To the Memory of Agner K. Erlang

ORIGINATOR OF TELETRAFFIC THEORY

This article is dedicated to the memory of the Danish scientist Agner Krarup Erlang, who during his professional lifetime made numerous pioneering and fundamental contributions in the area of teletraffic theory development. Going back to the history, it is interesting to remember some significant dates. More then 60 years ago, in October 1946, CCIF (CCITT and ITU forerunner) decided to adopt 1 Erlang as the international unit for telephone traffic. Also, more than nine decades ago (1917), one of the most significant work applying telephone traffic theory for the derivation of the loss formula, as well as the delay formula, was published by Erlang. Just over a century has passed since Erlang published (in 1909) his first work, a landmark contribution dealing with the theory of the probability application in resolving problems connected to telephone traffic. In that way, Poisson probability distribution was proved for telephone calls arrival and traffic.

Fig. 1. *A. K. Erlang at the age of 15.*

This Scanning Our Past article from Europe takes a look at the professional accomplishments of Danish scientist Agner Krarup Erlang, who made pioneering contributions in teletraffic theory development.

I. THE LIFE OF A. K. ERLANG

Agner Krarup Erlang was born more than 130 years ago (1 January 1878) in Lonborg, Denmark. He was the son of a schoolmaster, and a descendant of the prominent mathematician Thomas Fincke on his mother's side. Agner spent his early school days at his father's schoolhouse together with his mother, brother, and two sisters [1].

At age 14, A. K. Erlang passed the Preliminary Examination of the University of Copenhagen with distinction, after receiving dispensation to take it because he was younger than the usual minimum age (Fig. 1). The special entrance permission was granted. After that, Erlang returned to Lonborg and become assistant teacher at his father's school for two years. He taught alongside his father. When he reached the age of 16, his father thought that he ought to continue his studies. The Frederiksborg Grammar-School had two scholarships, and A. K. Erlang attained one of these. He studied mathematics and natural sciences at the University of Copenhagen. His mathematical education was greatly influenced by Prof. H. G. Zeuthen's and Prof. C. Juel's lectures. All his life he maintained and cultivated his interest in geometrical problems.

He finished his university studies in January 1901 by receiving the degree of *candidatus magisterii* with mathematics as principal subject and astronomy, physics, and chemistry as secondary subjects. For some years Erlang worked as a teacher at various schools. He proved to be in possession of excellent pedagogical qualities, even though his natural predilection was for scientific research rather than for teaching. He managed to find time for attempting the mathematical prize essay of the University for 1902–1903 on Huygens' solutions of infinitesimal problems, for which attempt he was rewarded with an "accessit" in 1904. He had then

Fig. 2. *A. K. Erlang from the period of scientific activity (age 32).*

already taken up the study of the theory of probabilities which later came to be the subject of his principal works.

In 1908, the Copenhagen Telephone Company engaged A. K. Erlang as scientific collaborator and head of its laboratory. In this position, which gave him an opportunity to develop and utilize his great gifts and considerable knowledge, he worked for the rest of his life. He set to work at applying the theory of probabilities to problems of telephone traffic, the field of interest that was to make his name widely known. In 1909 he published his first work on this subject, the above-mentioned "Theory of Probabilities and Telephone Con-

versations," which proves that the Poisson distribution applies to random telephone traffic and gives the exact solution of the delay problem in the special case of only one operator being available to handle the calls (Fig. 2).

Next, in 1917 A. K. Erlang published his most important work "Solution of Some Problems in the Theory Probabilities of significance in Automatic Telephone Exchanges," which contains his classic formulae for loss and waiting time developed on the basic of the principle of statistical equilibrium. These well-known formulae are of fundamental importance to the theory of telephone traffic. He was also an expert in the

history and calculation of the numerical tables of mathematical functions, particularly logarithms. He devised new calculation methods for certain forms of tables. In the course of the years, A. K. Erlang published several others valuable works on the theory of telephone traffic and some works with reference to other mathematical domains. His works on the theory of telephone traffic won recognition and understanding all over the world. For example, a few years after its appearance, his formula for the probability of loss was accepted by the British Post Office as the basic for calculations respecting circuit facilities.

A. K. Erlang published some articles dealing with the physicotechnical side of telephony that were of considerable importance at the time of their appearance. For instance, he constructed the so-called "Erlang's Complex Compensator," which represented an improvement compared with similar types of measuring instruments of earlier date (Fig. 3). Other works deal with a wide variety of problems within mathematics, theoretical physics, genetics, and population statistics. Particularly noteworthy is the proof of Maxwell's law.

A. K. Erlang was a member of the Danish Mathematicians' Association (TBMI). Through this met amateur mathematician Johan Jensen, the Chief Engineer of the Copenhagen Telephone Company (CTC), an offshoot of the International Bell
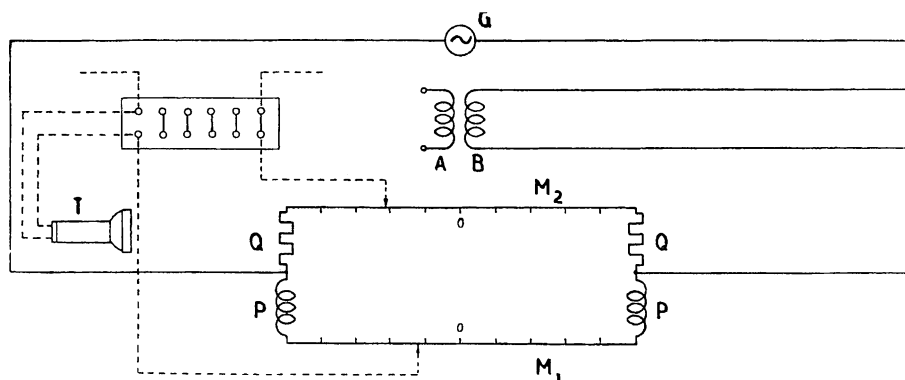

Fig. 3. *Essential parts of the apparatus for telephonic measurements, from 1913.*

Fig. 4. *The portrait of A. K. Erlang, which originally appeared in the Journal des Télécommunications, December 1947.*

Telephone Company. He worked for the CTC for almost 20 years. Besides being a member of Matematisk Forening, the meetings of which he attended regularly, Erlang was an associate of the British Institution of Electrical Engineers. A. K. Erlang died 1929 in Copenhagen, only 51 years old. Fig. 4 presents his most known portrait.

II. EFFECTS OF ERLANG'S THEORETICAL RESULTS

Erlang's results in the domain of telephone traffic theory are the best represented by the First Erlang Formula (EF), called the loss formula, or Erlang B Formula. The Second EF, often called the Delay formula, or Erlang C Formula, was completed and published in 1920, while two years later, we have the Third Erlang formula, named the Interconnection formula, or Erlang D Formula, in the published form.

In 1925, Erlang illustrated distributions of importance for holding time (Fig. 5), which is so-called Type III of Pearson's distributions. They represent a convolution of more exponential distributions, today known as Erlang's distributions.

Starting from this period, a list of researchers working in the fields of teletraffic and queueing theory became larger and larger. In the next 15 years, the scientific work of T. O. Engset, G. F. O'Dell, T. C. Fry, E. C. Molina, F. Pollaczek, C. D. Crommelin, C. Palm, A. Kolmogorov, A. Khinchin, etc., must be pointed out. In 1953, D. G. Kendall introduced the type of designation A/B/C for queueing models.

The second half of the 20th century is characterized by an extremely high number of theoretical works dealing with the development of circuit and packet switching networks. Also, we have the development
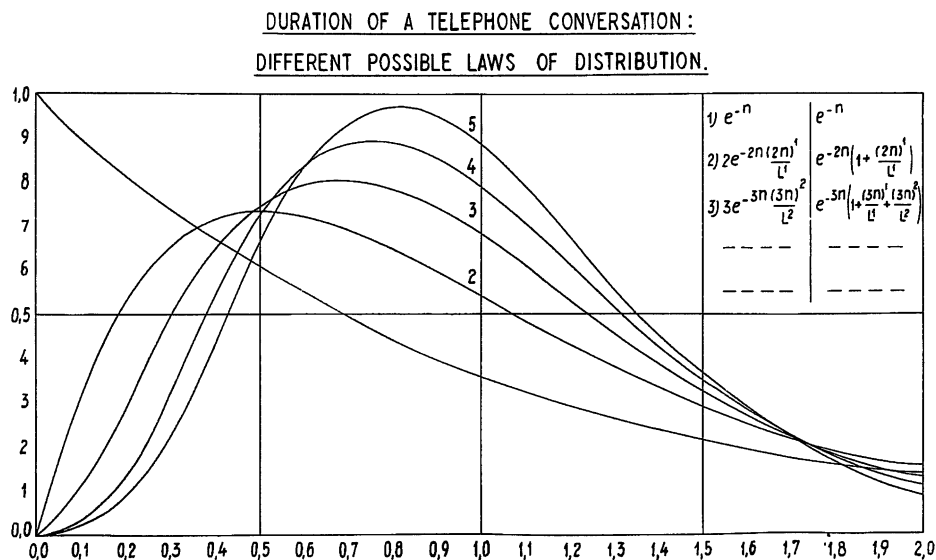

Fig. 5. *Erlang's distributions.*

of complex models for the need of the other types of traffic, based on the Erlang's formulae. Until 15 years ago, models solutions based on the process with dependence in the short range (Markov process, regression models) were dominant. The last decade of the 20th century brings the deep analysis of packet communication. It was shown that there exists multifractal nature of this traffic. The fractal analysis of teletraffic became one of the hottest research topics. There exist a great number of papers and books influenced by Erlang's theoretical results in teletraffic and queueing theory [2].

III. IMPORTANCE OF FIRST AND THIRD ERLANG FORMULAE

Erlang's theoretical results based on the statistical equilibrium are represented in the best way as a birth and a death process. The system (trunk group, bundle, capacity) consisting of channels (trunks, slots, servers, resources, capacity units, tasks, packets), with Poisson arrival demands (calls, customers, clients, tasks, packets). The serving time distribution is exponential. The system is with losses when all available channels are occupied (First and Third EF) or with the queueing possibility in an infinite queue (Second EF).

The theoretical works of Erlang have opened the field for wide applications. Starting from the First Formula, the following contributions are evident:

- The common EF dealing with multiple arrivals.
- The extended EF, which permits to one part of nonserved demands to be addressed for queueing.
- The general EF, which treats the problem of stream losses for general independent (GI, renewal) arrival distributions.
- The Integral EF, which treats explicitly noninteger trunk group.
- The Palm-Jacobaeus Formula (PJF), which is directly re-

lated to the EF for losses. This formula treats the probability of fixed trunks occupancy and the loss probability in the trunk group part with sequential hunting (overflow traffic loss).
- The Modified PJF, which is used in the case of trunk group with limited availability.
- The Fredericks-Hayward formula, suitable for providing the losses determination in the case of nonpoisson traffic distribution. Here, the parameters in EF are normalized with peakedness factor.

Generalization of the classical teletraffic theory was carried out with the development of Integrated Service Digital Network (ISDN) and Broadband-ISDN systems. Each class of service corresponds to a traffic stream, several of them are offered to the same trunk group. The classical multidimensional EF is an example of a reversible Markov process used in those systems. More general loss models and strategies include service protection and multislot allocation.

The Third EF is the result of the trunk group with the limited availability consideration, while for combinatory homogeneous grading, if was possible to take into account a system through macro state, i.e., to define the call arrival probability from the group with busy channels i.e. losses probability. It should be noted that with the application of Frederics-Haywards technique, Third EF can be used in the case of non-Poisson traffic serving in a limited available trunk group.

IV. POSSIBILITIES OF SECOND ERLANG FORMULA

A fully available trunk group with offered Poisson stream demands, with exponential serving time distribution and with a waiting possibility to be served in a queue with infinite number of places $(M/M/s)$ represents the basic model for the queueing system. Time losses, or waiting probability, repre-

sents the Second EF, which is in relation with First EF. For this queueing system, it is possible simple to define a lot of parameters, such as average number of demands in queue, average waiting time, average number of demand in system, average time in queue for those who wait, waiting time distribution, probability of waiting time longer then set time, and so on.

The elegant Erlang's solution for queueing system enables treating of different queueing disciplines. Customers in a queue waiting to be served can be selected for service according to many different principles. The classical queueing disciplines are:

- FCFS (First Come-First Served), also called a fair queue or an ordered queue, and is often preferred in real-life when customers are human beings. It is also denoted as FIFO (First In-First Out), refers to the queue only, not to the total system.
- LCFS (Last Come-First Served) corresponds to the stack principle. It is for instance used in storages, on shelves of shops etc. This is also denoted as LIFO (Last In-First Out).
- SIRO (Service In Random Order) when all customers waiting in the queue have the same probability of being chosen for service. This is also called RS (Random Selection).

The first two disciplines only take arrival times into considerations, while the third does not consider any criteria at all and so does not require any memory. They can be implemented in simple technical systems. For the three above-mentioned disciplines, the total waiting time for all customers is the same. The queueing discipline only decides how waiting time is allocated to the individual customers.

In a program-controlled queueing system, there are more complicated queueing disciplines. For computer system we often try to reduce the total waiting time by using the service time

as criterion: Shortest Job First (SJF), Shortest Job Next (SJN), Shortest Processing time First (SPF). It is assumed that we know the service time in advance and it minimizes the total waiting time for all customers.

The above-mentioned disciplines take into account either the arrival times or the service times. A compromise between these disciplines is obtained by the following:

- RR (Round Robin), where customer served is given at most a fixed service time (time slice or slot). If the service is not completed during this interval, the customer returns to the queue which is FCFS.
- PS (Processor Sharing), where all customers share the service capacity equally.
- FB (Foreground-Background) discipline tries to implement SJF without knowing the service times in advance. The server will offer service to the customer who so far has received the least amount of service. When all customers have obtained the same amount of service, FB becomes identical with PS.

The last mentioned disciplines are dynamic as the queueing disciplines depend on the amount of time spent in the queue. In real life, customers are often divided into priority classes, and we distinguish between two types of priority: nonpreemptive and preemptive.

In queueing literature we meet many other strategies and symbols.

The behavior of customers (Balking, Reneging, and Jockeying) is also subject to modeling. Thus, there are many different possible models. System $M/D/s$ was also considered by Erlang. A system with a parameter of impatience is Palm's $M/M/s + M$ model, with a solution often called Erlang-A (abandonment) formula. It is possible to explicitly treat a system with one part of nonaccepted demands which are lost.

Note that there is a great number of models with waiting and more complex arriving distribution as well as serving time. Usually, one channel models are considered. Solution for a system with general time serving distribution $M/G/1$ is more complex. Thus, very often we are satisfied with the results obtained using Little theorem as well as a formula Pollaczek-Khintchine. The case of general independent distribution of arriving stream $GI/G/1$ is of theoretical interest and there exists a great number of possibilities to obtain the corresponding solution. Traffic in packets network has got a "self-similar" character. This traffic exhibits a long range dependency and uses models with "long (heavy) tailed" distribution (Pareto, hiperexponential, log-normal), ON/OFF processes, or autoregressive models.

V. INFLUENCE ON TODAY'S TELETRAFFIC DEVELOPMENT

Telecommunication engineering methods applied in the modern networks development (ATM, BISDN, Internet, Mobile, Wireless, and Fourth Generation—4G) are very complex. They use beside queueing theory, simulation, operational research, graph theory and so on. The basic criteria for the traffic efficiency becomes quality of service (QoS) instead of classical grade of service (GoS).

Although Internet traffic is far too complicated to be modeled using the techniques developed for the telephone network, and the modeling tools cannot ignore real traffic characteristics, traditional techniques and classical results have their application and can shed light on the impact of possible networking evolutions.

The field of teletraffic has gone through major changes since the last decades, and so has the entire discipline of teletraffic engineering. Agner Krarup Erlang—through his brilliant contributions—will be long remembered as one who set the direction and pace of that change. ∎

ZORAN BOJKOVIC, *Senior Member IEEE*
Faculty of Transport and Traffic
Engineering University of Belgrade
Belgrade, Serbia

MIODRAG BAKMAZ, *Member IEEE*
Faculty of Transport and Traffic
Engineering University of Belgrade
Belgrade, Serbia

BOJAN BAKMAZ, *Student Member IEEE*
Faculty of Transport and Traffic
Engineering University of Belgrade
Belgrade, Serbia

REFERENCES

[1] E. Brockmeyer, H. L. Halstrøm, and A. Jensen, "The life and works of A. K. Erlang," *Trans. Danish Acad. Tech. Sci.*, no. 2, p. 277, 1948, Copenhagen.

[2] *Handbook: Teletraffic Engineering*, ITU-D, Study Group 2, Question 16/2, rev. Jun. 2006.