ent load components and to predict how likely it is to occur. In the procedure developed here the unavoidable subjective estimations and assumptions are made at a lower level, i.e., at the level of load components. This permits a closer logical control on one's forecastings about the system loading.

REFERENCES

[1] E. D. Farmer, "A method of prediction for non-stationary processes and its application to the problem of load estimation," *Trans. IFAC*, 1963.
[2] W. R. Christiaanse, "Short-term load forcasting using general exponential smoothing," *IEEE Trans. Power App. Syst.*, vol. 90, no. 2, Mar./Apr. 1971.
[3] F. D. Galiana, E. Handschin, and A. R. Fiechter, "Identification of stochastic electric load models from physical data," *IEEE Trans. Automat. Contr.*, vol. 19, no. 9, Decmber 1974.
[4] R. P. Thompson, "Weather sensitive electric demand and energy analysis on a large geographically diverse power system—application to short-term hourly electric demand forcasting," *IEEE Trans. Power App. Syst.*, vol. 95, no. 1, Jan./Feb. 1976.
[5] G. J. Berg, "Power system load representation," *Proc. IEEE*, vol. 120, no. 3, Mar. 1973.
[6] M. H. Kent, W. R. Schmus, F. A. McCrackin, and L. M. Wheeler, "Dynamic modeling of loads in stability studies," in *Stability of Large Electric Power Systems*, R. T. Byerly and E. W. Kimbark, Eds., New York: IEEE Press, 1974, pp. 139-146.
[7] R. B. Adler and C. C. Mosher, "Steady-state voltage power characteristics for power system loads," in *Stability of Large Electric Power Systems*, R. T. Byerly and E. W. Kimbark, Eds. New York: IEEE Press, 1974, pp. 147-153.
[8] E. Economakos, "Distribution system voltage studies using simulation techniques," *Proc. IEE*, vol. 125, no. 2, Feb. 1978.
[9] R. Jain, "Decision making in the presence of fuzzy variables," *IEEE Trans. Syst., Man, Cybern.*, vol. 6, no. 10, Oct. 1976.
[10] L. A. Zadeh, "Outline of a new approach to the analysis of complex systems and decision processes," *IEEE Trans. Syst., Man, Cybern.*, vol. 3, no. 1, Jan. 1973.
[11] W. H. Morgan, G. Hoyland, and M. Tuckett, "Transformer load monitoring from consumer billing information ···," IRE/ERA Conf. on Distribution, Publication no. 73.
[12] R. Juricic, "Estimation des puissances appelées à partir des consomation d'energie en basse tension," Rev. Gen. Elect., tome 80, no. 12, Déc. 1971, pp. 932-934.
[13] A. Kaufmann, *Introduction to the Theory of Fuzzy Subsets*." New York: Academic, 1975, vol. 1.

Symmetric Summation: A Class of Operations on Fuzzy Sets

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***Abstract*—Some natural applications of fuzzy set theory require a type of set combination different from those based on traditional set theory. A class of set combination laws which are symmetric under complementation is proposed.**

I. COMPLEMENTARY SETS

Fuzzy set theory is often used in cases where it is desired to partition a universal set into two subsets. If there is some ambiguity in the partitioning criterion used, definition of the subsets as fuzzy sets is a natural way to avoid dealing with an "excluded middle." In such cases both sets may have comparable significance, so that symmetry exists between the two complementary sets. This suggests that appropriate set operations should be defined in such a way that it does not matter whether we deal with a set or with its complement.

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The combination of two sets is of particular importance in this regard. Suppose, for example, that we wish to divide a set of objects into "good" and "bad" subsets, these being complementary fuzzy sets. We may begin by classifying the objects on the basis of single attributes to obtain various pairs of sets and then combining these sets to obtain a final classification. The rule of combination involves an element of choice; for example, if we decide that a "good" object can have only "good" attributes, then the final "good" set is the intersection of all the single-attribute "good" sets and the "bad" set is the union of all sets having a single "bad" attribute. This rule is clearly not symmetric under complementation. Given the usual definitions of intersection and union [1], the final decision is based entirely on the worst attribute of each object. A more balanced approach would be to look at the entire spectrum of attributes in such a way that a number of fairly good attributes could balance a very bad attribute and vice versa.

II. SET COMBINATION

Let $F_1 \square F_2$ represent the combination of two fuzzy sets F_1 and F_2 under some rule, and let F' be the complement of F. If μ is the membership function for F (the argument of the function representing the object will be suppressed for simplicity), then the membership function for F' is $\mu' = 1 - \mu$ [1]. The requirement that the rule of combination be independent of whether we deal with sets or their complements is equivalent to the condition

$$(F_1 \square F_2)' = F'_1 \square F'_2. \tag{1}$$

If the membership function μ_{12} for the set $F_1 \square F_2$ is given by an equation of the form $\mu_{12} = C(\mu_1, \mu_2)$, then (1) is equivalent to

$$1 - C(\mu_1, \mu_2) = C(1 - \mu_1, 1 - \mu_2), \tag{2}$$

or

$$C(\mu_1, \mu_2) + C(1 - \mu_1, 1 - \mu_2) = 1. \tag{3}$$

The various rules of combination proposed for fuzzy sets have recently been catalogued by Umano *et al.* [2], and it can be verified that none of them always satisfy (3). In particular, we find that

union:

$$\mu_1 \vee \mu_2 + (1 - \mu_1) \vee (1 - \mu_2) = 1, \qquad \text{iff } \mu_1 = \mu_2$$

intersection:

$$\mu_1 \wedge \mu_2 + (1 - \mu_1) \wedge (1 - \mu_2) = 1, \qquad \text{iff } \mu_1 = \mu_2$$

product:

$$\mu_1 \mu_2 + (1 - \mu_1)(1 - \mu_2) = 1, \qquad \text{iff } \mu_1 = \mu_2 = 0 \text{ or } 1$$

algebraic sum:

$$\mu_1 + \mu_2 - \mu_1\mu_2 + (1 - \mu_1) + (1 - \mu_2) - (1 - \mu_1)(1 - \mu_2) = 1, \qquad \text{iff } \mu_1 = \mu_2 = 0 \text{ or } 1$$

bounded sum:

$$1 \wedge (\mu_1 + \mu_2) + 1 \wedge (1 - \mu_1 + 1 - \mu_2) = 1, \qquad \text{iff } \mu_1 = \mu_2 = 0 \text{ or } 1.$$

It can also be seen that for ordinary sets ($\mu = 0$ or 1) these are equivalent to the ordinary union (union, algebraic sum, bounded sum) or intersection (intersection, product). Thus there exists no rule of combination for fuzzy sets that satisfies the symmetry condition represented by (1).

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III. Symmetric Sums

Although symmetric combination rules do not appear to have been developed in the theory of fuzzy sets, it is not difficult to find operations which satisfy the symmetry condition. Given any nonnegative function $g(\mu_1, \mu_2)$, the rule of combination defined by

$$C(\mu_1, \mu_2) = \frac{g(\mu_1, \mu_2)}{g(\mu_1, \mu_2) + g(1 - \mu_1, 1 - \mu_2)} \tag{4}$$

automatically satisfies (1) and thus generates what will be called a symmetric sum (the case of a vanishing denominator will be discussed later). Furthermore, any symmetric sum can be represented in this way since (4) is automatically true if we let $g(\mu_1, \mu_2) = C(\mu_1, \mu_2)$. Therefore any symmetric sum can be represented by a generating function $g(\mu_1, \mu_2)$ through (4), and every generating function defines a symmetric sum.

It is desirable to place restrictions on symmetric summation so as to obtain practical and meaningful results. Only those generating functions will be considered which are nondecreasing functions of their arguments and are symmetric under their interchange; this latter condition insures that symmetric summation is commutative. It then follows from (3) or (4) that $C(\mu, 1 - \mu) = \frac{1}{2}$.

From the original concept of symmetric summation as a rule of combination for fuzzy sets, it is evident that if μ_1 and μ_2 are both equal to zero then μ_{12} should also equal zero; in other words, $C(0, 0) = 0$, and similarly $C(1, 1) = 1$. Both of these conditions are automatically met if we require that $g(0, 0) = 0$. However, a problem arises if $g(0, 1) = g(1, 0) = 0$ since in this case the value of $C(0, 1)$ is not defined by (4). We obtain $C(0, 1) = \frac{1}{2}$ directly from (3) and the condition that symmetric summation be commutative, but this means that the function $C(\mu_1, \mu_2)$ is discontinuous at the points (0, 1) and (1, 0). For example, if we use $g(\mu_1, \mu_2) = \mu_1 \mu_2$ or $g(\mu_1, \mu_2) = \mu_1 \wedge \mu_2$ we get

$$\lim_{\mu \to 0} C(\mu, 1 - \mu) = \tfrac{1}{2}$$

$$\lim_{\mu \to 0} C(\mu, 1) = 1$$

$$\lim_{\mu \to 0} C(0, 1 - \mu) = 0.$$

Because of this it is often useful to restrict the definition of symmetric summation to strictly fuzzy sets, namely those for which the membership function is restricted to the open interval (0, 1). Any function $g(\mu_1, \mu_2)$ which is positive and continuous for all $\mu_1, \mu_2 > 0$ generates a well-defined symmetric sum which is continuous for strictly fuzzy sets, and in the balance of this correspondence only with sums which are continuous in this restricted sense will be considered.

Similarly, a rule of combination can best be considered associative even if it is associative only for strictly fuzzy sets. For example, $g(\mu_1, \mu_2) = \mu_1 \mu_2$ generates an associative sum in this sense, as can be seen by explicit calculation, but

$$C(\mu_1, C(0, 1)) = C(\mu_1, \tfrac{1}{2}) = \mu_1$$

while

$$C(C(\mu_1, 0), 1) = C(0, 1) = \tfrac{1}{2},$$

so the sum is not associative on [0, 1]. It can be seen that this breakdown of associativity always occurs for generating functions such that $g(0, \mu) = 0$ for all μ. For this reason we call a rule of combination associative if

$$C(\mu_1, C(\mu_2, \mu_3)) = C(C(\mu_1, \mu_2), \mu_3) \tag{5}$$

for μ_1, μ_2, and μ_3 in the open interval (0, 1).

It should be pointed out that associativity of the generating function is not sufficient for associativity of the symmetric sum, even in the above restricted sense. For example, $g(\mu_1, \mu_2) = \mu_1 \wedge \mu_2$ is associative, but the corresponding rule of combination is not.

The restriction of associativity to strictly fuzzy sets means that some aspects of the theory of symmetric summation are not applicable to ordinary set theory. This does not represent any loss of generality since the symmetric sum of two ordinary sets is always a fuzzy set (except when the two sets are identical). This follows directly from the commutativity condition, which is sufficient to determine that $C(0, 1) = \frac{1}{2}$. The only real difficulty with the restriction of the theory to strictly fuzzy sets is that the conditions $C(0, 0) = 0$ and $C(1, 1) = 1$ discussed previously are no longer appropriate. These can, however, be replaced by the condition that the limiting value of g as μ_1 and μ_2 approach zero is independent of the approach path and is equal to zero, which will be referred to as the consistency condition. This requires that g be continuous.

To summarize, a proper generating function is defined as one having the following properties: $g(\mu_1, \mu_2)$ is a proper generating function if

1) $g(\mu_1, \mu_2) > 0$
2) $g(\mu_1, \mu_2)$ is continuous
3) $g(\mu_1, \mu_2) = g(\mu_2, \mu_1)$
4) $\lim_{\mu_1 \to 0, \mu_2 \to 0} g(\mu_1, \mu_2) = 0$
5) $g(\mu_1, \mu_2)$ is a nondecreasing function of μ_1 and μ_2 for all $\mu_1, \mu_2 \in (0, 1)$.

These conditions are sufficient to ensure that $C(\mu_1, \mu_2)$ is uniquely defined, is continuous, and that symmetric summation is commutative and obeys the consistency condition.

The definition of symmetric summation leads directly to the result that $C(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$, and the consistency condition ensures that $C(0, 0) = 0$ and $C(1, 1) = 1$ (the appropriate limits can be used when dealing with strictly fuzzy sets). These three equalities may be considered special cases of the relationship $C(\mu, \mu) = \mu$, or $F \square F = F$, which in turn may be considered a necessary but not sufficient condition for the property of stability; a symmetric sum will be considered stable if and only if

$$\mu_1 \wedge \mu_2 \leq C(\mu_1, \mu_2) \leq \mu_1 \vee \mu_2$$

for all $\mu_1, \mu_2 \in (0, 1)$. Surprisingly, there is only one rule of combination which is both stable and associative. This is $C(\mu_1, \mu_2) = \text{med}\,(\mu_1, \mu_2, \frac{1}{2})$, where med means the median of the three values.

Proof: For any stable associative symmetric sum, $C(\mu_1, \mu_2) = \frac{1}{2}$ if $\mu_1 \leq \frac{1}{2} \leq \mu_2$. From (3) we know that $C(\mu, 1 - \mu) = \frac{1}{2}$. But stability requires that $C(\mu, \mu) = \mu$, so

$$C(C(\mu, \mu), 1 - \mu) = C(\mu, C(\mu, 1 - \mu)) = C(\mu, \tfrac{1}{2}) = \tfrac{1}{2}.$$

Thus if we pick μ such that $\mu \leq \mu_1 \leq \frac{1}{2} \leq \mu_2 \leq 1 - \mu$, the requirement that $C(\mu_1, \mu_2)$ be a nondecreasing function of its arguments means that

$$C(\mu, \tfrac{1}{2}) \leq C(\mu_1, \mu_2) \leq C(\tfrac{1}{2}, 1 - \mu),$$

so $C(\mu_1, \mu_2) = \frac{1}{2}$.

Next consider the case $\mu_1 \leq \mu_2 \leq \frac{1}{2}$. Since $C(\mu_1, \mu_1) = \mu_1$ and $C(\mu_1, \frac{1}{2}) = \frac{1}{2}$, it follows from the continuity of C that for any value of $\mu_2 \in [\mu_1, \frac{1}{2}]$ there exists some $\mu \in [\mu_1, \frac{1}{2}]$ such that $C(\mu_1, \mu) = \mu_2$.

Thus

$$\begin{aligned} C(\mu_1, \mu_2) &= C(\mu_1, C(\mu_1, \mu)) \\ &= C(C(\mu_1, \mu_1), \mu) \\ &= C(\mu_1, \mu) = \mu_2 \\ &= \mu_1 \vee \mu_2. \end{aligned}$$

The same result clearly applies if $\mu_2 \leq \mu_1 \leq \frac{1}{2}$, and the corresponding proof for $\mu_1, \mu_2 \geq \frac{1}{2}$ is virtually identical.

Combining these results we see that for a continuous rule of combination which is both stable and associative, the value of $C(\mu_1, \mu_2)$ must be $\frac{1}{2}$ when $\mu_1 \leq \frac{1}{2} \leq \mu_2$ or $\mu_2 \leq \frac{1}{2} \leq \mu_1$, $\mu_1 \vee \mu_2$ when $\mu_1, \mu_2 \leq \frac{1}{2}$, and $\mu_1 \wedge \mu_2$ when $\mu_1, \mu_2 \geq \frac{1}{2}$; this is equivalent to $C(\mu_1, \mu_2) = \text{med}\ (\mu_1, \mu_2, \frac{1}{2})$. Q.E.D.

IV. Examples

The simplest possible generating function is $g =$ constant, which leads to $C(\mu_1, \mu_2) = \frac{1}{2}$ for all values of μ_1 and μ_2. Although this trivial case does not merit lengthy consideration, it is worth noting that it represents a class of generating functions which violate the requirement that $C(0, 0) = 0$ and $C(1, 1) = 1$.

A less trivial but still simple generating function is $g(\mu_1, \mu_2) = \mu_1 + \mu_2$, which leads to $C(\mu_1, \mu_2) = (\mu_1 + \mu_2)/2$. This function is stable but not associative. Other stable nonassociative summation rules are generated by the min and max functions $\mu_1 \wedge \mu_2$ and $\mu_1 \vee \mu_2$, which have the rules of combination

$$C(\mu_1, \mu_2) = \frac{\mu_1 \wedge \mu_2}{1 - |\mu_1 - \mu_2|}$$

and

$$C(\mu_1, \mu_2) = \frac{\mu_1 \vee \mu_2}{1 + |\mu_1 - \mu_2|}.$$

In addition, the generating function $g(\mu_1, \mu_2) = \text{med}\ (\mu_1, \mu_2, \frac{1}{2})$, which is equal to $\mu_1 \wedge \mu_2$ if $\mu_1 \vee \mu_2 \geq \frac{1}{2}$, to $\mu_1 \vee \mu_2$ if $\mu_1 \wedge \mu_2 \geq \frac{1}{2}$, and to $\frac{1}{2}$ if $\mu_1 \wedge \mu_2 \leq \frac{1}{2} \leq \mu_1 \vee \mu_2$, generates a unique stable and associative rule of combination with $C(\mu_1, \mu_2) = g(\mu_1, \mu_2)$.

A large and interesting class of summation rules arises from factorizable generating functions, and a rule for combination will be called factorizable if $g(\mu_1, \mu_2)$ can be written in the form

$$g(\mu_1, \mu_2) = G(\mu_1)G(\mu_2) \tag{6}$$

even though $C(\mu_1, \mu_2)$ itself is not factorizable. For a factorizable generating function we obtain from (4)

$$\begin{aligned} \frac{\mu_{12}}{1 - \mu_{12}} &= \frac{C(\mu_1, \mu_2)}{C(1 - \mu_1, 1 - \mu_2)} = \frac{g(\mu_1, \mu_2)}{g(1 - \mu_1, 1 - \mu_2)} \\ &= \frac{G(\mu_1)}{G(1 - \mu_1)} \frac{G(\mu_2)}{G(1 - \mu_2)} = f\left(\frac{\mu_1}{1 - \mu_1}\right) f\left(\frac{\mu_2}{1 - \mu_2}\right) \end{aligned} \tag{7}$$

where

$$f(x) \equiv G\left(\frac{x}{1 + x}\right) \Big/ G\left(\frac{1}{1 + x}\right) \tag{8}$$

is well-defined so long as g is a proper generating function on strictly fuzzy sets. Thus although the rule for combination for the membership function itself may not be factorizable, the ratio $\mu/(1 - \mu)$, which might be called the inclusion/exclusion ratio, is given by a factorizable expression. The significance of this is evident if we take into consideration the symmetry between a fuzzy set and its complement with which the development of the concept of symmetric summation began, and note that the corresponding equation for the complementary sets is

$$\frac{\mu'_{12}}{1 - \mu'_{12}} = \frac{1 - \mu_{12}}{\mu_{12}} = f\left(\frac{\mu'_1}{1 - \mu'_1}\right) f\left(\frac{\mu'_2}{1 - \mu'_2}\right) \tag{9}$$

where use is made of the identity

$$f(1/x) = 1/f(x) \tag{10}$$

which is a consequence of the defining equation for f, (8). For example, consider the class of symmetric sums defined by $f(x) = x^a$, where a is a positive constant, which correspond to the generating functions $g(\mu_1, \mu_2) = (\mu_1 \mu_2)^a$. If $a = \frac{1}{2}$ the symmetric sum thus defined is stable but not associative, while if $a = 1$ the sum is associative but not stable; this latter case arises naturally when the sets F_1 and F_2 are defined by independent probabilistic events [3]. The corresponding combination rules are

$$C(\mu_1, \mu_2) = \frac{(\mu_1 \mu_2)^a}{(\mu_1 \mu_2)^a + (1 - \mu_1 - \mu_2 + \mu_1 \mu_2)^a} \tag{11}$$

or

$$C(\mu_1, \mu_2) = \frac{\mu_1 \mu_2}{1 - \mu_1 - \mu_2 + 2\mu_1 \mu_2} \tag{12}$$

if $a = 1$.

V. Summary

The symmetric sum of two fuzzy sets has the property that the sum of complements is the complement of the sum. This and other reasonable conditions, such as commutativity and monotonicity, restrict the choice of possible forms for the summation but do not specify it uniquely.

It is also possible to define the symmetric sum to be associative, so $F_1 \square (F_2 \square F_3) = (F_1 \square F_2) \square F_3$, or to be stable so that $F \square F = F$. The properties which are appropriate in a particular case depend on the application, but it is likely that associativity would be required for most applications. There is only one continuous symmetric sum which is both stable and associative.

Perhaps the most interesting feature of the symmetric sum is that it is truly an operator on fuzzy sets and cannot be applied in ordinary set theory, since the symmetric sum of two ordinary sets is a fuzzy set (except for the sum of a set with itself). This suggests that investigation of the properties of fuzzy sets under symmetric summation may lead to results which do not correspond to any in ordinary set theory.

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References

[1] L. A. Zadeh, "Outline of a new approach to the analysis of complex systems and decision processes," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-3, pp. 28–44, Jan. 1973.
[2] M. Umano, M. Mizumoto, and K. Tanaka, "FSTDS system: A fuzzy-set manipulation system," *Inform. Sci.*, vol. 14, pp. 115–159, 1978.
[3] W. Silvert, "Prediction," to be published.