Comments on "Systems with Infinite-Dimensional State Space: The Hilbert Space Approach"

JAMES A. DYER

Professor Helton has written a very interesting expository paper[1] on the applications of Hilbert space techniques to systems having infinite-dimensional state spaces. I find it, however, to be rather surprising that quantum mechanics, which is the best example in all mathematical physics of the applications of Hilbert space techniques to the study of infinite-dimensional systems, is not mentioned.

Not only does quantum mechanics provide excellent examples of infinite-dimensional systems, the language currently used by many physicists to describe a quantum system is the same as that used by a systems engineer to describe an engineering system from the state viewpoint. To illustrate this, consider a single-particle quantum mechanical system. Such a system is completely described by a probability amplitude function $\phi(x, y, z, t)$. For fixed $\bar{t}$, $\phi(\bar{t}) = \phi(x, y, z, \bar{t})$ is required to be an element of the unit sphere of the Hilbert space $L_2(R^3)$. For fixed $\bar{t}$, this function is usually called the state of the system even by physicists. The propagation in time of the state is given by Schrödinger's equation

$$\frac{d\phi}{dt} = -\frac{i}{\hbar} \mathcal{H} \phi$$

$$\phi(0) = \phi_0$$

where $i = \sqrt{-1}$, $\hbar$ is $(2\pi)^{-1}$ times Planck's constant, and $\mathcal{H}$ known as the Hamiltonian operator, is a self-adjoint unbounded linear operator defined on a dense subspace of $L_2(R^3)$. It will be observed that this equation has the form of the usual state propagation equation of linear systems theory. It will be observed from this equation also that quantum mechanics concerns itself only with zero-input solutions. The quantities which a systems engineer would call outputs of this system would be called observables by a quantum physicist. An observable y of a single-particle system is related to the state of the system by the equation

$$y = \langle \phi, \mathcal{L} \phi \rangle$$

where $\langle , \rangle$ denotes the inner product on $L_2(R^3)$ and $\mathcal{L}$ is a linear self-adjoint unbounded operator defined on a dense subspace of $L_2(R^3)$. For example, if $\mathcal{H}$ is the Hamiltonian operator then the observable $E = \langle \phi, \mathcal{H} \phi \rangle$ is the energy of the system. Again, one finds here a standard type of output equation for a system having an infinite-dimensional state space and a one-dimensional output. An excellent elementary introduction to quantum mechanics from the state viewpoint is given in [1].

In the Feynman integral approach to quantum mechanics one also makes use of the concept of state transition operator. Here one writes the state $\phi(t)$ as

$$\phi(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(t, \hat{x}, \hat{y}, \hat{z})\, \phi_0(\hat{x}, \hat{y}, \hat{z})\, d\hat{x}\, d\hat{y}\, d\hat{z},$$

where K is an L_2 valued function known as the propagator. In fact K is the impulse response of the quantum system and is the solution to Schrödinger's equation for the initial state $\phi_0(\hat{x}, \hat{y}, \hat{z}) = \delta(x - \hat{x}) \cdot \delta(y - \hat{y})\, \delta(z - \hat{z})$. Since Schrödinger's equation does not have a Green's function in the usual mathematical sense, K is determined as the value of an abstract integral known as a Feynman integral. An elementary introduction to this approach to quantum theory is given in [2].

The reason for the parallelism between the usual formulations of quantum mechanics and the present formulations of systems theory is that both subjects have a common parent in classical Hamiltonian mechanics. These origins are discussed in detail in [3], part II. Since both systems theory and quantum mechanics have followed independent but approximately parallel paths since their origins it is possible that cross-fertilization between the two subjects could prove beneficial to both.

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Reply[2] by J. W. Helton

It is pleasing to hear that my article seemed natural enough to be termed expository, since sections II, III, and IV were new. Professor Dyer gives a nice discussion of how basic quantum mechanics itself can be described as an infinite-dimensional system. Quantum mechanics was indeed not mentioned in the paper and so here are some comments on the scattering theory aspects of it.

The paper introduces a particular definition of infinite-dimensional A, B, C, D-type linear systems and in Section III proves that it is equivalent to Lax-Phillips scattering. Now their theory contains quantum mechanical scattering, although the examples they emphasize are classical. This correspondence is explained in Chapter VI, Section 4, of their book. Roughly, to a Schrodinger equation scattering problem with s-matrix S there is associated a scattering problem for a wave equation with scattering matrix Γ. Then $S(z) = \Gamma(\sqrt{z})$. Functions with branch cuts are not pseudomeromorphic in the sense of Section IV-D of the paper and so $S(z)$ is not pseudomeromorphic. Also note that most results described in the exposition, Section V, hold not only for classical but for quantum mechanical scattering. We should emphasize that when we say quantum mechanical scattering we automatically mean very special input and output operators B, C and so these comments do not address many systems based on Schrodinger's equation.

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Comments on "Geometric Progression Ladder *RC* Networks"

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***Abstract*—It is shown that a large class of nonrecurrent *RC* ladder networks including, as a special case, the geometric ladder networks discussed by Bhattacharyya and Swamy can be represented as an equivalent *RCG* ladder network.**

Bhattacharyya and Swamy's [1] conjecture that other nonrecurrent ladder networks may be represented in terms of recurrent ladder networks is readily answered with reference to the author's earlier publications [2]-[3].

The application of Kirchhoff's law to the nth node of the ladder network of Fig. 1 shows that the voltage $V(n)$ satisfies the difference equation

$$\frac{V(n+1)}{Z(n+1)} - V(n)\left[Y(n) + \frac{1}{Z(n+1)} + \frac{1}{Z(n)}\right] + \frac{V(n-1)}{Z(n)} = 0, \qquad n \geq 1 \tag{1}$$

where for a nonrecurrent RC network the series impedance $Z(n) = R(n)$ and the shunt admittance $Y(n) = sC(n)$.

In the special case of a recurrent ladder network, $Z(n)$ and $Y(n)$ are invariant with respect to n; in particular, if $Z(n) = R_u$ and $Y(n) = G_u + sC_u$, then replacing $V(n)$ by $V_u(n)$, the solution of the difference equation (1) is given by

$$V_u(n) = C_1 e^{n\theta} + C_2 e^{-n\theta}$$

where

$$\cosh\theta = 1 + \frac{R_u(G_u + sC_u)}{2}. \tag{2}$$

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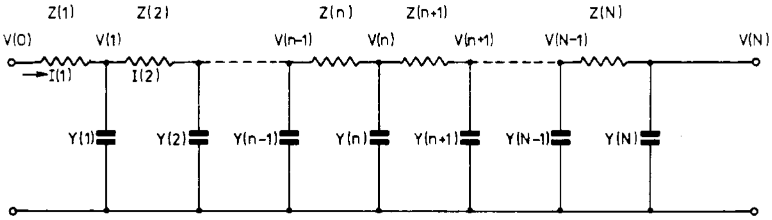

Fig. 1. The generalized *RC* ladder network.

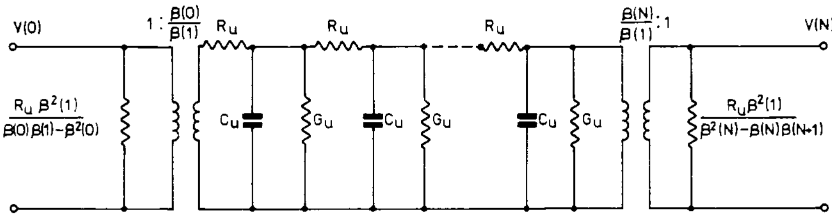

Fig. 2. Equivalent circuit of the generalized *RC* ladder network in terms of an *RCG* recurrent ladder.

We consider a second-order linear difference equation with variable coefficients

$$h\beta(n+1)\beta(n)V(n+1) - (a_1 s + b_1)h\beta^2(n)V(n) + h\beta(n)\beta(n-1)V(n-1) = 0 \quad (3)$$

where $\beta(n)$ is a yet unspecified function of n while a_1, b_1, and h are as yet unspecified constants and s is the frequency variable.

Comparison of (1) and (3) suggests that

$$Z(n) = 1/[h\beta(n)\beta(n-1)] \quad (4)$$

$$Y(n) = ha_1 s\beta^2(n) \quad (5)$$

$$h\beta(n+1)\beta(n) - hb_1\beta^2(n) + h\beta(n)\beta(n-1) = 0. \quad (6)$$

Hence, if $b_1 \neq 2$, $\beta(n) = C_3 e^{n\psi} + C_4 e^{-n\psi}$, where

$$\cosh \psi = b_1/2 \quad (7)$$

while

$$\beta(n) = C_5 + C_6 n, \qquad \text{if } b_1 = 2. \quad (8)$$

The solution (3) is readily reduced to that of solving a linear difference equation with constant coefficients by letting

$$\beta(n)V(n) = V_u(n). \quad (9)$$

Equation (9) is seen to be a generalization of Bhattacharyya and Swamy's [1] transformation (3).

With reference to (1) and (3),

$$a_1 s + b_1 = 2 + R_u(G_u + sC_u)$$

$$a_1 = R_u C_u$$

$$b_2 = 2 + G_u R_u. \quad (10)$$

Referring to (2), (7), and (8), $C_i (i = 1 - 6)$ are arbitrary constants, and, assuming that $R(1) = R$ and $C(1) = C = C_u$, it follows that subject to (4) and (5)

$$a_1 = CR\frac{\beta(0)}{\beta(1)} \qquad h = \frac{1}{R_u\beta^2(1)} \qquad R_u = R\frac{\beta(0)}{\beta(1)}. \quad (11)$$

Equations (7) and (8) specify the ladder taper and it is apparent that, using the notation of [1], the geometric taper represents a special case obtained by letting $C_3 = 0$, $C_4 = 1$, $\beta(n) = a^{-n/2}$, and, hence,

$$b_1 = a^{1/2} + a^{-1/2} \quad \text{and} \quad a_1 = a^{1/2}RC.$$

The equivalent *RCG* ladder network may be derived along the lines of [1] and is shown in Fig. 2 for the network described by (4) and (5).

The *ABCD* parameters have been derived earlier by the author [2], [3] (allowing for some differences in notation and one additional series impedance), a special case applicable to a geometrically tapered ladder network having been quoted by Pang [4]. For reference, the *ABCD* parameters of the network shown in Fig. 1 and described by (4) and (5) are

$$A = \frac{\beta(N)}{\beta(0)}\frac{\sinh (N+1)\theta}{\sinh\theta} - \frac{\beta(N+1)}{\beta(0)}\frac{\sinh N\theta}{\sinh\theta} \quad (12)$$

$$B = R\frac{\beta(1)}{\beta(N)}\frac{\sinh N\theta}{\sinh\theta} \quad (13)$$

$$C = \frac{1}{R}\left[\frac{\beta(N)}{\beta(0)}\frac{\sinh (N+1)\theta}{\sinh\theta} - \frac{\beta(N)}{\beta(1)}\frac{\sinh N\theta}{\sinh\theta} - \frac{\beta(N+1)}{\beta(0)}\frac{\sinh N\theta}{\sinh\theta} + \frac{\beta(N+1)}{\beta(1)}\frac{\sinh (N-1)\theta}{\sinh\theta}\right] \quad (14)$$

$$D = \frac{\beta(1)}{\beta(N)}\frac{\sinh N\theta}{\sinh\theta} - \frac{\beta(0)}{\beta(N)}\frac{\sinh (N-1)\theta}{\sinh\theta} \quad (15)$$

where

$$\cosh\theta = \frac{a_1 s + b_1}{2}. \quad (16)$$

It may be mentioned in passing that the *ABCD* parameters can be alternatively derived by first finding either the open-circuit or the short-circuit parameters with the aid of the appropriate discrete Green's function [5].

A dual case is obtained if, in place of voltage equation (1), the corresponding current equation is derived. Retracing the above steps, it will be found that $Z(n) = ha_1\beta^2(n)$ while $Y(n)$ has a form corresponding to that given by (4); the determination of the equivalent *RCG* circuit can be undertaken in an analogous manner and the *ABCD* parameters calculated with the aid of [2].

In conclusion, it has been shown that it is possible to represent nonrecurrent ladder networks in terms of recurrent ladder networks whenever either the appropriate voltage or current equation can be reduced to a linear difference equation with constant coefficients by means of a suitable transformation; the transformation includes the geometric ladder network as a special case.

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