On Weakened Sufficiency Requirements for the Multiplicative Form of Value Function

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Abstract—**New sufficiency conditions are given for the multiplicative form of value function, the use of which alleviates a good bit of the effort for verification, using existing procedures, of an additive value function form. These conditions result in increased sensitivity of the scoring form for many applications. These are possible since the new results are weaker sufficient conditions than mutual preferential independence of attributes, the existing sufficiency condition for the multiplicative form of value function. An example illustrating use of the new sufficiency conditions for an aircraft retrofit problem is presented, as is a concise summary of a suggested assessment procedure.**

I. Introduction

Mutual preferential independence of attributes is generally used as a sufficiency condition for the multiplicative form of value function. Often the effort to verify existence of this condition is considerable. In this correspondence new and weakened sufficiency conditions are given for the multiplicative form of value function. Use of these new conditions will generally require much less effort than use of the mutual preferential independence conditions. Also the new sufficiency conditions will often present very desirable measurement sensitivity conditions.

Section II discusses the motivation for use of multiplicative form of value function. Section III presents and proves the existence of weaker conditions which allow use of the multiplicative function while alleviating much of the verification process. Finally, section IV presents a numerical application which concerns aircraft retrofit requirements.

II. The Additive Value Function

The popularity of the additive form of measurable value functions is warranted because of the facility of assessment of the constituent value functions and ease of evaluation of the scaling constants [1] and [6]. Unfortunately, two drawbacks should, in practice, often discourage the widescale use of the additive form of value function. The additive form of value function is

$$v(x) = \sum_{i=1}^{n} k_i \cdot v_i(x_i) \tag{1}$$

where $v(x)$ is a scalar signifying the score for a specific action with attribute levels x_i, $v_i(x_i)$ is a component value statement for attribute level x_i, x is an n vector of attribute levels, and k_i is a scaling constant. It is proper to use this additive form only when the attributes x_i are mutually preferentially independent (MPI) [6], and this can be difficult for the gerent to verify. It is a sizable task to show MPI of attributes where the dimension of the value attributes n is large.

Another shortcoming of the additive form is that it tends to be insensitive to individual attribute levels. Additive forms are compensatory in the sense that an increase in one attribute can compensate for a decrease in any other attribute. This means that large increases or decreases in any one attribute may be offset by changes in other attribute levels and consequently may have little effect on the scoring value of the total model. Huber and Johnson [3] have pointed out that this compensatory characteristic may be undesirable in many applications.

III. The Multiplicative Value Function

A form of value function that alleviates these difficulties is the multiplicative form. While the requirement for MPI of attributes is a sufficient condition for a multiplicative form of value function, it is not a necessary condition as it was for the additive form. Dyer and Sarin [1], building on the work of Fishburn [2] and Keeney and Raiffa [6], have determined a sufficient condition for the multiplicative form of the general multilinear form. If attributes $X_1, X_2, \cdots, X_n$ are mutually weak difference independent (MWDI), then the form of the value function is

$$\begin{aligned} v(x) = &\sum_{i=1}^{n} k_i v_i(x_i) \\ &+ K \sum_{i=1, j>i}^{n} k_i k_j v_i(x_i) v_j(x_j) \\ &+ K^2 \sum_{i=1, j>i, l>j}^{n} k_i k_j k_l v_i(x_i) v_j(x_j) v_l(x_1) \\ &+ \cdots + K^{n-1} \prod_{i=1}^{n} k_i v_i(x_i) \end{aligned} \tag{2}$$

or

$$1 + Kv(x) = \prod_{i=1}^{n} [1 + Kk_i v_i(x_i)]. \tag{3}$$

To continue the development, it is desirable to define weak difference independence (WDI).[1]

Definition: Attribute X_i is WDI of $X_{\bar{i}}$ if given any $a_i, b_i, c_i, d_i \in X_i$, such that

$$v(a_i, a_{\bar{i}}) - v(b_i, a_{\bar{i}}) \geq v(c_i, a_{\bar{i}}) - v(d_i a_{\bar{i}})$$

for some $a_{\bar{i}} \in X_{\bar{i}}$, then it is required that

$$v(a_i b_{\bar{i}}) - v(b_i b_{\bar{i}}) \geq v(c_i b_{\bar{i}}) - v(d_i b_{\bar{i}})$$

for any $b_{\bar{i}} \in X_{\bar{i}}$.

Thus, WDI means that the ordering of preference differences depends only on the value differences associated with attribute X_i and not on the fixed value of all other attributes of $X_{\bar{i}}$. The attribute X_i is WDI of $X_{\bar{i}}$ if the value function is of the form

$$v(X_i, X_{\bar{i}}) = g(X_{\bar{i}}) + h(X_{\bar{i}}) v(X_i, X'_{\bar{i}}) \tag{4}$$

for all X_i, $X_{\bar{i}}$, and $X'_{\bar{i}}$, where $g(X_{\bar{i}})$ and $h(X_{\bar{i}}) > 0$ are functions which depend only on $X_{\bar{i}}$.

A physical interpretation of this definition of WDI will now be presented for an equipment retrofit design effort. It is assumed that only two attributes are important to the problem, X_1 which represents volume and $X_2 = X_{\bar{i}}$ which represents cost. The object is to minimize the cost and the volume of the retrofit equipment. To check if X_1 is WDI of $X_{\bar{1}}$, we first choose levels of $a_1, b_1, c_1, d_1 \in X_1$ and $a_{\bar{1}} \in X_{\bar{1}}$, so that the exchange of the combination of attributive levels $(b_1, a_{\bar{1}})$ for the pair $(a_1, a_{\bar{1}})$ is preferred to the exchange of pair $(d_1, a_{\bar{1}})$ for $(c_1, a_{\bar{1}})$. If this preference order is

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[1] The symbol $Y_{\bar{i}}$ is used to indicate all components of Y not contained in Y_i.

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preserved for other levels of attribute X_{T}, that is to say if the exchange of (b_1, b_{T}) for the combination (a_1, b_{T}) is preferred to the exchange of pair (d_1, b_{T}) for (c_1, b_{T}) for any $b_{\mathrm{T}} \in X_{\mathrm{T}}$, then X_1 is WDI of X_{T}. In this case let

$$a_1 = 4 \text{ m}^3, \quad b_1 = 5 \text{ m}^3, \quad c_1 = 2 \text{ m}^3, \quad d_1 = 3 \text{ m}^3,$$

and $a_{\mathrm{T}} = \$10^6$, and assume that 4 m^3 equipment volume fills the allotted space and that 5 m^3 of equipment means a cutback of a co-located equipment function. To satisfy the above requirement at a single attribute level, the decisionmaker (DM) must prefer the exchange of a system with characteristics of (5 m^3, $\$10^6$) for a configuration (4 m^4, $\$10^6$) over the exchange of a configuration (3 m^3, $\$10^6$)for a configuration (2 m^3, $\$10^6$). WDI requires that this preference order for exchanges must apply for other levels of X_{T} such as $b_{\mathrm{T}} = \$1.5 \times 10^6$ in each system configuration. Thus the DM must still prefer the exchange of system (5 m^3, $\$1.5 \times 10^6$) for system (4 m^3, $\$1.5 \times 10^6$) over the exchange of system (3 m^3, $\$1.5 \times 10^6$) for system (2 m^3, $\$1.5 \times 10^6$). If the DM expresses the same preference order responses for a number of different quadruples of levels X_1 and fixed levels of X_{T}, then it can be deduced that X_1 is WDI of X_{T}. A caveat should be issued in order to insure that the DM is stating preferences concerning exchanges of configurations and outcomes and not stating preference for the configurations or outcomes themselves. For instance a rational DM who prefers an exchange of the pair ($100, 1 oz of gold) for ($105, 1 oz of gold) to an exchange of the attribute pair ($40, 1 oz of gold) for ($50, 1 oz of gold) is obviously not concentrating on the exchange itself (if it is assumed that the DM has a linearly increasing monetary value preference curve) because this DM is irrationally preferring an increase in cash of $5 to an increase of $10. A DM who incorrectly stated this would be erroneously establishing the first condition for WDI at the single attribute level. This exchange preference idea agrees with Kahneman and Tversky [4] in their "prospect theory" which accounts for the reference effect of the asset position of the DM in rational choices.

The verification of the appropriateness of the multiplicative form of value function requires checking all subsets of attributes of WDI. We present the following to make this task simpler and less time-consuming.

Theorem 1: Given attributes $X_1, X_2, \cdots, X_n$, the following are equivalent.

a) Attributes $X_1, X_2, \cdots, X_n$ are mutually weak difference independent (MWDI).
b) X_i is weak difference independent of $X_{\bar{i}}$, and $(X_i, X_j), j \neq i$, is preferentially independent (PI); $j = 1, 2, 3, \cdots, n, n \geq 3$.

The result of this theorem allows a much reduced effort in order to verify sufficient conditions for the validity of a product form of measurable value function. The proof of this theorem requires a fundamental relationship between PI and WDI. This relationship follows the same reasoning presented in Keeney [5] and Keeney and Raiffa [6] for weak sufficiency associated with mutual utility independence.

Our proof is simplified by consideration first of the three attribute cases described by Lemma 1.

Lemma 1: Given a set of attributes A, B, and C; if A is weak difference independent of $\bar{A}$, and if (A, B) is preferentially independent of $(\bar{A}, \bar{B})$, then (A, B) is WDI of $(\bar{A}, \bar{B})$.

The proof of this Lemma proceeds as follows. We let $\bar{A} = B \odot C$ where $\odot$ signifies a Cartesian product space. The case where (A, B) is WDI of $(\bar{A}, \bar{B})$ for all pairs of attributes is a sufficient condition for the proof of Theorem 1 in the three attribute cases, and this can be extended to the n attribute case, for

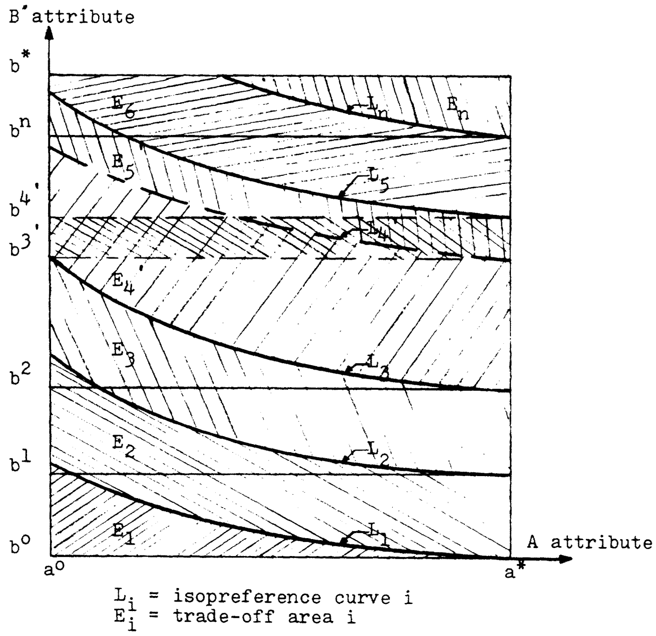

Fig. 1. Conceptual illustration of requirements for difference independence.

mutual weak difference independence (MWDI) [1]. The condition where A is WDI of $\bar{A}$ can be represented by

$$v(a, b, c) = g(a^0, b, c) + h(a^0, b, c)v(a, b^0, c^0), \qquad h > 0 \tag{5}$$

where a, b, and c represent levels of attributes A, B, and C, respectively. We assume that the function $g(\cdot)$ is also a measurable value function with the same mapping as $v(\cdot)$. Therefore, for simplicity, we replace $g(a, b, c)$ by $v(a, b, c)$. The function $h(\cdot)$ is defined as a positive value function similar to $v(\cdot)$. Since (A, B) is PI of C, we know that

$$v(a^0, b^0, c^0) \geq v(a^1, b^1, c^0)$$

implies

$$v(a^0, b^0, c^1) \geq v(a^1, b^1, c^1), \qquad \text{for all } c \in C. \tag{6}$$

Proof of the above lemma requires one to ascertain that (A, B) is WDI of C.

Constructs motivating our proof can be illustrated graphically as in Fig. 1. Here a and b are scalar attributes. It is first shown that the condition of (A, B) being WDI of C holds for all c and (a, b) pairs in E_1. Then because a horizontal line $b = b^1$ intercepts indifference line L_1 and regions E_1 and E_2, this allows the WDI condition to hold for all pairs (a, b^1). Now other (a, b) pairs in E_2 are indifferent to the pair (a, b^1). This extends the WDI concept to region E_2. This same procedure is repeated over and over again until all of the attribute space is covered. Then it is shown that all pairs $(a, b) \in (A, B)$ are WDI of $c \in C$ and use of the multiplicative form of value function given by (2) or (3) is justified.

Formal proof of Lemma 1 is now a relatively simple matter. A pair (a, b) in E_1 is defined by

$$E_1 = \{(a, b, c^0) \colon v(a, b, c^0) \leq v(a^0, b^0, c^0)\}, \tag{7}$$

we assume there exists an a^1 such that

$$v(a, b, c^0) = v(a^1, b^0, c^0), \qquad \text{for all } (a, b) \in E_1. \tag{8}$$

Now from (6) and (7), it follows that

$$v(a, b, c) = v(a^1, b^0, c), \qquad \text{for all } c \in C, (a, b) \in E_1. \tag{9}$$

Substituting (9) into (5) results in

$$v(a, b, c) = v(a^0, b^0, c) + h(a^0, b^0, c)v(a^1, b^0, c^0), \quad \text{for all } c \in C, (a, b) \in E_1. \tag{10}$$

We now combine (8) and (10) to eliminate $v(a^1, b^0, c^0)$ and this yields the desired result

$$v(a, b, c) = v(a^0, b^0, c) + h(a^0, b^0, c)v(a, b, c^0), \quad \text{for all } c \in C, (a, b) \in E_1. \tag{11}$$

As (11) shows, the WDI condition is shown for (A, B) worth independent of C in the region E_1. To extend this for all possible (a, b) pairs in space $A \odot B$, we shall next move into space E_2. There, we choose b^1 such that

$$v(a^0, b^0, c^0) < v(a^0, b^1, c^0) < v(a^*, b^0, c^0) \tag{12}$$

Since $(a^0, b^1) \in E_1$, we may replace a by a^0 and b by b^1 in (11) to obtain

$$v(a^0, b^1, c) = v(a^0, b^0, c) + h(a^0, b^0, c)v(a^0, b^1, c^0), \quad \text{for all } c \in C. \tag{13}$$

We rewrite (5) using the levels b^1 and c^0 levels as

$$v(a, b^1, c^0) = v(a^0, b^1, c^0) + h(a^0, b^1, c^0)v(a, b^0, c^0), \quad \text{for all } a \in A. \tag{14}$$

Now we set $b = b^1$ in (11) to obtain

$$v(a, b^1, c) = v(a^0, b^0, c) + h(a^0, b^0, c)v(a, b^1, c^0), \quad \text{for all } (a, b^1) \in E_1. \tag{15}$$

This result can now be combined with (14) to yield

$$\begin{aligned} v(a, b^1, c) &= v(a^0, b^0, c) + h(a^0, b^0, c)[v(a^0, b^1, c^0) \\ &\quad + h(a^0, b^1, c^0)v(a, b^0, c^0)] \\ &= v(a^0, b^1, c) + h(a^0, b^0, c)h(a^0, b^1, c^0)v(a, b^0, c^0), \end{aligned}$$
$$\text{for all } c \in C, (a, b^1) \in E_1. \tag{16}$$

We now use the inequality of (12) to define and restrict b^1. There exists an a with $v(a, b^0, c^0) > v(a^0, b^0, c^0)$ which satisfies (16). We compare (16) to (5) with $b = b^1$ and this shows that

$$h(a^0, b^1, c) = h(a^0, b^0, c)h(a^0, b^1, c^0), \quad \text{for all } c \in C. \tag{17}$$

Substituting (13) and (17) into (5) with $b = b^1$ results in

$$\begin{aligned} v(a, b^1, c) &= v(a^0, b^1, c) + h(a^0, b^1, c)v(a, b^0, c^0) = v(a^0, b^0, c) \\ &= v(a^0, b^0, c) + h(a^0, b^0, c)[v(a^0, b^1 c^0) \\ &\quad + h(a^0, b^1, c^0)v(a, b^0, c^0)]. \end{aligned} \tag{18}$$

Now we combine (18) and (14) to obtain

$$v(a, b^1, c) = v(a^0, b^0, c) + h(a^0, b^0, c)v(a, b^1, c^0). \tag{19}$$

Region E_2 is defined by

$$E_2 = \{(a, b, c^0): v(a^*, b^0, c^0) < v(a, b, c^0) \le v(a^*, b^1, c^0)\}. \tag{20}$$

For any $(a, b) \in E_2$, there exists an a^2 such that

$$v(a, b, c^0) = v(a^2, b^1, c^0), \quad \text{for all } c \in C, (a, b) \in E_2, \tag{21}$$

consequently from (6), it follows that

$$v(a, b, c) = v(a^2, b^1, c), \quad (a, b) \in E_2. \tag{22}$$

Now we evaluate the right side of (22) using (19) to obtain

$$v(a, b, c) = v(a^0, b^0, c) + h(a^0, b^0, c)v(a^2, b^1, c^0), \quad \text{for all } c \in C \tag{23}$$

when combined with (21) the foregoing yields

$$v(a, b, c) = v(a^0, b^0, c) + h(a^0, b^0, c)v(a, b, c^0), \quad (a, b) \in E_2. \tag{24}$$

This shows the desired result that (A, B) is WDI for regions E_1 and E_2. This same process can be and is repeated until the entire attribute space is covered. Additional isopreference lines may need to be inserted to allow overlap of the indifference regions for given attribute levels of the attribute space. This is shown by interaction of L_4', $b^{3'}$, and $b^{4'}$ in Fig. 1. The continuity assumption on the measurable value function v, and the nonsatiation assumption that more is better than less of a desired attribute; or that less is better than more of an undesired attribute, and the assumption that $h(a, b, c)$ is positive allows one to show that (A, B) is WDI of C.

The preceding weakened conditions for establishment of MWDI (and resulting verification of the multiplicative form of value function) ameliorate drawbacks associated with verifying the additive form of value function. Fig. 2 illustrates an assessment procedure to establish MWDI of the attributes. This lessens the time commitment required of the DM and analyst. Also it provides a less compensatory scoring form, because of the product terms that allows for more sensitivity to the attributes. This is essential in certain applications.

IV. Numerical Example

In the application of the approach suggested here to an electronic warfare equipment selection situation, a proposed measurable value function, which can serve as a criterion for a multiobjective optimization approach, can take the form

$$v(x) = f(X_1, X_2, \cdots, X_n) \tag{25}$$

where the system effectiveness attribute vector takes the product form

$$f(X_1) = v_1(X_1) = k_1 X_{1a} + k_2 X_{1b} + k_{12} X_{1a} x_{1b} \tag{26}$$

where X_{1a} is the number of threats covered by alternatives and X_{1b} is the degree of effectiveness produced by alternatives. Assuming that MWDI has been established for the two attributes, the product form above reflects the required sensitivity to alternatives with either an excess or lack of either attribute. Past use of additive value criteria in the electronic warfare area have not been sufficiently able to penalize inferior alternatives. This has led to unjustified confidence in the capability of operating systems as has been pointed out by Peterson, Hays, and O'Connor [7].

To illustrate this we consider two alternatives A and B with the following pertinent characteristics

alternative A:

*8 out of 10 critical threats are affected by the system (score $8/10 = 0.8$),

*3 out of 10 of the critical threats are covered sufficiently to assume them countered (score $3/10 = 0.3$):

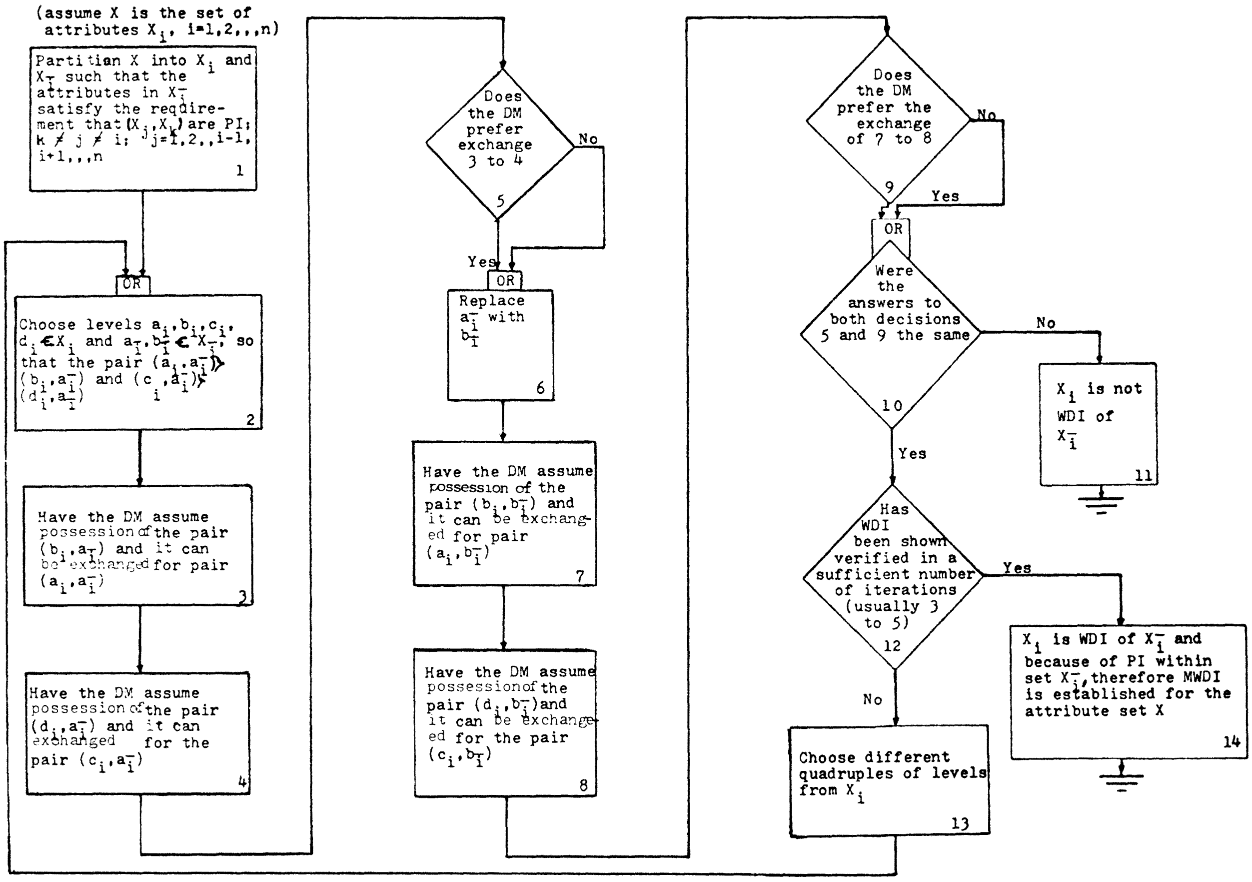

Fig. 2. MWDI assessment procedure.

alternative *B*:

*6 out of 10 critical threats are affected by the system (score $6/10 = 0.6$),

*5 out of 10 critical threats are sufficiently countered (score $5/10 = 0.5$).

Let an additive value function, assuming MPI is also established, of the form

$$v_{11}(X_1) = k_{11}X_{1a} + k_{22}X_{1b} \tag{27}$$

be used as a comparison with the multiplicative form of (26). Assume the scaling constants have been assessed to be $k_{11} = k_{22} = 0.5$ and $k_1 = k_2 = k_{12} = 0.333$ so that v_1 and v_{11} are normalized to equal 1.0 when both X_{1a} and X_{1b} are at their maximum value of 1.0 and $v_1 = v_{11} = 0.0$ when both X_{1a} and X_{1b} are at their minimum value of 0.0. Now inserting the values above (for alternative A; $X_{1a} = 0.8$, $X_{1b} = 0.3$ and for alternative B; $X_{1a} = 0.6$, $X_{1b} = 0.5$), the following results are obtained:

alternative A: $v_{11} = 0.55$ and $v_1 = 0.45$
alternative B: $v_{11} = 0.55$ and $v_1 = 0.47$.

The results show that the additive form of v_{11} evaluate both alternatives identically where as the multiplicative form of v_1 differentiates in favor of alternative *B*. As a matter of interest, all of the DM's polled, preferred alternative *B* which indicates that there is a strong basis for pursuing the multiplicative criteria form in certain application efforts where sensitivity is paramount.

V. Summary

Alternative sufficient conditions which can be used to verify MWDI of attributes for the valid use of the multiplicative form of value function have been obtained. These conditions allow for reduced efforts in this verification process which is of value to decisionmakers and analysts. A summary of a recommended assessment procedure has been presented. The increased sensitivity of the product form of value function makes this form particularly useful in certain applications as is evidenced by the numerical example.

The investigation of other implications of the product form to a DM, such as time, information requirements and alternate decision situation models, are interesting topics for current investigation.

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