right]^- \overset{(t)}{*} E^B_k - \left[H^B_{m,r}\right]^- \overset{(t)}{*} E^A_r \tag{53}$$

represents the transfer of field interaction

$$\begin{aligned} U^{AB} = {} & E^B_k \overset{(t)}{*} \left(\epsilon^A_{k,r} - \epsilon^B_{r,k}\right) \overset{(t)}{*} E^A_r \\ & - \left[H^B_{i,j}\right]^- \overset{(t)}{*} \left(\mu^{-;A}_{i,j,p,q} - \mu^{-;B}_{p,q,i,j}\right) \overset{(t)}{*} \left[H^A_{p,q}\right]^- \end{aligned} \tag{54}$$

yields the contrast-in-media interaction, and

$$\begin{aligned} W^{AB} = -\Bigg(& E^B_k \overset{(t)}{*} J^A_k - E^A_r \overset{(t)}{*} J^B_r - \left[H^B_{i,j}\right]^- \overset{(t)}{*} \left[K^A_{i,j}\right]^- \\ & + \left[H^A_{p,q}\right]^- \overset{(t)}{*} \left[K^B_{p,q}\right]^- \Bigg) \end{aligned} \tag{55}$$

represents the field/source interaction.

Upon integrating (52) over a bounded domain $\mathcal{D} \subset \mathbb{R}^N$ (Figure 4) and applying Gauss' theorem, we arrive at the global time convolution reciprocity relation (for the domain $\mathcal{D}$) as

$$\int_{\partial\mathcal{D}} \nu_m S^{AB}_m \, dA + \partial_t \int_{\mathcal{D}} U^{AB} \, dV = \int_{\mathcal{D}} W^{AB} \, dV \tag{56}$$

in which $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$ and ν_m is the unit vector along the outward normal to $\partial\mathcal{D}$.

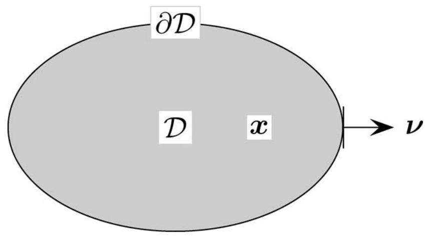

Fig. 4. ***Configuration of application of the global time-convolution reciprocity relation.***

The further discussion of corollaries of (56) goes along the same lines as in [2, Sec. 28.2]. In computational electromagnetics, (56) provides an important check on the consistency of the pertaining numerical codes.

Field Interaction With a Kirchhoff Circuit: In a Kirchhoff circuit, the propagation time for the field to traverse the maximum diameter of the circuit is negligible with respect to the spatial extent of pertaining pulsed field. Then, on some closed surface $\mathcal{S}^K$ surrounding the circuit in such a way that, upon it, the Kirchhoff circuit description in terms of the voltages $\{V_k(t); k = 1, \ldots, K\}$ across and electric currents $\{I_k(t); k = 1, \ldots, K\}$ fed into its K accessible ports holds, the (Maxwell) field/(Kirchhoff) circuit interaction integral over $\mathcal{S}^K$ is expressed as

$$\int_{\partial\mathcal{D}^K} \nu_m S^{AB}_m \, dA = \sum_{k=1}^{K}\left(I^A_k \overset{(t)}{*} V^B_k - I^B_k \overset{(t)}{*} V^A_k\right) \tag{57}$$

where the unit vector ν_m along the normal points toward the circuit. This relation is one of the basic ones in the electromagnetic interference (EMI) analysis of the "emitter"/"susceptor" interaction.

VIII. TIME-CORRELATION RECIPROCITY

The time-correlation reciprocity relation is most easily arrived at by writing the time correlation of two of the pertaining quantities as their time convolution, where the second of the two quantities is replaced by its *time-reversed* one. Denoting the operation of time reversal by the superscript *, we start from the field equations applying to state A as

$$\partial_m\left[H^A_{m.k}\right]^- + \partial_t\left(\epsilon^A_{k,r} \overset{(t)}{*} E^A_r\right) = -J^A_k \tag{58}$$

$$\left[\partial_i E^A_j\right]^- + \partial_t\left(\mu^{-;A}_{i,j,p,q} \overset{(t)}{*} \left[H^A_{p,q}\right]^-\right) = -\left[K^A_{i,j}\right]^-. \tag{59}$$

State B applies to the time-reversed field that satisfies

$$\partial_m \left[H^{B*}_{m.r}\right]^- - \partial_t \left(\epsilon^{B*}_{r,k} \overset{(t)}{*} E^{B*}_k\right) = -J^{B*}_r \tag{60}$$

$$\left[\partial_p E^{B*}_q\right]^- - \partial_t \left(\mu^{-;B*}_{p,q,i,j} \overset{(t)}{*} \left[H^{B*}_{i,j}\right]^-\right) = -\left[K^{B*}_{p,q}\right]^-. \tag{61}$$

Upon carrying out the operation

$$(58) \overset{(t)}{*} E^{B*}_k + (59) \overset{(t)}{*} \left[H^{B*}_{i,j}\right]^- + (60) \overset{(t)}{*} E^A_r + (61) \overset{(t)}{*} \left[H^A_{p,q}\right]^-$$

we arrive at the *local form* of the *time-convolution reciprocity relation*

$$\partial_m S^{AB*}_m + \partial_t U^{AB*} = W^{AB*} \tag{62}$$

in which

$$S^{AB*}_m = \left[H^A_{m,k}\right]^- \overset{(t)}{*} E^{B*}_k + \left[H^{B*}_{m,r}\right]^- \overset{(t)}{*} E^A_r \tag{63}$$

represents the transfer of field interaction

$$\begin{aligned} U^{AB*} = {} & E^{B*}_k \overset{(t)}{*} \left(\epsilon^A_{k,r} - \epsilon^{B*}_{r,k}\right) \overset{(t)}{*} E^A_r \\ & + \left[H^{B*}_{i,j}\right]^- \overset{(t)}{*} \left(\mu^{-;A}_{i,j,p,q} - \mu^{-;B*}_{p,q,i,j}\right) \overset{(t)}{*} \left[H^A_{p,q}\right]^- \end{aligned} \tag{64}$$

yields the contrast-in-media interaction, and

$$\begin{aligned} W^{AB*} = -\Big(& E^{B*}_k \overset{(t)}{*} J^A_k + E^A_r \overset{(t)}{*} J^{B*}_r + \left[H^{B*}_{i,j}\right]^- \overset{(t)}{*} \left[K^A_{i,j}\right]^- \\ & + \left[H^A_{p,q}\right]^- \overset{(t)}{*} \left[K^*_{p,q}B\right]^- \Big) \end{aligned} \tag{65}$$

represents the field/source interaction.

Upon integrating (62) over a bounded domain $\mathcal{D} \subset \mathbb{R}^N$ (Fig. 5) and applying Gauss' theorem, we arrive at the global time-correlation reciprocity relation (for the domain $\mathcal{D}$) as

$$\int_{\partial\mathcal{D}} \nu_m S^{AB*}_m \, dA + \partial_t \int_{\mathcal{D}} U^{AB*} \, dV = \int_{\mathcal{D}} W^{AB*} \, dV \tag{66}$$

in which $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$ and ν_m is the unit vector along the outward normal to $\partial\mathcal{D}$.

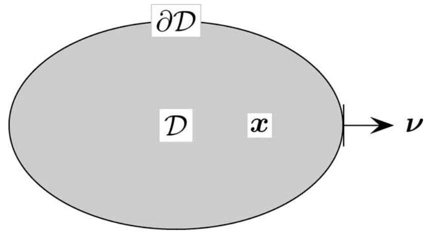

Fig. 5. ***Configuration of application of the global time-correlation reciprocity relation.***

The further discussion of corollaries of (66) goes along the same lines as in [2, Sec. 28.3]. In computational electromagnetics, (66) provides an important check on the consistency of the pertaining numerical codes, as well.

Field Interaction With a Kirchhoff Circuit: Similar to Section VII, the (Maxwell) field/(Kirchhoff) circuit interaction integral over $\mathcal{S}^K$ is now expressed as

$$\int_{\partial\mathcal{D}^K} \nu_m S^{AB*}_m \, dA = \sum_{k=1}^{K} \left(I^A_k \overset{(t)}{*} V^{B^*}_k + I^{B*}_k \overset{(t)}{*} V^A_k \right) \tag{67}$$

where the unit vector ν_m along the normal points toward the circuit. This relation is one of the basic ones in EMI analysis in case time-correlation properties are discussed.

IX. GREEN'S TENSORS AND THE DIRECT SOURCE PROBLEM

The time-convolution reciprocity relation (56) leads in a straightforward manner to the introduction of the electromagnetic field Green's tensors by applying the relation to appropriate point-source solutions. In the majority of configurations considered in computational electromagnetics, the relevant Green's tensors apply to unbounded domains. For this, the standard provisions for such an application are taken, with the consequence that the contribution from the surface integral over the sphere $\mathcal{S}_0$ with center at the origin $\mathcal{O}$ and radius R_0 vanishes in the limit $R_0 \to \infty$. Furthermore, we take in the entire configuration

$$\epsilon^A_{k,r} = \epsilon^B_{r,k} \tag{68}$$

$$\mu^{-;A}_{i,j,p,q} = \mu^{-;B}_{p,q,i,j} \tag{69}$$

i.e., the media properties are each other's *adjoints*. Then, (56) leads to

$$\int_{\mathcal{D}^{J;B}} E_r^A \overset{(t)}{*} J_r^B\, dV - \int_{\mathcal{D}^{K;B}} \left[H_{p,q}^A\right]^- \overset{(t)}{*} \left[K_{p,q}^B\right]^- dV = \int_{\mathcal{D}^{J;A}} E_k^B \overset{(t)}{*} J_k^A\, dV - \int_{\mathcal{D}^{K;A}} \left[H_{i,j}^B\right]^- \overset{(t)}{*} \left[K_{i,j}^A\right]^- dV \quad (70)$$

where $\mathcal{D}^{J,K;A,B}$ denotes the spatial support of the pertaining source distributions.

From this relation the *electric-current source Green's tensors* for the medium in state B arise by taking

$$J_r^B = a_r^B \delta(\boldsymbol{x} - \boldsymbol{x}^B, t) \quad \text{and} \quad \left[K_{p,q}^B\right]^- = 0 \quad (71)$$

where $\delta(\boldsymbol{x} - \boldsymbol{x}^B, t)$ is the Dirac distribution operative at $\boldsymbol{x} = \boldsymbol{x}^B$ and $t = 0$, and writing the corresponding field quantities as

$$E_k^B = G_{k,r}^{EJ;B}(\boldsymbol{x}, \boldsymbol{x}^B, t) a_r^B \quad (72)$$

$$\left[H_{i,j}^B\right]^- = G_{i,j,r}^{HJ;B}(\boldsymbol{x}, \boldsymbol{x}^B, t) a_r^B. \quad (73)$$

Similarly, the *magnetic-current source Green's tensors* for the medium in state B arise by taking

$$J_r^B = 0 \quad \text{and} \quad \left[K_{p,q}^B\right]^- = b_{p,q}^B \delta(\boldsymbol{x} - \boldsymbol{x}^B, t) \quad (74)$$

with $b_{p,q}^B = -b_{q,p}^B$, and writing the corresponding field quantities as

$$E_k^B = G_{k,p,q}^{EK;B}(\boldsymbol{x}, \boldsymbol{x}^B, t) b_{p,q}^B \quad (75)$$

$$\left[H_{i,j}^B\right]^- = G_{i,j,p,q}^{HK;B}(\boldsymbol{x}, \boldsymbol{x}^B, t) b_{p,q}^B. \quad (76)$$

Further, we need the corresponding results for the medium in state A, i.e., taking

$$J_k^A = a_k^A \delta(\boldsymbol{x} - \boldsymbol{x}^A, t) \quad \text{and} \quad \left[K_{i,j}^A\right]^- = 0 \quad (77)$$

writing the corresponding field quantities as

$$E_r^A = G_{r,k}^{EJ;A}(\boldsymbol{x}, \boldsymbol{x}^A, t) a_k^A \quad (78)$$

$$\left[H_{p,q}^A\right]^- = G_{p,q,k}^{HJ;A}(\boldsymbol{x}, \boldsymbol{x}^A, t) a_k^B \quad (79)$$

and, finally, taking

$$J_k^A = 0 \quad \text{and} \quad \left[K_{i,j}^A\right]^- = b_{i,j}^A \delta(\boldsymbol{x} - \boldsymbol{x}^A, t) \quad (80)$$

with $b_{i,j}^A = -b_{j,i}^A$, and writing the corresponding field quantities as

$$E_r^A = G_{r,i,j}^{EK;A}(\boldsymbol{x}, \boldsymbol{x}^A, t) b_{i,j}^A \quad (81)$$

$$\left[H_{p,q}^A\right]^- = G_{p,q,i,j}^{HK;A}(\boldsymbol{x}, \boldsymbol{x}^A, t) b_{i,j}^A. \quad (82)$$

Substituting these expressions in (70) and observing that the result holds for arbitrary values of a_k^B, a_r^A, $b_{i,j}^B$, and $b_{p,q}^A$, we first arrive at the *reciprocity relations for the Green's tensors*

$$G_{k,r}^{EJ;B}(\boldsymbol{x}^A, \boldsymbol{x}^B, t) = G_{r,k}^{EJ;A}(\boldsymbol{x}^B, \boldsymbol{x}^A, t) \quad (83)$$

$$G_{i,j,r}^{HJ;B}(\boldsymbol{x}^A, \boldsymbol{x}^B, t) = -G_{r,i,j}^{EK;A}(\boldsymbol{x}^B, \boldsymbol{x}^A, t) \quad (84)$$

$$G_{k,p,q}^{EK;B}(\boldsymbol{x}^A, \boldsymbol{x}^B, t) = -G_{p,q,k}^{HJ;A}(\boldsymbol{x}^B, \boldsymbol{x}^A, t) \quad (85)$$

$$G_{i,j,p,q}^{HK;B}(\boldsymbol{x}^A, \boldsymbol{x}^B, t) = G_{p,q,i,j}^{HK;A}(\boldsymbol{x}^B, \boldsymbol{x}^A, t) \quad (86)$$

and, second, using these Green's tensors reciprocity relations, at the field representations in the medium of state A

$$E_r^A(\boldsymbol{x}, t) = \int_{\mathcal{D}^{J;A}} G_{r,k}^{EJ;A}(\boldsymbol{x}, \boldsymbol{x}^A, t) \overset{(t)}{*} J_k^A(\boldsymbol{x}^A, t)\, dV(\boldsymbol{x}^A) + \int_{\mathcal{D}^{K;A}} G_{r,i,j}^{EK;A}(\boldsymbol{x}, \boldsymbol{x}^A, t) \overset{(t)}{*} \left[K_{i,j}^A\right]^-(\boldsymbol{x}^A, t)\, dV(\boldsymbol{x}^A) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^N, \quad t \in \mathbb{R} \quad (87)$$

$$\left[H_{p,q}^A\right]^-(\boldsymbol{x}, t) = \int_{\mathcal{D}^{J;A}} G_{p,q,k}^{HJ;A}(\boldsymbol{x}, \boldsymbol{x}^A, t) \overset{(t)}{*} J_k^A(\boldsymbol{x}^A, t) dV(\boldsymbol{x}^A) + \int_{\mathcal{D}^{K;A}} G_{p,q,i,j}^{HK;A}(\boldsymbol{x}, \boldsymbol{x}^A, t) \overset{(t)}{*} \left[K_{i,j}^A\right]^-(\boldsymbol{x}^A, t) dV(\boldsymbol{x}^A) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^N, \quad t \in \mathbb{R}. \quad (88)$$

These relations quantify all sorts of wave propagation from source to receiver, with the intrachip and interchip digital wireless signal transfer as a recent subject of investigation.

X. FIELD REPRESENTATIONS IN A SUBDOMAIN OF $\mathbb{R}^N$, EQUIVALENT SURFACE SOURCES, HUYGENS' PRINCIPLE, AND THE OSEEN–EWALD EXTINCTION THEOREM

Proceeding as in Section IX, but applying the time-convolution reciprocity relation (56) to the bounded domain $\mathcal{D} \subset \mathbb{R}^N$ with piecewise smooth boundary surface $\partial\mathcal{D}$ (Fig. 6), we obtain

$$\begin{aligned}\int_{\mathcal{D}} \Big[& G^{EJ;A}_{r,k}(\boldsymbol{x},\boldsymbol{x}^A,t) \overset{(t)}{*} J^A_k(\boldsymbol{x}^A,t) \\ & + G^{EK;A}_{r,i,j}(\boldsymbol{x},\boldsymbol{x}^A,t) \overset{(t)}{*} \left[K^A_{i,j}\right]^-(\boldsymbol{x}^A,t)\Big] dV(\boldsymbol{x}^A) \\ + \int_{\partial\mathcal{D}} \Big[& G^{EJ;A}_{r,k}(\boldsymbol{x},\boldsymbol{x}^A,t) \overset{(t)}{*} \partial J^A_k(\boldsymbol{x}^A,t) \\ & + G^{EK;A}_{r,i,j}(\boldsymbol{x},\boldsymbol{x}^A,t) \overset{(t)}{*} \left[\partial K^A_{i,j}\right]^-(\boldsymbol{x}^A,t)\Big] dA(\boldsymbol{x}^A) \\ & = \{1, 1/2, 0\} E^A_r(\boldsymbol{x},t), \\ & \qquad \text{for } \boldsymbol{x} \in \{\mathcal{D}, \partial\mathcal{D}, \mathcal{D}'\}, \qquad t \in \mathbb{R} \end{aligned} \tag{89}$$

$$\begin{aligned}\int_{\mathcal{D}} \Big[& G^{HJ;A}_{p,q,k}(\boldsymbol{x},\boldsymbol{x}^A,t) \overset{(t)}{*} J^A_k(\boldsymbol{x}^A,t) \\ & + G^{HK;A}_{p,q,i,j}(\boldsymbol{x},\boldsymbol{x}^A,t) \overset{(t)}{*} \left[K^A_{i,j}\right]^-(\boldsymbol{x}^A,t)\Big] dV(\boldsymbol{x}^A) \\ + \int_{\partial\mathcal{D}} \Big[& G^{HJ;A}_{p,q,k}(\boldsymbol{x},\boldsymbol{x}^A,t) \overset{(t)}{*} \partial J^A_k(\boldsymbol{x}^A,t) \\ & + G^{HK;A}_{p,q,i,j}(\boldsymbol{x},\boldsymbol{x}^A,t) \overset{(t)}{*} \left[\partial K^A_{i,j}\right]^-(\boldsymbol{x}^A,t)\Big] dA(\boldsymbol{x}^A) \\ & = \{1, 1/2, 0\} \left[H^A_{p,q}\right]^-(\boldsymbol{x},t), \\ & \qquad \text{for } \boldsymbol{x} \in \{\mathcal{D}, \partial\mathcal{D}, \mathcal{D}'\}, \qquad t \in \mathbb{R} \end{aligned} \tag{90}$$

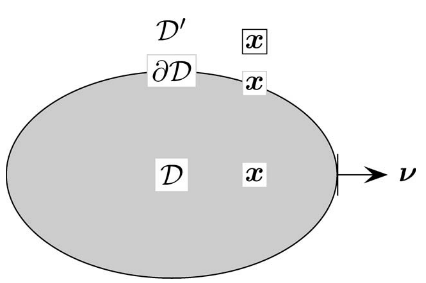

Fig. 6. *Field representations in a subdomain of space.*

in which $\mathcal{D}'$ is the domain exterior to $\partial\mathcal{D}$

$$\partial J^A_k(\boldsymbol{x}^A,t) = \nu_m [H_{m,k}]^- \tag{91}$$

is the *equivalent surface density of electric current* on $\partial\mathcal{D}$, and

$$\left[\partial K^A_{i,j}\right]^-(\boldsymbol{x}^A,t) = [\nu_i E_j]^- \tag{92}$$

is the *equivalent surface density of magnetic current* on $\partial\mathcal{D}$.

The result for $\boldsymbol{x} \in \mathcal{D}$ is representative for *Huygens' principle* [12], [13]. The result for $\boldsymbol{x} \in \partial\mathcal{D}$, in which the integrals have to be interpreted as their Cauchy principal values, is the basis for the *boundary integral-equation method* of computation, while the result for $\boldsymbol{x} \in \mathcal{D}'$ is representative for the Oseen–Ewald extinction theorem [14], [15] and forms the basis for the computational *null-field method* [16].

XI. THE CALDERÓN IDENTITIES

Upon applying (89) and (90) to a source-free domain exterior to the bounded closed surface $\mathcal{S}$ and taking the fields to be causally related to the action of sources interior to $\mathcal{S}$, the relation for $\boldsymbol{x} \in \mathcal{S}$ can, in operator form, be written as

$$(1/2)\begin{bmatrix}\partial J \\ \partial K\end{bmatrix} = \begin{bmatrix}\Gamma^{JJ} & \Gamma^{JK} \\ \Gamma^{KJ} & \Gamma^{KK}\end{bmatrix}\begin{bmatrix}\partial J \\ \partial K\end{bmatrix} \tag{93}$$

where the operators contain the (singular and hypersingular) integrals of the relevant Green's tensors over $\mathcal{S}$. In the boundary integral-equation computational method, these relations are sufficient conditions for the surface expansions of ∂J and ∂K to satisfy the condition of being associated with outwardly radiating fields. However, the integrations involved are burdened with numerical difficulties. As has been demonstrated in [17]–[19], the numerical difficulties can be reduced by using the *Calderón identities* associated with (93)

$$(1/4)\begin{bmatrix}\partial J \\ \partial K\end{bmatrix} = \begin{bmatrix}\Gamma^{JJ} & \Gamma^{JK} \\ \Gamma^{KJ} & \Gamma^{KK}\end{bmatrix}\begin{bmatrix}\Gamma^{JJ} & \Gamma^{JK} \\ \Gamma^{KJ} & \Gamma^{KK}\end{bmatrix}\begin{bmatrix}\partial J \\ \partial K\end{bmatrix} \tag{94}$$

that lead to

$$\Gamma^{JJ}\Gamma^{JJ} + \Gamma^{JK}\Gamma^{KJ} = \mathrm{I} \tag{95}$$
$$\Gamma^{JJ}\Gamma^{JK} + \Gamma^{JK}\Gamma^{KK} = 0 \tag{96}$$
$$\Gamma^{KJ}\Gamma^{JJ} + \Gamma^{KK}\Gamma^{KJ} = 0 \tag{97}$$
$$\Gamma^{KJ}\Gamma^{JK} + \Gamma^{KK}\Gamma^{KK} = \mathrm{I} \tag{98}$$

in which I denotes the $(N+1)$-space-time identity operator. For the case of $(3+1)$-space-time operators, an extensive discussion on their application can be found in [17]–[19].

XII. THE SPACE-TIME-INTEGRATED FIELD EQUATIONS METHOD OF COMPUTATION

The space-time-integrated field equations method of computation is a local discretization method in which the spatial domain of computation is discretized into a union of *simplices* and the time window of observation into a sequence of adjacent intervals. Within this space-time discretization, the field and source quantities are expanded in linear functions of space and time and the constitutive parameters are taken to be piecewise constant. In each of the spatial elements of discretization, the relevant expansion coefficients are taken to be associated with the boundary values of the components of the field quantities that are continuous upon crossing an interface of jump discontinuities in the constitutive parameters, which expansions are continued linearly into the interior of the element. In each of the time intervals, a linear expansion is used. These expansions are substituted in the space-time-integrated versions of the field equations upon the application of simplicial integration rule [the $(N+1)$-space-time generalization of the trapezoidal rule]. The advantage of the method is that, in view of the definition of the Riemann integral (as based on the simplicial rule), the method converges for differentiable functions at decreasing mesh sizes to the pertaining differential equations, while the interface continuity conditions remain satisfied in machine precision (i.e., no artificial surface sources turn up). The numerical disadvantage of the method is that it is not explicit in time.

Let the spatial domain of computation be discretized into the union of simplices $\cup_{I\Sigma=1}^{N\Sigma}\Sigma_{I\Sigma}$ and the time coordinate into the union of adjacent time intervals $\cup_{I\mathcal{T}=1}^{N\mathcal{T}}\mathcal{T}_{I\mathcal{T}}$. The space-time-integrated field equations associated with $\Sigma_{I\Sigma}\times\mathcal{T}_{I\mathcal{T}}$ are then (cf. Section III)

$$\int_{\mathcal{T}_{I\mathcal{T}}} dt \int_{\partial\Sigma_{I\Sigma}} \nu_m[H_{m,k}]^- \, dA + \left[\int_{\Sigma_{I\Sigma}} \epsilon_{k,r} \overset{(t)}{*} E_r \, dV\right]\Bigg|_{\partial\mathcal{T}_{I\mathcal{T}}} = -\int_{\mathcal{T}_{I\mathcal{T}}} dt \int_{\Sigma_{I\Sigma}} J_k \, dV \tag{99}$$

$$\int_{\mathcal{T}_{I\mathcal{T}}} dt \int_{\partial\Sigma_{I\Sigma}} [\nu_i E_j]^- \, dA + \left[\int_{\Sigma_{I\Sigma}} \mu^-_{i,j,p,q} \overset{(t)}{*} [H_{p,q}]^- \, dV\right]\Bigg|_{\partial\mathcal{T}_{I\mathcal{T}}} = -\int_{\mathcal{T}_{I\mathcal{T}}} dt \int_{\Sigma_{I\Sigma}} [K_{i,j}]^- \, dV \tag{100}$$

where $\partial\Sigma_{I\Sigma}$ is the boundary of $\Sigma_{I\Sigma}$ and $\partial\mathcal{T}_{I\mathcal{T}}$ is the boundary of $\mathcal{T}_{I\mathcal{T}}$ and Gauss' theorem has been applied. To accommodate the condition of outgoing radiation, the method can be coupled to the boundary integral-equation method (implemented via the Calderón identities) outlined in Section XI. This, however, is in essence a mutually incompatible approach. A compatible approach to handle the condition of outgoing radiation is to surround the domain of interest with a perfectly matched absorbing embedding via the *time-domain, causality preserving Cartesian coordinate stretching method*. This method will briefly be discussed in the next section.

XIII. THE TIME-DOMAIN, CAUSALITY PRESERVING, CARTESIAN COORDINATE STRETCHED PERFECTLY MATCHED EMBEDDING

A substantial number of field computations is requested to handle the radiation of the field into unbounded free space. Now, by computational necessity, any domain of computation must, however, be of bounded support and, hence, has to be terminated by boundaries upon which boundary conditions are specified in accordance with the uniqueness of the pertaining problem. Several types of "absorbing boundary conditions" are in use for this purpose. By far the most superior one is the *time-domain, causality preserving Cartesian coordinate stretching procedure*. Here, a "target region" (in the shape of an N-rectangle) is selected that contains the sources that excite the field and in which the field equations are taken as they are. Next, the target region is surrounded by a source-free "perfectly matched embedding" in which the field equations are modified in such a manner that their solution shows an adjustable (user-defined) delay in time and an adjustable (user-defined) decay in space with increasing distance from the target region, while leaving the field values in the target region unaltered [20]. At some finite "thickness," the embedding is terminated and on the boundaries of the resulting computational N-rectangle *periodic boundary conditions* are prescribed. This process of termination gives rise to "spurious reflections" that manifest themselves in the target region. In view of the periodicity of the generated solution, the spurious contributions are generated by the periodically repeated sources outside the target region and have thus undergone at least twice the time delay and twice the decay associated with the field propagation across the embedding's "layer." Proper adjustment of the delay and decay parameters of the embedding lead the "truncation error" to be within user-defined bounds. For the case of radiation generated in a homogenous, isotropic, lossless medium the relevant procedure can be analyzed analytically [21], as will be shown below.

The basic step in the time-domain Cartesian coordinate stretching method is the introduction, along each of the

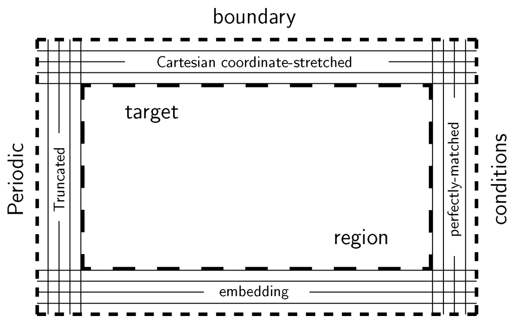

Fig. 7. ***Cartesian-coordinate stretched, causality preserving, perfectly matched embedding, terminated with periodic boundary conditions.***

axes of the chosen reference frame in $\mathbb{R}^N$, of an appropriate, causality preserving coordinate stretching function in the time Laplace-transform domain. This replacement is taken to be of the type

$$x_m \longrightarrow \hat{\hat{x}}_m = \int_{\xi_m=0}^{x_m} \hat{\chi}^{[m]}(\xi_m, s)\, d\xi_m, \qquad \text{for } m = 1, \dots, N \tag{101}$$

in which $\hat{\chi}_m(\xi_m, s)$ is an analytic function of s in the right half of the complex s-plane and positive for real, positive values of s and a piecewise continuous function of x_m, while it is subject to the condition that $\hat{\chi}_m(\xi_m, s) = 1$ for $\xi_m \in \{\text{target region}\}$ (Fig. 7). Evidently, in the target region, we have $\hat{\hat{x}}_m = x_m$. Denoting the derivative with respect to $\hat{\hat{x}}_m$ by $\hat{\hat{\partial}}_m$, it follows that, under the replacement, we have

$$\partial_m \longrightarrow \hat{\hat{\partial}}_m = \frac{1}{\hat{\chi}^{[m]}(x_m, s)} \partial_m, \qquad \text{for } m = 1, \dots, N. \tag{102}$$

Correspondingly, the s-domain field equations applying to the radiation problem considered in Section VI are replaced with

$$\hat{\hat{\partial}}_m [\hat{H}_{m,k}]^- + s\epsilon \hat{E}_k = -\hat{J}_k \tag{103}$$

$$[\hat{\hat{\partial}}_i \hat{E}_j]^- + s\mu^- [\hat{H}_{i,j}]^- = -[\hat{K}_{i,j}]^-. \tag{104}$$

Now, in this form, the equations are not directly amenable to the computational handling by any of the spatial discretization methods applied in the target region. To overcome this difficulty, we multiply (103), noting that in the summation the term with $m = k$ is missing, by

$$\hat{\Pi}_\chi^{[k]}(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_N, s) = \hat{\chi}^{[1]}(x_1, s) \cdots \hat{\chi}^{[k-1]}(x_{k-1}, s) \times \hat{\chi}^{[k+1]}(x_{k+1}, s) \cdots \hat{\chi}^{[N]}(x_N, s) \tag{105}$$

and (104) by $\hat{\chi}^{[i]}(x_i, s)\hat{\chi}^{[j]}(x_j, s)$. Furthermore, introducing the notation

$$\begin{aligned} \hat{\Pi}_\chi^{[k,m]}&(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{m-1}, x_{m+1}, \dots, x_N, s) \\ &= \hat{\chi}^{[1]}(x_1, s) \cdots \hat{\chi}^{[k-1]}(x_{k-1}, s)\hat{\chi}^{[k+1]}(x_{k+1}, s) \\ &\quad \cdots \hat{\chi}^{[m-1]}(x_{m-1}, s)\hat{\chi}^{[m+1]}(x_{m+1}, s) \cdots \hat{\chi}^{[N]}(x_N, s) \\ &\quad \text{for } k \neq m \end{aligned} \tag{106}$$

the field equations (103) and (104) can be rewritten as

$$\partial_m [\hat{\hat{H}}_{m,k}]^- + s\hat{\hat{D}}_k = -\hat{\hat{J}}_k \tag{107}$$

$$[\partial_i \hat{\hat{E}}_j]^- + s[\hat{\hat{B}}_{i,j}]^- = -[\hat{\hat{K}}_{i,j}]^- \tag{108}$$

together with the constitutive relations

$$\hat{\hat{D}}_k = \epsilon \left(\hat{\Pi}_\chi^{[k]} / \chi^{[k]} \right) \hat{\hat{E}}_k \tag{109}$$

$$[\hat{\hat{B}}_{i,j}]^- = \mu^- \left(\hat{\Pi}_\chi^{[i,j]} / \chi^{[i]}\chi^{[j]} \right) [\hat{\hat{H}}_{i,j}]^- \tag{110}$$

and

$$\hat{\hat{J}}_k = \hat{\Pi}_\chi^{[k]} \hat{J}_k \tag{111}$$

$$[\hat{\hat{K}}_{i,j}]^- = \hat{\Pi}_\chi^{[i,j]} [\hat{K}_{i,j}]^-. \tag{112}$$

In the target region, these relations are identical to the ones for the original field. In the total domain of computation (target region plus embedding), the time-domain equivalents of (107) and (108) simply are

$$\partial_m [\bar{H}_{m,k}]^- + \partial_t \bar{D}_k = -\bar{J}_k \tag{113}$$

$$[\partial_i \bar{E}_j]^- + \partial_t [\bar{B}_{i,j}]^- = -[\bar{K}_{i,j}]^- \tag{114}$$

which can be handled by, for example, the space-time-integrated field equations method discussed in Section XII.

The time-domain equivalents of (109) and (110) depend on the choice of the coordinate stretching functions. A convenient class of them is the one that introduces an excess time delay and an excess absorption of the type that has been considered in [21], i.e.,

$$\hat{\chi}^{[m]}(x_m,s) = \left[1+N^{[m]}(x_m)\right] + s^{-1}\sigma^{[m]}(x_m) \quad \text{for } m=1,\ldots,N \tag{115}$$

in which $N^{[m]}(x_m)$ is the *excess time-delay profile*, and $\sigma^{[m]}(x_m)$ is the *excess absorption profile*. Both are taken to be piecewise continuous functions of x_m. Sufficient conditions for $\hat{\chi}^{[m]}(x_m,s) > 0$ for $s \in \mathbb{R}$, $\mathrm{Re}(s) > 0$, to hold are $N^{[m]}(x_m) \geq -1$ and $\sigma^{[m]}(x_m) > 0$. The time-domain equivalent of (115) is

$$\chi^{[m]}(x_m,t) = \left[1+N^{[m]}(x_m)\right]\delta(t) + \sigma^{[m]}(x_m)H(t) \quad \text{for } m=1,\ldots,N \tag{116}$$

where $H(t)$ denotes the Heaviside unit step function. For this class of stretching functions, (111) and (112) reduce to repeated time integrations or to ordinary differential equations in time.

As has been demonstrated in Section VI, the main wave propagation features are exhibited by the scalar Green's function $G(\boldsymbol{x},\boldsymbol{x}';t)$ that satisfies the scalar wave equation (33). In the coordinate-stretched configuration this function satisfies in the time Laplace-transform domain

$$\hat{\bar{\partial}}_m(\hat{\bar{\partial}}_m G) - (s^2/c^{-2})G = -\delta(\hat{\bar{\boldsymbol{x}}} - \hat{\bar{\boldsymbol{x}}}'). \tag{117}$$

For the case $N=3$, the solution to this equation is given by

$$\hat{\bar{G}} = \frac{\exp\left[-(s/c)\hat{\bar{R}}\right]}{4\pi\hat{\bar{R}}}, \qquad \text{for } \hat{\bar{R}} \neq 0 \tag{118}$$

in which

$$\hat{\bar{R}} = \left[\left(\hat{\bar{x}}_m - \hat{\bar{x}}'_m\right)\left(\hat{\bar{x}}_m - \hat{\bar{x}}'_m\right)\right]^{1/2} \tag{119}$$

with

$$\hat{\bar{x}}_m - \hat{\bar{x}}'_m = \int_{x'_m}^{x_m} \hat{\chi}^{[m]}(\xi_m,s)\,d\xi_m. \tag{120}$$

For the class of stretching functions (115), then [21]

$$\hat{\bar{G}} = \frac{s}{c}\,\frac{\exp\left\{-T\left[(s+\alpha)^2+\Omega^2\right]^{1/2}\right\}}{4\pi T\left[(s+\alpha)^2+\Omega^2\right]^{1/2}} \tag{121}$$

in which

$$T = (T_m T_m)^{1/2} \tag{122}$$

with

$$T_m = \frac{1}{c}\int_{x'_m}^{x_m}\left[1+N^{[m]}(\xi_m)\right]d\xi_m \tag{123}$$

$$\alpha = T_m\Gamma_m/T^2 \tag{124}$$

with

$$\Gamma_m = \frac{1}{c}\int_{x'_m}^{x_m}\sigma^{[m]}(\xi_m)\,d\xi_m \tag{125}$$

and

$$\Omega = [\Gamma^2/T^2 - \alpha^2]^{1/2}. \tag{126}$$

The corresponding time-domain result is [22, eq. 29.3.92]

$$\bar{G} = \partial_t\left[\frac{\exp(-\alpha t)}{4\pi cT}J_0\left[\Omega(t^2-T^2)^{1/2}\right]H(t-T)\right] \tag{127}$$

where J_0 is the Bessel function of the first kind and order zero. From this it is clear that T is the travel time of the coordinate-stretched wave function from the source point to the point of observation, α determines the attenuation that the wave undergoes during its passage, while Ω is the angular frequency of oscillation induced by the coordinate stretching procedure (α and Ω both vanish for vanishing excess absorption).

Some specific stretching profiles have been studied in [21], which shows their flexibility as to adjusting user-defined values of the maximum truncation error with periodic boundary conditions. Note that, contrary to what is sometimes stipulated in the literature, the stretching profiles in our analysis need only be piecewise continuous and may have jump discontinuities of any size.

XIV. ONE-DIMENSIONAL PULSED ELECTRIC-CURRENT/MAGNETIC-CURRENT EXCITED WAVE FIELDS IN A LAYERED MEDIUM: AN IEEE-WEBSITE DEMONSTRATOR IN MATLAB

In this section, the different computational features associated with the space-time field integrations method discussed in Section XII, the use of the Cartesian coordinate time-domain stretching procedure for constructing perfectly matched embeddings in accordance with the excess time delay and absorption profiles studied in Section XIII, and the termination of such an embedding with periodic boundary conditions are illustrated for the 1-D pulsed electric-current/magnetic-current excited wave fields in a layered medium in 3-D space. Each of the layers can be assigned an electric permittivity ϵ, an electric conductivity σ, a magnetic permeability μ, and a magnetic loss coefficient κ. Let the wave propagation take place in the direction of x_3 normal to the layering and let the electric source current have a component along x_1 only (linear polarization) and the magnetic source current only a component along the combination x_1, x_3, then, with

$$E_r = E\delta_{r,1} \tag{128}$$
$$H_{p,q} = H\delta_{p,3}\delta_{q,1} \tag{129}$$
$$J_k = J\delta_{k,1} \tag{130}$$
$$K_{i,j} = K\delta_{i,3}\delta_{j,1} \tag{131}$$

the pertaining field equations are

$$\partial_3 H + (\sigma + \epsilon\partial_t)E = -J \tag{132}$$
$$\partial_3 E + (\kappa + \mu\partial_t)H = -K. \tag{133}$$

The 1-D Green's function in the stretched x_3-coordinate is

$$\hat{\bar{G}} = \frac{c}{2s}\exp(-sT_3 - \alpha_3) \tag{134}$$

with

$$T_3 = \frac{1}{c}\int_{x_3'}^{x_3}\left[1 + N^{[3]}(\xi_3)\right]d\xi_3 \tag{135}$$
$$\alpha_3 = \frac{1}{c}\int_{x_3'}^{x_3}\sigma^{[3]}(\xi_3)d\xi_3. \tag{136}$$

The corresponding time-domain result is

$$G = \exp(-\alpha_3)H(t - T_3). \tag{137}$$

A Matlab program has been written that handles user-defined input data as regards the properties of the layered configuration, the source signatures of the exciting current distributions, and the type of desired plot output.[2]

XV. CONCLUSION

A modern time-domain tensor/array approach to electromagnetic field theory is shown to lead to considerable simplifications in the presentation. Through its structure, the standard vector calculus proves to be a completely superfluous vehicle and even the right-handedness of the coordinate systems employed is not a necessity. Only elementary mathematical operations are needed to formulate the theory, which enables its generalization to $(N+1)$-space-time. The structure introduces magnetic currents and their associated magnetic charges in a manner that deviates from what is standard, with the particular outcome that the magnetic charge is not a scalar (as it is treated in Dirac's theory), but a cyclically symmetric tensor of rank three. In its turn, this has consequences for string theory in quantum electrodynamics and theoretical cosmology. As far as computation is concerned, the basic steps of the space-time field integration method, combined with the construction of perfectly matched embeddings and the application of periodic boundary conditions, are indicated. The 1-D pulsed wave propagation across a stack of homogeneous, isotropic layers demonstrates a variety of features. ■

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[2]Information about the Matlab files can be obtained from the author (e-mail: a.t.dehoop@tudelft.nl).

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Adrianus T. de Hoop (Member, IEEE) was born on December 24, 1927, in Rotterdam, The Netherlands. He received the M.Sc. degree in electrical engineering and the Ph.D. degree in technological sciences, both with the highest distinction, from Delft University of Technology, Delft, The Netherlands, in 1950 and 1958, respectively.

He served Delft University of Technology as an Assistant Professor (1950–1957), Associate Professor (1957–1960), and Full Professor in Electromagnetic Theory and Applied Mathematics (1960–1996). Since 1996, he has been Lorentz Chair Emeritus Professor in the Faculty of Electrical Engineering, Mathematics and Computer Science of Delft University of Technology. In 1970, he founded the Laboratory of Electromagnetic Research, at Delft University of Technology, which has developed into a world-class center for electromagnetics, having a huge impact on the world's electromagnetic community and on electromagnetic research and education in The Netherlands. His research interests are in the broad area of wavefield modeling in acoustics, electromagnetics, and elastodynamics. His interdisciplinary insights and methods in this field can be found in his seminal *Handbook of Radiation and Scattering of Waves* (London, U.K.: Academic, 1995), with wavefield reciprocity serving as one of the unifying principles governing direct and inverse scattering problems and wave propagation in complex (anisotropic and dispersive) media. He spent a year (1956–1957) as a Research Assistant with the Institute of Geophysics, University of California at Los Angeles, where he pioneered a modification of the Cagniard technique for calculating impulsive wave propagation in layered media, later to be known as the "Cagniard-DeHoop technique." This technique is currently considered as a benchmark tool in analyzing time-domain wave propagation. During a sabbatical leave at Philips Research Laboratories, Eindhoven, The Netherlands (1976–1977), he was involved in research on magnetic recording theory. Since 1982, he has been, on a regular basis, Visiting Scientist with Schlumberger-Doll Research, Ridgefield, CT (now at Cambridge, MA), where he contributes to research on geophysical applications of acoustic, electromagnetic, and elastodynamic waves. Grants from the "Stichting Fund for Science, Technology and Research" (founded by Schlumberger Limited) supported his research at Delft University of Technology.

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