Similarly, the functions f_β, g_β, and h_β are separable so that (B3) decouples into n_β independent subproblems

$$\max_{x_{\beta i}} f_{\beta i}(x_{\beta i}, y_\beta) \text{ subject to } g_{\beta i}(x_{\beta i}, y_\beta) = 0$$
$$h_{\beta i}(x_{\beta i}) = y_{\beta i} \qquad \text{(B5)}$$

when the levels of the interactions y_β are fixed. As in the linear case, the interaction levels in one decomposition are fixed by the values of the decision variables selected in the other decomposition, that is,

$$y_\beta = h_\beta(P_{\beta\alpha}x_\alpha) \text{ and } y_\alpha = h_\alpha(P_{\alpha\beta}x_\beta).$$

The following analysis is in Hilbert spaces. We assume f is a real-valued function on a Hilbert space $\mathscr{X}$ and the functions g, h_α, and h_β are transformations between Hilbert spaces. Also, P_α and P_β are bounded linear transformations and bijections between Hilbert spaces. Direct substitution shows that the problems

$$\max_x f(x) \text{ subject to } g(x) = 0 \text{ and } h_\alpha(P_\alpha x) = y_\alpha^n \qquad \text{(B6)}$$

and

$$\max_x f(x) \text{ subject to } g(x) = 0 \text{ and } h_\beta(P_\beta x) = y_\beta^n \qquad \text{(B7)}$$

are equivalent as defined in Appendix A to the problems (B2) and (B3), respectively, with the interaction levels fixed at $y_\alpha = y_\alpha^n$ and $y_\beta = y_\beta^n$. It should be apparent that Properties 1 and 2 are directly applicable to the nonlinear case.

Property 1': If the strategy (S1)–S4)) is begun with a feasible value of x, that is, $g(x) = 0$, then at each stage n of the iteration process the problems (B6) and (B7) have nonempty feasible domains.

Property 2': Let $f^{\text{opt}} = \max f(x)$ subject to $g(x) = 0$, and suppose the sequences $\{\xi_{\alpha n}\}$ and $\{\xi_{\beta n}\}$ are such that

$$\xi_{\alpha n} \text{ maximizes } f(x) \text{ subject to } T_\alpha(x) = T_\alpha(\xi_{\beta,n-1})$$
$$\xi_{\beta n} \text{ maximizes } f(x) \text{ subject to } T_\beta(x) = T_\beta(\xi_{\alpha n}) \qquad \text{(B8)}$$

for all $n > 0$ where

$$T_\alpha(x) \equiv \begin{pmatrix} g(x) \\ h_\alpha(P_\alpha x) \end{pmatrix} \text{ and } T_\beta(x) \equiv \begin{pmatrix} g(x) \\ h_\beta(P_\beta x) \end{pmatrix},$$

then both conditions i) and ii) of (A3) are valid.

Property 3': Assume f is concave and both f and g are Frechet differentiable and the constraints of both problems (B6) and (B7) are regular. Furthermore, assume

$$\mathscr{N}(dg(x)) = \mathscr{N}(dT_\alpha(x)) + \mathscr{N}(dT_\beta(x)) \qquad \text{(B9)}$$

where $\mathscr{N}(\cdot)$ denotes the null space of $(\cdot)$ and $dg(x)$, $dT_\alpha(x)$, and $dT_\beta(x)$ are the Frechet differentials of g, T_α, and T_β at x. If the sequences $\{\xi_{\alpha n}\}$ and $\{\xi_{\beta n}\}$ satisfy (B8), then any accumulation point ξ^* of either sequence solves (B1).

Proof: Let ξ^* be an accumulation point of $\{\xi_{\alpha n}\}$ where the sequences $\{\xi_{\alpha n}\}$ and $\{\xi_{\beta n}\}$ satisfy (B8). Define the real-valued functions F_α and F_β on $\mathscr{X}$:

$$F_\alpha(x) \equiv \max\{f(x): T_\alpha(x) - T_\alpha(\xi) = 0\}$$
$$F_\beta(x) \equiv \max\{f(x): T_\beta(x) - T_\beta(\xi) = 0\}.$$

Then both F_α and F_β are lower semicontinuous on $\mathscr{X}$, and it follows (see proof of Property 3) that ξ^* maximizes $f(x)$ subject to $T_\alpha(x) - T_\alpha(\xi^*) = 0$ and $T_\beta(x) - T_\beta(\xi^*) = 0$. Consequently, $\nabla f(\xi^*)$ is perpendicular to both the null space of $dT_\alpha(\xi^*)$ and $dT_\beta(\xi^*)$. Thus under the assumption (B9), $\nabla f(\xi^*)$ is perpendicular to the null space of $dg(\xi^*)$. And since f is concave, ξ^* maximizes $f(x)$ subject to $g(x) = 0$. Q.E.D.

Acknowledgment

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Stabilization of Linear Time-Invariant Interconnected Systems Using Local State Feedback

M. EROL SEZER AND ÖZAY HÜSEYİN

Abstract—**The stability of composite systems formed by an arbitrary linear interconnection of linear time-invariant subsystems is investigated. It is shown that the composite system can be made stable using local state feedback with moderate gains around the subsystems (but not from one to another), provided that each subsystem is controllable.**

I. Introduction

Recently there has been a lot of work on the stability of composite systems. Some of these works aim completely at analysis and give necessary and/or sufficient conditions in terms of lower order subsystems and their interconnections for the composite system to be stable (see, for example, [1] and [2]). Some others aim at design and investigate the possibility of stabilizing a composite system using constant or dynamic output feedback or state feedback (see, for example, [3]–[6]). Among these, Davison [5] has shown in an elegant way that an arbitrary interconnection of nonlinear time-varying unknown single-input–single-output systems having a particular structure can be stabilized using high-gain local state feedback. However, as in the case of all nonlinear time-varying problems, his procedure leads to extremely high feedback gains that are naturally much more than necessary for a linear time-invariant system. Except for the above references, [7] and [8] include various studies on the stability and stabilization of interconnected systems.

In this correspondence, the stability of a composite system formed by interconnecting a number of linear time-invariant multivariable systems is considered. It is shown that such a composite system can be stabilized using local state feedback with moderate gains provided that all the subsystems are controllable. Although it is possible to stabilize the composite system using state feedback by considering the whole system as a single controllable multivariable system, it is usually desirable to control each system without requiring any knowledge about the other systems. The procedure offered here not only stabilizes the composite system, but also guarantees system stability under any possible

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The authors are with the Department of Electrical Engineering, Middle East Technical University, Ankara, Turkey.

failure conditions that might occur in the interconnections. Moreover, if a composite system formed by single-input–single-output subsystems is considered and the results are compared with those of Davison [5], it is observed that the procedure of this paper leads to much smaller feedback gains (see example 2), which are just sufficient to stabilize the composite system.

II. Problem Statement

Consider the collection of N subsystems represented by

$$S_i\colon \dot{x}_i = A_i x_i + B_i u_i,$$
$$y_i = C_i x_i, \qquad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i \in \mathscr{R}^{n_i}$ is the state, $u_i \in \mathscr{R}^{m_i}$ is the input, and $y_i \in \mathscr{R}^{r_i}$ is the output of the ith subsystem S_i. The matrices A_i, B_i, and C_i ($i = 1, \cdots, N$) are constant and of appropriate dimensions. Let these subsystems be interconnected according to

$$u_i = v_i + \sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij} y_j, \qquad i = 1, 2, \cdots, N, \tag{2}$$

where $v_i \in \mathscr{R}^{m_i}$ is the external control input to S_i, $G_{ij} \in \mathscr{R}^{m_i \times r_j}$ are the interconnection matrices, and the second term on the right side of (2) represents the aggregate interaction inputs to S_i from other subsystems. The composite system formed by the above interconnection can then be described by

$$\dot{x} = (A + BGC)x + Bv \tag{3}$$

where

$$\begin{aligned} x &= \text{column } [x_1, \cdots, x_N] \\ v &= \text{column } [v_1, \cdots, v_N] \\ A &= \text{block diag } [A_1, \cdots, A_N] \\ B &= \text{block diag } [B_1, \cdots, B_N] \\ C &= \text{block diag } [C_1, \cdots, C_N] \end{aligned}$$

and

$$G = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1N} \\ G_{21} & 0 & \cdots & G_{2N} \\ \vdots & \vdots & & \vdots \\ G_{N1} & G_{N2} & \cdots & 0 \end{bmatrix}.$$

The problem considered in this correspondence is to find local state feedback rules such that if

$$v_i = w_i + F_i x_i \tag{4}$$

where $w_i \in \mathscr{R}^{m_i}$ is the reference input of S_i and $F_i \in \mathscr{R}^{m_i \times n_i}$ is the local feedback matrix, then the resulting composite system

$$\dot{x} = (A + BF + BGC)x + Bw \tag{5}$$

is stable, where

$$w = \text{column } [w_1, \cdots, w_N]$$
$$F = \text{block diag } [F_1, \cdots, F_N].$$

The block diagram of a composite system consisting of $N = 2$ subsystems is shown in Fig. 1 together with the local compensating state feedback matrices.

III. Main Result

Theorem: Let each subsystem S_i described by (1) be controllable. Then it is possible to choose the local state feedback matrices F_i in (4) such that the resulting composite system described by (5) is stable.

To prove the theorem the following lemmas are needed.

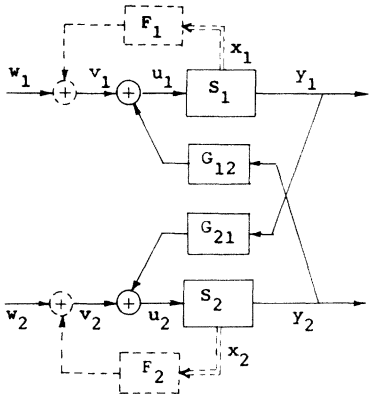

Fig. 1. General composite system consisting of two subsystems with local state feedback compensation.

Lemma 1: Let $a(s)$ and $p(s)$ be monic polynomials where the zeros of $p(s)$ are all in the open left-half complex plane $\mathscr{C}^-$. Let $q(s)$ be any polynomial such that

$$\deg\{a(s)\} + \deg\{p(s)\} > \deg\{q(s)\}.$$

Then the coefficients of $a(s)$ can be chosen so that all the zeros of $h(s) = a(s)p(s) + q(s)$ are in $\mathscr{C}^-$.

Proof: Let

$$\begin{aligned} a(s) &= s^n + a_1 s^{n-1} + \cdots + a_n \\ &= s^n + a_1 \alpha(s) \end{aligned}$$

where $\alpha(s)$ is a monic polynomial with $\deg\{\alpha(s)\} = n - 1$. Then

$$h(s) = s^n p(s) + q(s) + a_1 \alpha(s) p(s).$$

Now consider the case when $a_1 \to \infty$. Since

$$\deg\{s^n p(s) + q(s)\} = \deg\{\alpha(s)p(s)\} + 1,$$

the well-known result of the classical root-locus technique [9] indicates that one zero of $h(s)$ approaches to $-\infty$, while the remaining zeros approach to those of $\alpha(s)p(s)$. Since the zeros of $p(s)$ are in $\mathscr{C}^-$, choosing the zeros of $\alpha(s)$ in $\mathscr{C}^-$ and making a_1 sufficiently large, the result follows.

Lemma 2: The theorem is true for $N = 2$.

Proof: For $N = 2$, the composite system in (5) takes the form

$$\dot{x} \doteq \begin{bmatrix} A_1 + B_1 F_1 & B_1 G_{12} C_2 \\ B_2 G_{21} C_1 & A_2 + B_2 F_2 \end{bmatrix} x + \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix} w. \tag{6}$$

Let F_1 be chosen such that all the eigenvalues of $\hat{A}_1 = A_1 + B_1 F_1$ are in $\mathscr{C}^-$ [10].

Let the coordinate transformation $\tilde{A}_2 = T_2^{-1} A_2 T_2$, $\tilde{B}_2 = T_2^{-1} B_2$, $\tilde{C}_2 = C_2 T_2$ bring the pair $(\tilde{A}_2, \tilde{B}_2)$ into Luenberger canonical form [11], i.e., if $p_{21} \geq p_{22} \geq \cdots \geq p_{2m_2}$ are the controllability indices of the pair (A_2, B_2), then

$$\tilde{A}_2 = \begin{bmatrix} \tilde{A}_{211} & \cdots & \tilde{A}_{21m_2} \\ \vdots & & \vdots \\ \tilde{A}_{2m_21} & \cdots & \tilde{A}_{2m_2m_2} \end{bmatrix}, \qquad \tilde{B}_2 = \begin{bmatrix} \tilde{B}_{21} \\ \vdots \\ \tilde{B}_{2m_2} \end{bmatrix} D_2,$$
$$\tilde{C}_2 = [\tilde{C}_{21} \quad \cdots \quad \tilde{C}_{2m_2}] \tag{7}$$

where

$$\tilde{A}_{2jj} = \begin{bmatrix} 0 \;|\; I \\ \hline -\tilde{a}_{2jj}^T \end{bmatrix} \in \mathscr{R}^{p_{2j} \times p_{2j}}, \tag{8a}$$

$$\tilde{A}_{2jk} = \begin{bmatrix} 0 \\ -\tilde{a}_{2jk}^T \end{bmatrix} \in \mathscr{R}^{p_{2j} \times p_{2k}}, \tag{8b}$$

$$\tilde{B}_{2j} = \begin{bmatrix} 0 \\ e_j^T \end{bmatrix} \in \mathscr{R}^{p_{2j} \times m_2}. \tag{8c}$$

$D_2 \in \mathscr{R}^{m_2 \times m_2}$ is a nonsingular matrix, $\tilde{a}_{2jk}^T$ are row vectors, and e_j^T is the jth row of the unit matrix I_{m_2}. Let

$$F_2 = D_2^{-1}\tilde{F}_2 T_2^{-1} \tag{9}$$

where $\tilde{F}_2$ is partitioned as

$$\tilde{F}_2 = [\tilde{F}_{21} \cdots \tilde{F}_{2m_2}]. \tag{10}$$

Now choose $\tilde{F}_{2k}$ such that

$$\tilde{A}_{2jk} + \tilde{B}_{2j}\tilde{F}_{2k} = 0, \tag{11}$$

i.e., $-\tilde{a}_{2jk}^T + e_j^T\tilde{F}_{2k} = 0$, for all $j \neq k$, $k = 1, \cdots, m_2$. Thus all rows of $\tilde{F}_{2k}$ are fixed except the kth row. Let

$$-\tilde{a}_{2kk}^T + e_k^T\tilde{F}_{2k} = -\hat{a}_{2kk}^T, \qquad k = 1, \cdots, m_2. \tag{12}$$

Then using (7)–(12) in (6), it follows that

$$A + BGC + BF \sim \begin{bmatrix} \hat{A}_1 & B_1G_{12}\tilde{C}_{21} & \cdots & B_1G_{12}\tilde{C}_{2m_2} \\ \tilde{B}_{21}D_2G_{21}C_1 & \hat{A}_{211} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{B}_{2m_2}D_2G_{21}C_1 & 0 & \cdots & \hat{A}_{2m_2m_2} \end{bmatrix} \tag{13}$$

where $\sim$ denotes matrix similarity and the $\hat{A}_{2jj}$ have the same form as $\tilde{A}_{2jj}$ except that the $-\tilde{a}_{2jj}^T$ are replaced by $-\hat{a}_{2jj}^T$, $j = 1, \cdots, m_2$. Now define

$$P_j = \begin{vmatrix} \hat{A}_1 & B_1G_{12}\tilde{C}_{21} & \cdots & B_1G_{12}\tilde{C}_{2j} \\ \tilde{B}_{21}D_2G_{21}C_1 & \hat{A}_{211} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{B}_{2j}D_2G_{21}C_1 & 0 & \cdots & \hat{A}_{2jj} \end{vmatrix}, \qquad j = 1, \cdots, m_2, \tag{14}$$

and

$$d_j(s) = |sI - P_j|, \qquad j = 1, \cdots, m_2. \tag{15}$$

Note that $P_{m_2} \sim A + BGC + BF$. It can be shown using elementary column operations that

$$d_j(s) = \begin{bmatrix} sI - \hat{A}_1 & -B_1G_{12}\tilde{c}_{21}(s) & \cdots & -B_1G_{12}\tilde{c}_{2j}(s) \\ -e_1^TD_2G_{21}C_1 & \hat{a}_{211}(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -e_j^TD_2G_{21}C_1 & 0 & \cdots & \hat{a}_{2jj}(s) \end{bmatrix} \tag{16}$$

where

$$\hat{a}_{2kk}(s) = s^{p_{2k}} + \hat{a}_{2kk}^T \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{p_{2k}-1} \end{bmatrix}, \qquad k = 1, \cdots, m_2,$$

and

$$\tilde{c}_{2k}(s) = \tilde{C}_{2k} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{p_{2k}-1} \end{bmatrix}, \qquad k = 1, \cdots, m_2.$$

It follows from (12) that the coefficients of $\hat{a}_{2kk}(s)$ can be chosen arbitrarily by selecting the kth row of $\tilde{F}_{2k}$ properly. We shall now show by induction that it is possible to choose the coefficients of the $\hat{a}_{2kk}(s)$, $k = 1, \cdots, m_2$, so that each $d_j(s)$, $j = 1, \cdots, m_2$, has zeros all in $\mathscr{C}^-$.

a) For $j = 1$,

$$d_1(s) = \begin{vmatrix} sI - \hat{A}_1 & -B_1G_{12}\tilde{c}_{21}(s) \\ -e_1^TD_2G_{21}C_1 & \hat{a}_{211}(s) \end{vmatrix} = \hat{a}_{211}(s)|sI - \hat{A}_1| + \begin{vmatrix} sI - \hat{A}_1 & -B_1G_{12}\tilde{c}_{21}(s) \\ -e_1^TD_2G_{21}C_1 & 0 \end{vmatrix}. \tag{17}$$

Then the result follows from Lemma 1 on identifying $\hat{a}_{211}(s) = a(s)$, $|sI - \hat{A}_1| = p(s)$, and the second term in (17) as $q(s)$.

b) Assume that the coefficients of $\hat{a}_{2jj}(s)$, $j = 1, \cdots, k - 1$, are chosen such that the zeros of $d_{k-1}(s)$ are all in $\mathscr{C}^-$. Now writing $d_k(s)$ as the sum of $\hat{a}_{2kk}(s)d_{k-1}(s)$ and a lower degree polynomial as in case (i), the proof follows from Lemma 1. This completes the induction step and the proof.

Proof of the Theorem: The proof is given by induction on N.

a) $N = 2$: Lemma 2.

b) Assume that any composite system formed by interconnecting $k - 1$ subsystems can be stabilized using local state feedback in the subsystems. Then a composite system formed by interconnecting k subsystems can be considered as an interconnection of any one of the subsystems with the subcomposite system formed by the remaining $k - 1$ subsystems, and the proof follows from Lemma 2.

Remarks

1) Consider the proof of Lemma 1. Suppose that the coefficients of $q(s)$ take values in finite known intervals, but are not known exactly. In this case, it is still possible to guarantee that the zeros of $h(s)$ are in $\mathscr{C}^-$ for all permissible values of the coefficients of $q(s)$, provided that a_1 is chosen sufficiently large. Since $q(s)$ takes into account the effect of interconnections in the proof of Lemma 2, it follows that the composite system can be stabilized independently of the interconnection gains provided that they are bounded. This ensures that some of the subsystems can safely be disconnected without driving the composite system into instability.

2) If all the subsystems are controllable, then clearly the composite system formed by the interconnection in (2) is also controllable. Hence it is possible to find a state feedback rule that stabilizes the composite system. But in this case, since the feedback matrix becomes dependent on the interconnection matrix G, the above robustness property of the composite system is lost. In fact, with this procedure each isolated subsystem is made stable. This might not be achieved by considering the whole composite system as a single system and using state feedback.

3) Although the proof of Lemma 2 provides an algorithm for the selection of the local feedback matrices, it presents difficulties when the interconnection gains are not known exactly. A separate algorithm based on a trial and error method is given in section IV.

4) Although there seems to be a relation between the above results and the results of Milne [12] on weak coupling, it is not necessary for the eigenvalues of the subsystems to differ in magnitude greatly as in weak coupling.

IV. An Algorithm for the Selection of the Local State Feedback Matrices

A careful inspection of the proofs of Lemmas 1 and 2 shows that the stabilization procedure is based on placing the eigenvalues of the subsystems sufficiently far from the imaginary axis so that the interconnections become ineffective in the compensated system characteristic polynomial. The following algorithm presented uses this fact, which first selects the compensated subsystem eigenvalues and then checks the effect of interconnections. If the interconnections are seen to be effective, a different choice for the compensated subsystem eigenvalues is made, and the procedure is repeated until satisfactory results are obtained.

The algorithm is based on a method developed by Paraskevopoulos for pole assignment using constant output feedback [13], [14]. A slight modification of this method shows that, if the pair (A, B) is in Luenberger controllable canonical form [11] with controllability indices $p_1 \geq p_2 \geq \cdots \geq p_m$ and if there exists a transformation such that

$$Q^{-1}(A + BF)Q = H \tag{18}$$

where H is any matrix, then Q is of the form

$$Q = \begin{bmatrix} q_1^T \\ \vdots \\ q_1^T H^{p_1 - 1} \\ \vdots \\ q_m^T \\ \vdots \\ q_m^T H^{p_m - 1} \end{bmatrix}. \tag{19}$$

Conversely, the state feedback matrix F that makes the matrix $A + BF$ similar to any given matrix H is given by

$$F = (B^T B)^{-1} B^T (QHQ^{-1} - A) \tag{20}$$

where Q is any nonsingular matrix of the form (19).

The Algorithm

1) Using the transformation $\tilde{A}_i = T_i^{-1} A_i T_i$, $\tilde{B}_i = T_i^{-1} B_i$, $\tilde{C}_i = C_i T_i$, $i = 1, \cdots, N$, bring the pair $(\tilde{A}_i, \tilde{B}_i)$ into Luenberger controllable canonical form. Let the controllability indices of the pair $(\tilde{A}_i, \tilde{B}_i)$ be $p_{i1} \geq \cdots \geq p_{im_i}$.

2) Choose

$$H_i = \begin{bmatrix} \lambda_{i1} & & & \\ & \lambda_{i2} & & \\ & & \ddots & \\ & & & \lambda_{in_i} \end{bmatrix}, \qquad i = 1, \cdots, N, \tag{21}$$

where λ_{ij}, $j = 1, \cdots, n_i$, are the eigenvalues of the compensated subsystem S_i, $i = 1, \cdots, N$.

3) Choose $q_{i1}^T, \cdots, q_{im_i}^T \in \mathscr{R}^{1 \times n_i}$, $i = 1, \cdots, N$, such that

$$Q_i = \begin{bmatrix} q_{i1}^T \\ \vdots \\ q_{i1}^T H^{p_{i1} - 1} \\ \vdots \\ q_{im_i}^T \\ \vdots \\ q_{im_i}^T H^{p_{im_i} - 1} \end{bmatrix}$$

are nonsingular.

4) Let $T = \operatorname{diag}[T_1, \cdots, T_N]$ and $Q = \operatorname{diag}[Q_1, \cdots, Q_N]$. Then,

$$\hat{A} = Q^{-1}T^{-1}(A + BGC + BF)TQ = \begin{bmatrix} H_1 & Q_1^{-1}\tilde{B}_1 G_{12} \tilde{C}_2 Q_2 & \cdots & Q_1^{-1}\tilde{B}_1 G_{1N} \tilde{C}_N Q_N \\ Q_2^{-1}\tilde{B}_2 G_{21} \tilde{C}_1 Q_1 & H_2 & \cdots & Q_2^{-1}\tilde{B}_2 G_{2N} \tilde{C}_N Q_N \\ \vdots & \vdots & & \vdots \\ Q_N^{-1}\tilde{B}_N G_{N1} \tilde{C}_1 Q_1 & Q_N^{-1}\tilde{B}_N G_{N2} \tilde{C}_2 Q_2 & \cdots & H_N \end{bmatrix} \tag{22}$$

5) Check if $\hat{A}$ in (22) is diagonal dominant [15] for all permissible values of the interconnection gains. If so, determine the local feedback matrices F_i from

$$\begin{aligned} F_i &= (\tilde{B}_i^T \tilde{B}_i)^{-1} \tilde{B}_i^T (Q_i H_i Q_i^{-1} - \tilde{A}_i) T_i^{-1} \\ &= D_i^{-1} (\bar{Q}_i Q_i^{-1} - \bar{A}_i) T_i^{-1} \end{aligned} \tag{23}$$

where

$$\bar{Q}_i = \begin{bmatrix} q_{i1}^T H_i^{p_{i1}} \\ \vdots \\ q_{im_i}^T H_i^{p_{im_i}} \end{bmatrix}, \qquad \bar{A}_i = \begin{bmatrix} -\tilde{a}_{i11}^T & \cdots & -\tilde{a}_{i1m_i}^T \\ \vdots & & \vdots \\ -\tilde{a}_{im_i1}^T & \cdots & -\tilde{a}_{im_im_i}^T \end{bmatrix} \tag{24}$$

and the $\tilde{a}_{ijk}^T$ are defined in (8a) and (8b) for $i = 2$. If $\hat{A}$ is not dominant, or cannot be made dominant by scaling its rows and columns, choose another set of H_i, and repeat steps 2)–5).

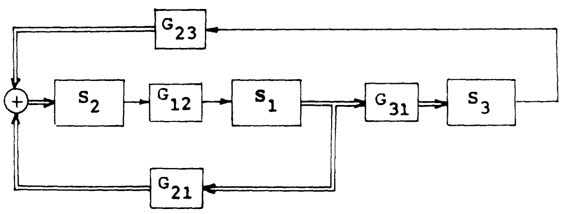

Fig. 2. Composite system of Example 1.

Remarks

A possible choice for H_i and $q_{ij}^T, j = 1, \cdots, m_i$, which guarantees the nonsingularity of Q_i and also simplifies calculations, is

$$H_i = \operatorname{diag}[H_{i1}, \cdots, H_{im_i}] \tag{25}$$

where each H_{ij}, $j = 1, \cdots, m_i$, is diagonal and has distinct eigenvalues, and

$$q_{ij}^T = [0 \ \cdots \ 0 \,\vdots\, \cdots \,\vdots\, \underbrace{1 \ 1 \ \cdots \ 1}_{\substack{j\text{th block having} \\ p_{ij}\ 1\text{'s}}} \,\vdots\, \cdots \,\vdots\, 0 \ \cdots \ 0]. \tag{26}$$

This choice of H_i and $q_{ij}^T, j = 1, \cdots, m_i$, results in a block diagonal Q_i that can be inverted much more easily than a full Q_i.

V. Examples

Example 1

Consider the composite system shown in Fig. 2 where the subsystems are described by

$$S_1: A_1 = \begin{bmatrix} -2 & 0 \\ -2 & -1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$S_2: A_2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -4 & -1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad C_2 = [1 \ \ 0 \ \ 1],$$

$$S_3: A_3 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad B_3 = I_2, \qquad C_3 = [1 \ \ 0],$$

and the nonzero interconnection matrices are given as

$$G_{12} = g_{121}, \qquad G_{21} = \begin{bmatrix} g_{211} & g_{212} \\ g_{213} & g_{214} \end{bmatrix},$$

$$G_{23} = \begin{bmatrix} g_{231} \\ g_{232} \end{bmatrix}, \qquad G_{31} = \begin{bmatrix} g_{311} & g_{312} \\ g_{313} & g_{314} \end{bmatrix}$$

where

$$|g_{ijk}| \leq 1. \tag{27}$$

Letting

$$T_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad T_3 = I_2,$$

the subsystems are transformed into

$$S_1: \tilde{A}_1 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{C}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$S_2: \tilde{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$\tilde{C}_2 = [1 \quad 0 \quad 0]$$

$$S_3: \tilde{A}_3 = A_3, \quad \tilde{B}_3 = B_3, \quad \tilde{C}_3 = C_3.$$

Letting

$$H_1 = \begin{bmatrix} -2 & \\ & -6 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 1 \\ -2 & -6 \end{bmatrix},$$

$$H_2 = \left[\begin{array}{cc|c} -5 & & \\ & -15 & \\ \hline & & -5 \end{array}\right], \quad Q_2 = \left[\begin{array}{cc|c} 1 & 1 & \\ -5 & -15 & \\ \hline & & 1 \end{array}\right],$$

$$H_3 = \left[\begin{array}{c|c} -15 & \\ \hline & -5 \end{array}\right], \quad Q_3 = I_2,$$

the composite system matrix in (22) becomes

$$\hat{A} = \left[\begin{array}{cc|ccc|cc} -2 & & \alpha_{121} & \alpha_{122} & 0 & & \\ & -6 & \alpha_{123} & \alpha_{124} & 0 & & \\ \hline \alpha_{211} & \alpha_{212} & -5 & & & \alpha_{231} & 0 \\ \alpha_{213} & \alpha_{214} & & -15 & & \alpha_{232} & 0 \\ \alpha_{215} & \alpha_{216} & & & -5 & \alpha_{233} & 0 \\ \hline \alpha_{311} & \alpha_{312} & & & & -15 & \\ \alpha_{313} & \alpha_{314} & & & & & -5 \end{array}\right]$$

where (27) guarantees that

$$|\alpha_{12i}| \le 0.25, \qquad i = 1, \cdots, 4,$$

$$|\alpha_{211}|, \ |\alpha_{213}| \le 1.5, \ |\alpha_{212}|, \ |\alpha_{214}| \le 5.1,$$

$$|\alpha_{215}| \le 5, \ |\alpha_{216}| \le 17,$$

$$|\alpha_{231}|, \ |\alpha_{232}| \le 0.3, \ |\alpha_{233}| \le 1,$$

$$|\alpha_{311}|, \ |\alpha_{313}| \le 5, \ |\alpha_{312}|, \ |\alpha_{314}| \le 17.$$

Multiplying 3rd, 5th, 6th, and 7th rows of $\hat{A}$ respectively by $\frac{1}{2}$, ε_1, $\frac{1}{2}$, and ε_2; and corresponding columns by their reciprocals, and letting $\varepsilon_1, \varepsilon_2 \to 0$, it is seen that $\hat{A}$ becomes row dominant.

Now the required local feedback matrices are obtained from (23) as

$$F_1 = [-10 \quad 5],$$

$$F_2 = \begin{bmatrix} -82 & -16 & -74 \\ 4 & 0 & 0 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} -15 & -1 \\ 1 & -3 \end{bmatrix}.$$

Note that robust stabilization is achieved without requiring excessively large local feedback gains.

Example 2

Consider the feedback connected single-loop systems shown in Fig. 3 where

$$S_1: A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c_1 = [1 \quad 1],$$

$$S_2: A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad c_2 = [1 \quad 1 \quad 1],$$

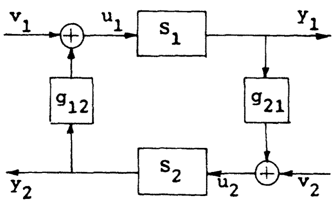

Fig. 3. Composite system of Example 2.

and $|g_{12}|, |g_{21}| \le 1$. Davison [5] assumes feedback matrices of the form

$$F_1 = [\rho^4 \quad 2\rho^2], \qquad F_2 = [\rho^3 \quad 3\rho^2 \quad 3\rho],$$

where a lower bound for the parameter ρ is obtained as $\rho > 486$, for the composite system to be stable. Using the algorithm of Section IV, it can be shown that

$$F_1 = [-3 \quad -4], \qquad F_2 = [-32 \quad -45 \quad -13],$$

which results in

$$H_1 = \begin{bmatrix} -1 & \\ & -4 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -1 & & \\ & -4 & \\ & & -8 \end{bmatrix},$$

are sufficient for stability. Note that the above feedback matrices involve much smaller gains than those obtained using Davison's results. The reason is quite clear: Davison's procedure does not differentiate between linear and nonlinear or time-varying and time-invariant systems and is based on forcing the system under consideration to behave as a model system. To achieve this, he assumes a predetermined form for the feedback gains, which does not depend on the actual system parameters.

Example 3

Consider the hydraulic system shown in Fig. 4, where

- h_j liquid level in the jth tank,
- q_j^i inflow rate to the jth tank,
- q_j^o rate of flow from the jth tank to the next,
- C_{hj} hydraulic capacitance of the jth tank, and
- R_{hj} hydraulic resistance of the jth valve.

The equations describing the dynamic behavior of the tanks can be obtained as

$$C_{hj}\, \Delta\dot{h}_j = \Delta q_j^i + \Delta q_{j-1}^o - \Delta q_j^o, \qquad j = 1, \cdots, N,$$

$$\Delta q_j^o = \frac{1}{R_{hj}} (\Delta h_j - \Delta h_{j+1}), \qquad j = 1, \cdots, N-1,$$

$$\Delta q_o^o = 0, \qquad \Delta q_N^o = \frac{\Delta h_N}{R_{hN}}$$

where Δh_j, Δq_j^i, and Δq_j^o denote incremental deviations from the steady-state values. Letting

$$y_j = x_j = \Delta h_j, \; v_j = \Delta q_j^i, \text{ and } u_j = v_j + \Delta q_{j-1}^o - \Delta q_j^o,$$

the system of Fig. 4 can be modeled as a composite system where the subsystems are described by

$$S_j: \dot{x}_j = \frac{1}{C_{hj}} u_j$$

$$y_j = x_j,$$

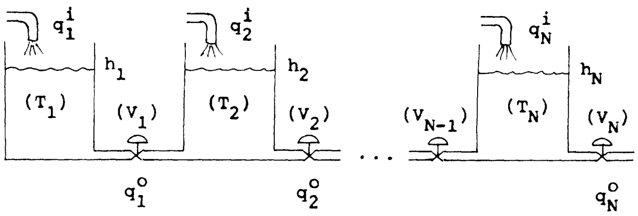

Fig. 4. Hydraulic composite system of Example 3.

and the interconnection gains are given by

$$G_{jj} = -\left(\frac{1}{R_{hj-1}} + \frac{1}{R_{hj}}\right), \qquad j = 2, \cdots, N,$$

$$G_{11} = -\frac{1}{R_{h1}},$$

$$G_{j,j-1} = \frac{1}{R_{h,j-1}}, \qquad j = 2, \cdots, N,$$

$$G_{j,j+1} = \frac{1}{R_{hj}}, \qquad j = 1, \cdots, N-1.$$

Note that the above model is slightly different from the composite system model in (5) in that it involves self-interaction terms G_{jj} that do not appear in (5). This, however, is not important in achieving robust stabilization as can be seen from the algorithm. The significance of G_{jj} terms is that they take into account some of the system parameters that may be changed during operation.

It can be shown that, the above system is always stable for all values of C_{hj} and R_{hj}. However, depending on the relative values of C_{hj} and R_{hj} there is a certain amount of interaction among the subsystems. Our aim is to reduce this interaction using high-gain local state feedback by making the $\hat{A}$ matrix highly dominant.

As a typical example, let $N = 3$ and the system parameters at the operating point be $C_{hj} = 1m^2$, $R_{hj} = 5$ min/m^2, $j = 1, 2, 3$. It can be shown that without any compensation, the steady-state changes in Δh_1, Δh_2, and Δh_3 for a step input applied to S_2 are in proportion to $1:1:\frac{1}{2}$. Letting

$$v_j = w_j - F_j x_j, \qquad F_j = 1 \text{ m}^2/\text{min}, \qquad j = 1, 2, 3,$$

the above ratios become $\frac{1}{6}:1:\frac{1}{7}$, for a step input w_2. Moreover, the above ratios do not change considerably as R_{hj} are changed from 5 min/m^2 to ∞.

VI. Conclusions

It is shown that any composite system formed by interconnecting a number of multivariable systems can be stabilized using local state feedback with relatively small gains. Moreover, proper selection of the feedback matrices guarantees system stability under any possible failure conditions in the interconnection matrices. An algorithm is given for the selection of local state feedback matrices and is applied to three examples. Although the algorithm is based on a trial and error approach, an experienced designer can reach a satisfactory result after few trials. The examples show that robust stabilization of the composite system can be achieved without using excessively high feedback gains.

The model chosen to represent a composite system is suitable for a nontrivial class of composite systems; an example is the hydraulic composite system considered in Example 3. Nevertheless, the method given in this correspondence can be used in stabilizing some other composite system models such as the one considered in [16].

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A Generalized Fuzzy-Set Theory

SATOSI WATANABE, FELLOW, IEEE

Abstract—**In the Zadehian version of the fuzzy-set theory, the exact values of the membership (or grade) functions play a crucial role. In the generalized fuzzy-set theory, the exact values of the membership function, in general, no longer matter much; yet their logical structure remains essentially the same as in the ordinary fuzzy-set theory. The generalized fuzziness must be considered just as undeterminable as in the ordinary fuzzy-set theory.**

I. Introduction

In addition to some other complaints [1], the following three points must be mentioned as shortcomings of the fuzzy-set theory as proposed by Zadeh [2]. 1) There is no way of determining the value of the membership function, either rationally or empirically. 2) In spite of this undecidedness, the implication relation[1] depends on a hair-splitting difference of the values of the two membership functions involved. 3) The minimum–maximum rules

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The author is with Sophia University, Tokyo, Japan, on leave from the University of Hawaii, Honolulu.

[1] It is called "containment" in Zadeh's paper. In his newer paper, another notion of implication is introduced, but this one depends on the so-called fuzzy-set theory of type 2 and therefore requires a separate scrutiny.

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