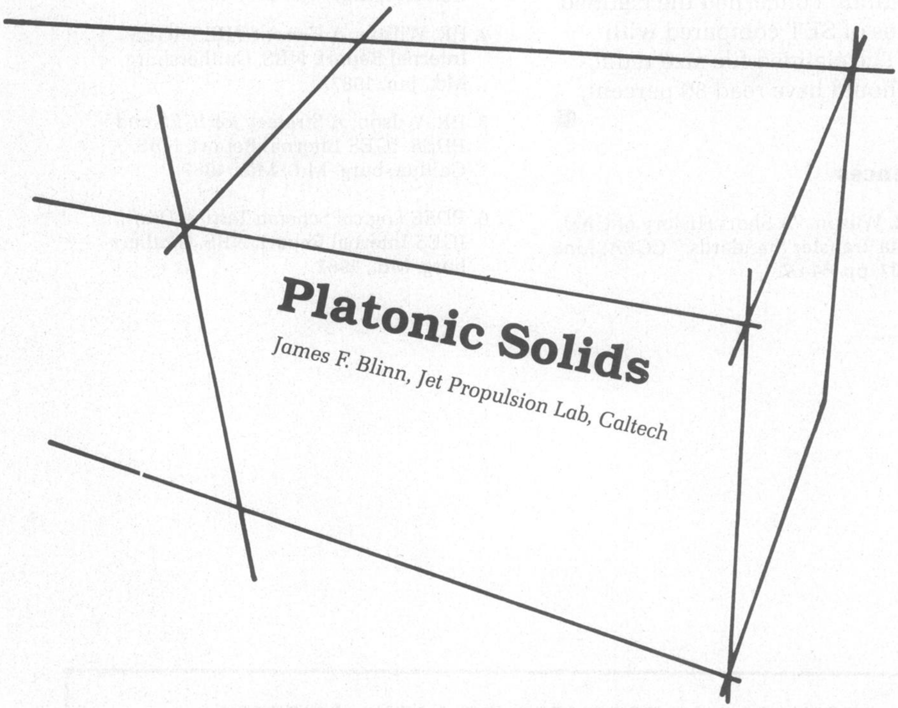

Platonic Solids

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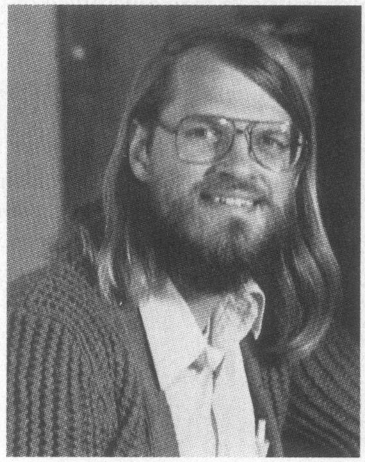

It is not often that major long-standing theories in geometry are overthrown. Since the time of Plato it has been thought that there were only five regular solids. Recently, however, James Arvo and David Kirk of Apollo have discovered a sixth. The new shape—the teapotahedron—is illustrated on the back cover of the SIGGRAPH 87 conference proceedings. Since I dealt with this shape a few columns ago, I thought I would share some observations on databases of the other five Platonic solids. Constructing a database for these shapes is a good basis for exploring the various sorts of symmetry they have. The main problem is to find explicit coordinates for the vertices. A cube or an octahedron uses pretty simple numbers. The other shapes, made of equilateral triangles or pentagons, might at first seem to require strange numbers as coordinates, but the messiness of the values depends on the orientations of the shapes. My object here is to find orientations that allow the vertex coordinates to be as simple as possible. It is possible to construct all five shapes using only the numbers 0, 1, and the golden ratio φ. This latter is defined by the equation

$$\frac{1}{\varphi} = \frac{\varphi}{1+\varphi}$$

which works out to

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618034$$

Now some notational conventions: Each point (vertex) is numbered, and the coordinates are declared by a line of the form

```
PNT  n, x, y, z
```

After the points are defined, the polygons (faces) are described by a sequence of point numbers denoted

```
POLY  n1, n2, n3, ...
```

Each polygon is carefully constructed so that the points are named going in a consistent clockwise order as seen from the outside (if you are left-handed) or counterclockwise order (if you are right-handed).

The cube

The cube is centered at the origin, has an edge length of 2, and uses only the numbers +1 and −1 as coordinates.

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```
PNT  1,  1.,  1.,  1.
PNT  2,  1.,  1., -1.
PNT  3,  1., -1.,  1.
PNT  4,  1., -1., -1.
PNT  5, -1.,  1.,  1.
PNT  6, -1.,  1., -1.
PNT  7, -1., -1.,  1.
PNT  8, -1., -1., -1.
POLY 2,1,3,4
POLY 5,6,8,7
POLY 1,2,6,5
POLY 4,3,7,8
POLY 3,1,5,7
POLY 2,4,8,6
```

A picture of this appears in Figure 1.

I am including decimal points in the coordinates, even though they happen to be integers, to emphasize the fact that they are floating-point numbers—a little readability trick. All the point numbers are integers and are essentially just names or labels. (In fact, many polygon modeling systems actually allow symbolic names here.)

The octahedron

This database is just as easy. Here we use the numbers +1, 0, and −1 (see Figure 2).

```
PNT  1,  1.,  0.,  0.
PNT  2, -1.,  0.,  0.
PNT  3,  0.,  1.,  0.
PNT  4,  0., -1.,  0.
PNT  5,  0.,  0.,  1.
PNT  6,  0.,  0., -1.
POLY 1,3,5
POLY 3,1,6
POLY 4,1,5
POLY 1,4,6
POLY 3,2,5
POLY 2,3,6
POLY 2,4,5
POLY 4,2,6
```

The octahedron is what is known as the "dual" shape of the cube. That is, each vertex of the octahedron lies at the center of a face of the cube, and each vertex of the cube corresponds to a face of the octahedron. (Scaling the cube uniformly by 1/3 makes its vertices lie exactly in the center of the octahedron's faces.) In fact, even though we are not numbering polygons explicitly, the polygons and points of the above databases have been carefully ordered so that the *j*th point of the octahedron lies at the center of the *j*th polygon of the cube. Likewise, the *j*th point of the cube (times 1/3) lies at the center of the *j*th polygon of the octahedron.

So all right...the octahedron is balancing on its nose. To get it to lie with one face on, say the $z = 0$ plane, you have to rotate and translate it. I have chosen to do this by selecting the final polygon (that is, points 2, 4, and 6) and solving for a rotation and translation matrix that makes all three points have a z coordinate of zero. First rotate by 45 degrees around *y* to get the points to have coordinates

2:	$-R$,	0,	$-R$
4:	0,	-1,	0
6:	R,	0,	$-R$

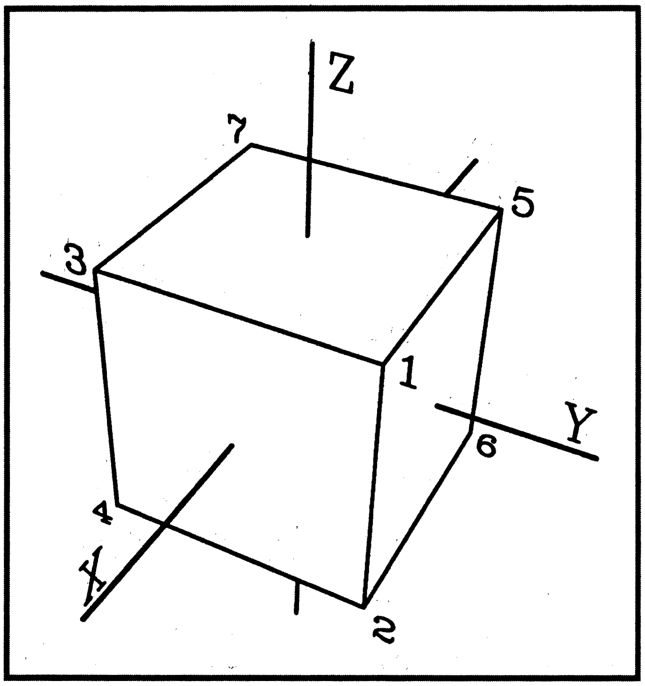

Figure 1.

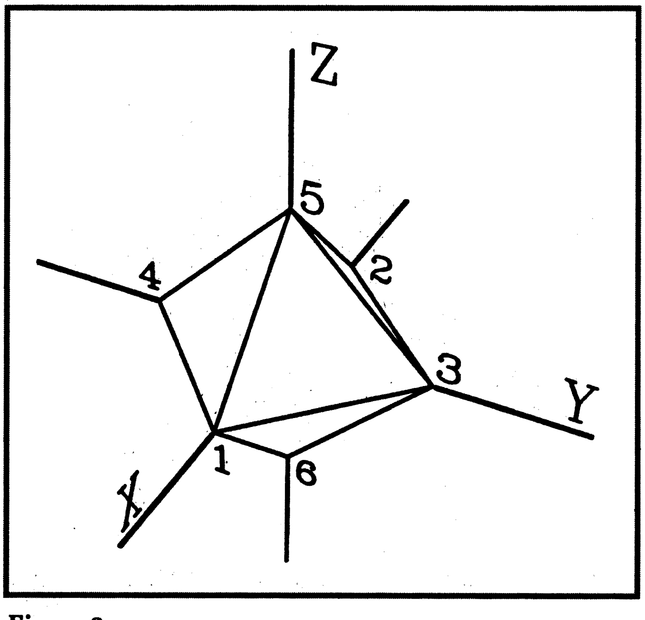

Figure 2.

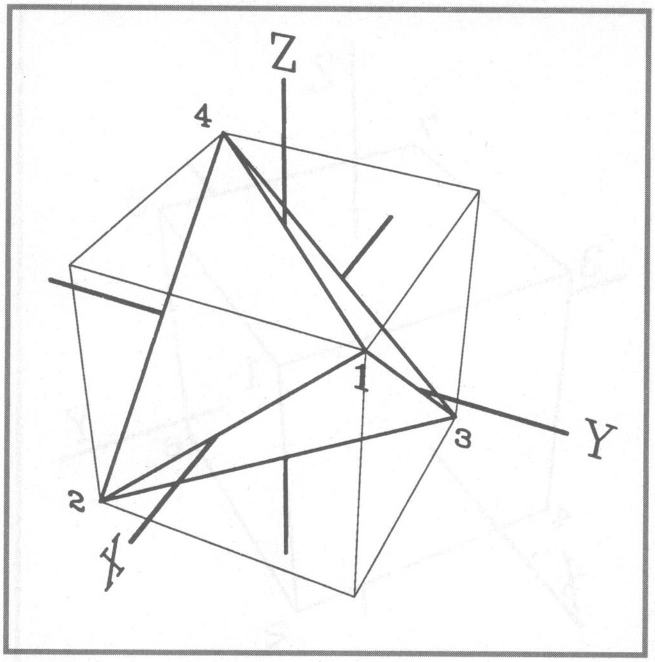

Figure 3.

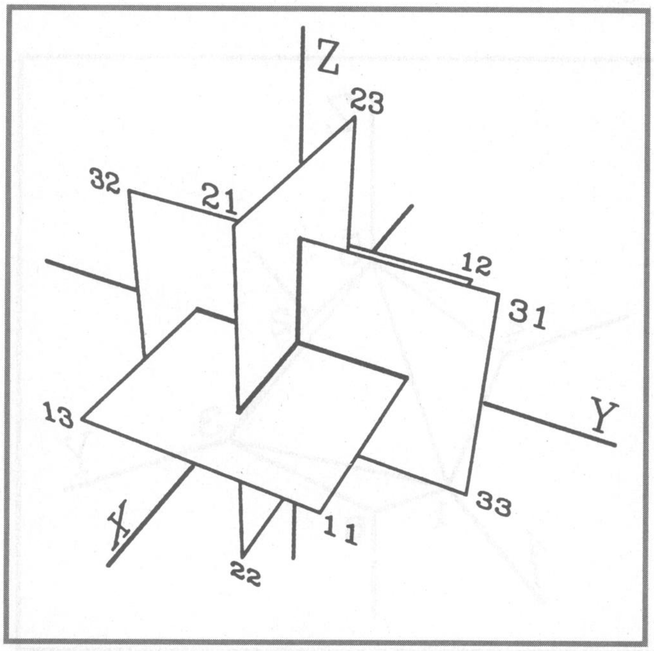

Figure 4.

where

$R = \sqrt{2}/2$

Then rotate about x by the angle φ, putting the points at

2:	$-R$,	$-R\sin\alpha$,	$-R\cos\alpha$
4:	0,	$-\cos\alpha$,	$\sin\alpha$
6:	R,	$-R\sin\alpha$,	$-R\cos\alpha$

To make the z coordinates the same, the angle and z value must be

$$\alpha = \arctan\frac{-\sqrt{2}}{2} \approx -35.2644^\circ$$
$$z = -\sqrt{3} \approx -.57735$$

Using the notation for transformations developed last month, the face-on octahedron is

```
PUSH
TRAN  0, 0, .57735
ROT  -35.2644, 1
ROT   45.0000, 2
DRAW OCTAHEDRON
POP
```

The tetrahedron

This is the first tricky one. The initial thought is to put one of the triangles in, say, the z=0 plane, but the coordinates are not obvious. It happens, though, that a tetrahedron can fit entirely inside of a cube, its edges lining up with the diagonals to the cube's faces (abracadabra). The database is

```
PNT 1,  1.,  1.,  1.
PNT 2,  1., -1., -1.
PNT 3, -1.,  1., -1.
PNT 4, -1., -1.,  1.
POLY 4,3,2
POLY 3,4,1
POLY 2,1,4
POLY 1,2,3
```

No muss, no fuss. See Figure 3.

Admittedly it is in a bit of a weird orientation, sitting on one edge instead of one face. You can make it face down by putting it through exactly the same transformation that made the octahedron face down (double abracadabra).

```
PUSH
TRAN  0, 0, .57735
ROT  -35.2644, 1
ROT   45.0000, 2
DRAW TETRAHEDRON
POP
```

The icosahedron

This can also be easily done edge on. In this case the 12 vertices of the icosahedron happen to lie at the corners of three golden rectangles that are symmetrically intertwined, as in Figure 4. A golden rectangle has a height-to-width ratio of 1:φ, where the aforementioned $\varphi \approx 1.618034$. Admittedly this is an irrational number, but it seems to be a popular one in nature. Generating the numbers and tying them together into polygons gives the database:

```
PNT 11,  1.618034,  1.,  0.
PNT 12, -1.618034,  1.,  0.
PNT 13,  1.618034, -1.,  0.
PNT 14, -1.618034, -1.,  0.

PNT 21,  1., 0.,  1.618034
PNT 22,  1., 0., -1.618034
PNT 23, -1., 0.,  1.618034
PNT 24, -1., 0., -1.618034

PNT 31,  0.,  1.618034,  1.
PNT 32,  0., -1.618034,  1.
PNT 33,  0.,  1.618034, -1.
PNT 34,  0., -1.618034, -1.

POLY 11, 31, 21   !  1
POLY 11, 22, 33   !  2
POLY 13, 21, 32   !  3
POLY 13, 34, 22   !  4
POLY 12, 23, 31   !  5
POLY 12, 33, 24   !  6
POLY 14, 32, 23   !  7
POLY 14, 24, 34   !  8

POLY 11, 33, 31   ! 11
POLY 12, 31, 33   ! 12
POLY 13, 32, 34   ! 13
POLY 14, 34, 32   ! 14

POLY 21, 13, 11   ! 21
POLY 22, 11, 13   ! 22
POLY 23, 12, 14   ! 23
POLY 24, 14, 12   ! 24

POLY 31, 23, 21   ! 31
POLY 32, 21, 23   ! 32
POLY 33, 22, 24   ! 33
POLY 34, 24, 22   ! 34
```

See Figure 5. The points have been number-named nonconsecutively in an attempt to show which golden rectangle they come from. The polygons are named in the comment field (after the "!") also in a nonconsecutive way, which will be useful later. Again, this orientation is unsatisfying if you want to set the shape on a table. To get it face down, let us rotate it about the x axis so that the vertices of polygon 34 (that is, points 34, 24, and 22) all have equal z coordinates. The angle and z turn out to be

$$\arctan \frac{-1}{\varphi^2} \approx -20.9051^\circ$$
$$z = -\varphi^2/\sqrt{3} \approx -1.51152$$

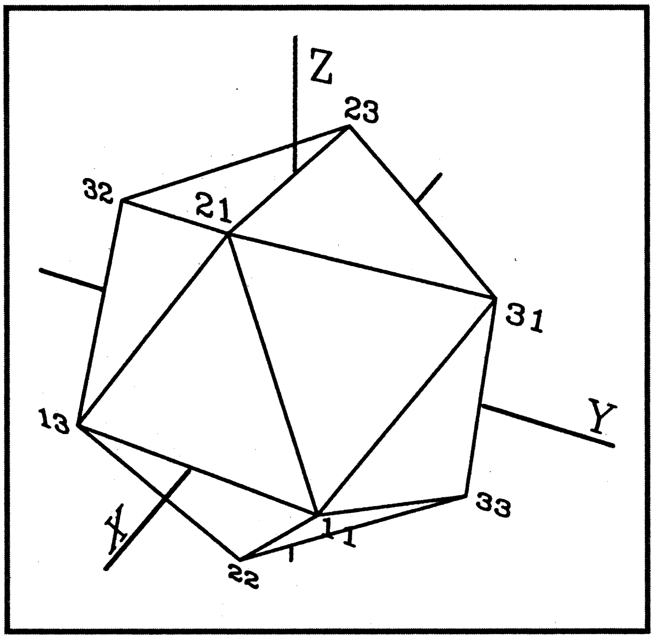

Figure 5.

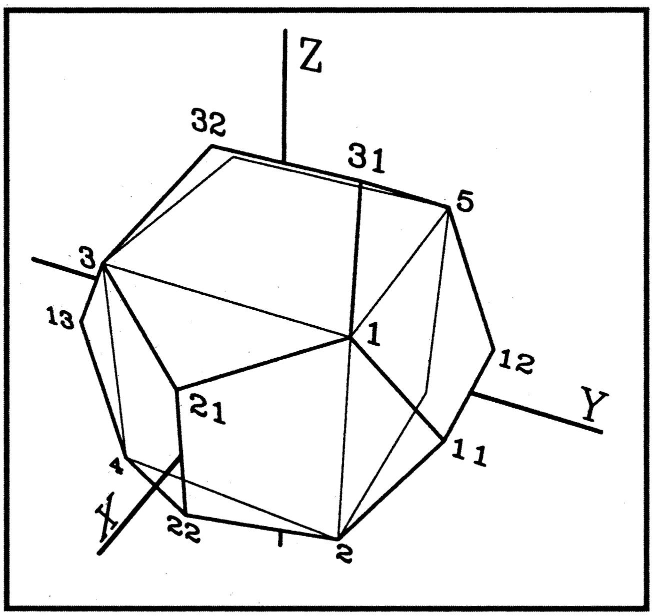

Figure 6.

The stable icosahedron is then

```
PUSH
TRAN 0., 0., 1.51152
ROT -20.9051,1
DRAW ICOSAHEDRON
POP
```

The dodecahedron

Once again an edge-on orientation gives the easiest coordination. Here, we need to generate 20 vertices.

The dodecahedron is the dual shape of the icosahedron. This means that each vertex of the dodecahedron can be calculated as the centroid of a face of the icosahedron. Doing this for each face yields coordinates that look a bit messy at first. But by using the following identities

$$
\begin{aligned}
1/\varphi &= \varphi - 1 \\
\varphi^2 &= \varphi + 1 \\
\varphi^3 &= 2\varphi + 1
\end{aligned}
$$

and scaling the whole shape by $1/\varphi^2$, you get points with coordinates that use only the values 0, 1, φ, and $1/\varphi$. This process reveals another interesting property of the dodecahedron: It has a cube embedded in it. Eight of the vertices are the same as the eight vertices of our cube. The other 12 come from a set of three rectangles, intertwined in a fashion similar to those in the icosahedron. In this case, however, the rectangles have aspect ratio $1/\varphi{:}\varphi$. The database is

```
PNT  1,   1.,  1.,  1.
PNT  2,   1.,  1., -1.
PNT  3,   1., -1.,  1.
PNT  4,   1., -1., -1.
PNT  5,  -1.,  1.,  1.
PNT  6,  -1.,  1., -1.
PNT  7,  -1., -1.,  1.
PNT  8,  -1., -1., -1.

PNT 11,   .618034,  1.618034, 0.
PNT 12,  -.618034,  1.618034, 0.
PNT 13,   .618034, -1.618034, 0.
PNT 14,  -.618034, -1.618034, 0.

PNT 21,  1.618034, 0.,  .618034
PNT 22,  1.618034, 0., -.618034
PNT 23, -1.618034, 0.,  .618034
PNT 24, -1.618034, 0., -.618034

PNT 31, 0.,  .618034,  1.618034
PNT 32, 0., -.618034,  1.618034
PNT 33, 0.,  .618034, -1.618034
PNT 34, 0., -.618034, -1.618034

POLY 2,11,1,21,22   ! 11
POLY 5,12,6,24,23   ! 12
POLY 3,13,4,22,21   ! 13
POLY 8,14,7,23,24   ! 14

POLY 3,21,1,31,32   ! 21
POLY 2,22,4,34,33   ! 22
POLY 5,23,7,32,31   ! 23
POLY 8,24,6,33,34   ! 24

POLY 5,31,1,11,12   ! 31
POLY 3,32,7,14,13   ! 32
POLY 2,33,6,12,11   ! 33
POLY 8,34,4,13,14   ! 34
```

Now the naming of the polygons of the icosahedron makes sense. Polygon *i* of the icosahedron corresponds to point *i* of the dodecahedron. Polygon *j* of the dodecahedron corresponds to point *j* of the icosahedron. How to get it face down? Rotate about x to make all points of face 34 have the same z coordinate. The angle and z are

$$
\begin{aligned}
\arctan(-\varphi) &\approx -58.2825^\circ \\
z = -\varphi^2/\sqrt{\varphi + 2} &\approx -1.37638
\end{aligned}
$$

The net transformation is

```
PUSH
TRAN 0., 0., 1.37638
ROT -58.2825, 1
DRAW DODECAHDERON
POP
```

Other ways

These databases have simple point coordinates but the solids need to be rotated to get them face down. It is still an interesting exercise to try to generate a database directly in another coordinate system, either face-on or vertex-on. I leave this as (ahem) an exercise for the reader. Another way to generate these highly symmetric shapes is to use a single triangular polygon and place rotated copies of it in space to form the shape. In a few months I will show the results of some such experiments.

Applications

What good is this? Hey, it's abstract mathematics; it doesn't have to be practical. But actually there are practical applications. Geodesic domes are based on the icosahedron. You need to calculate locations of the vertices to build the dome. Knowledge of these coordinates also proved useful in the making of *The Mechanical Universe*. Several of the programs dealt with electric field lines radiating from point charges in space. A symmetric placement of starting points for these field lines was necessary. The vertex coordinates of an icosahedron or a dodecahedron proved exactly what was needed. ■