weights are increased to represent the expected changes in the U.S. population as derived from U.S. Census projections. Income is projected to bring their values to the expected value for the year of interest. The inaccuracies which result from assuming no change in family composition or demographic characteristics of persons are small when projections for the near future are considered. Projections can be based on age, sex, race, or other demographic characteristics available in the survey data. Rates of change of weights are available from U.S. Census projections. For example, to project the number of families with white heads, the weight attached to white head families is projected using numbers derived from the U.S. Census projections; to obtain the amount of wages and salaries that a family or person will receive in 1975, the 1972 reported figure may be multiplied three times by a wages and salaries growth factor (compound rate of growth).

The methodology to project population and income does not alter the work history of the population. Thus the employment rate for the projected data equals the one of the original data base. Different unemployment rates are simulated using Monte Carlo techniques by allowing persons to become unemployed and/or their income and/or weeks worked reduced with probabilities that depend on the work experience, occupation, and income of the individual considered. The probabilities may be derived from historical data available for unemployment rates of different occupations and the individual characteristics of persons unemployed. Alternatively, one can simulate the impact of predetermined growth rates in population, income, or employment.

The data projected to a given year at a certain unemployment rate becomes the basis for the simulation of programs in that year. For example, 1973 data projected to 1975 at an estimated unemployment rate of 8 percent may be used to evaluate programs planned for operation in 1975.

The data file can be updated every year using the newly collected CPS. Growth rates for income and population can be modified according to the latest projections available.

The emphasis of simulation is on the evaluation of alternatives to existing or proposed income maintenance programs that would be implemented relatively soon. When simulation is attempted, the latest available data are likely to be one or two years old, and projections must be made to update it. Aside from sampling and response errors in the survey, short-term projections (less than five years) are quite accurate.

The validity of the simulation depends on the accuracy of the survey data used and the reasonableness of the assumptions that simulate regulations and legislation. The sensitivity of the model to variations in its parameters helps to identify areas where emphasis should be placed to improve the accuracy of the simulation. Known programs can be simulated for particular values of the parameters (population eligible, allowances, and offset tax rates). The simulation should be designed to match the benefits received by each person with control distributions derived from outside sources to the data base, such as administrative statistics on the distribution of payments made by government agencies, or other available surveys.

For example, one can simulate the current Federal income taxes paid by individuals. The results provided by the simulation are compared with the administrative statistics available on distributions of income taxes paid by individuals [6], or one can simulate the public assistance programs and compare the results with the administrative statistics about distributions of payments and demographic characteristics of public assistance recipients available from surveys.

Conclusions

Simulation will eventually become a major tool in policy analysis with a considerable increase in versatility and accuracy to evaluate cost-effectiveness tradeoffs among government programs. The Federal Government has taken over most or all of the state adult programs, and over a period of several years Aid to Families with Dependent Children may move from the State to the Federal Government. A comprehensive review of welfare programs is currently underway, and simulation could play a decisive role for policy making. This paper has shown how to simulate proposed changes in income maintenance legislation in order to compare the alternatives available. Furthermore, all the income transfer programs can be considered simultaneously, instead of on a piecemeal basis as it is currently done. During the process of simulating income transfer regulations, inconsistencies and redundancies are discovered. Therefore, regulations and legislation may be simplified and improved.

References

[1] Report of the Committee on Ways and Means on H.R. 16311. U.S. Government Printing Office, Washington, D.C. 1970.
[2] G. R. Wilensky, "An income transfer simulation model" in *The President's Commission on Income Maintenance Programs, Technical Studies*, U.S. Government Printing Office, 1970.
[3] 1972 Amendments, Chart Booklet: SSI. Pub. No. 43-73 Social Security Administration, DHEW, June 1973.
[4] For additional information, see N. McClung, J. Moeller, and E. Siguel, "Transfer income program evaluation," Working Paper 950-3, The Urban Institute, Washington, D.C., 1971.
[5] The CPS is described in Technical Publication No. 7, U.S. Bureau of the Census, Department of Commerce. The March CPS contains income in addition to demographic data.
[6] Such as the Statistics of Income published every year by the Internal Revenue Service.

Decisionmaking in the Presence of Fuzzy Variables

RAMESH JAIN

***Abstract*—In most complex decisionmaking problems, the states of the system are not known exactly. By assuming that the states are known statistically, one proceeds to find the optimal alternative by using the tools of statistical decision theory. However, in many practical cases the states are known imprecisely, rather than uncertainly, and hence in such cases statistics may not be useful. Moreover, it is not uncommon to know imprecisely the result of an alternative for the given state also. In all such cases, the imprecision may be represented using fuzzy sets. A decision method for systems in which the state of the system and/or the utilities of the alternative actions are known imprecisely is presented. By assuming that these imprecise quantities may be represented using fuzzy sets, a decision procedure is presented which results in the fuzzy set representing an optimal alternative. This set gives us the best alternative and the rating of other alternatives in comparison to the optimal alternative. The computation procedure is illustrated using some examples.**

I. Introduction

In most decisionmaking situations, the state of the system is seldom known exactly. If the state of the system is known exactly, then one may select that alternative action which results in maximum gain, or in other words, which has maximum utility. With inexact knowledge about the state of the system, the choice of the best alternative is not so simple. For handling this type of situation, statistical decision theory has been almost universally in use. The statistical theory assumes that this inexact

Manuscript received October 14, 1975; revised May 7, 1976.
The author is with the Institut für Informatik, Universität Hamburg, 2 Hamburg 13, West Germany, on leave from the Department of Electronics and Electrical Communication Engineering, Indian Institute of Technology, Kharagpur 721302, India.

knowledge about the state of the system may be attributed to uncertainty, and then the optimal decision may be taken using the tools of statistics. In many cases this inexact knowledge about the state of the system is correctly attributed to uncertainty, but there are systems where it is incorrect to consider this inexactness due to uncertainty. This inexactness may be due to impreciseness. Such situations are quite familiar to human beings, who are accustomed to using linguistic variables such as small, very small, medium, very large, etc. These terms give rise to inexactness which is due to ill-defined terms, rather than due to uncertainty. The application of statistical decision theory is not proper in such situations.

The situation is more complex if the utilities associated with various alternatives for a given state are also known imprecisely. This situation is very common in humanistic systems where one uses statements such as: if the state is x_j and alternative a_i is selected, then the utility is very high. In such situations the application of statistical decision theory is also doubtful. However, it is well known that every normal human being is capable of understanding the meaning of such ill-defined terms and of taking proper action if the situation is not too complex. However, if there are many states and many alternatives available, then some mechanized or systematic approach to decision making is needed. This paper presents such an approach.

Zadeh introduced the concept of fuzzy sets [1] for the representation of ill-defined terms. Since then, this concept has been applied to automata theory, system theory, decision theory, switching theory, pattern recognition, and to many other fields [2]–[15]. Bellman and Zadeh [3] presented a method for multistage decisionmaking, using the concept of goals and constraints, for finite-state fuzzy systems. This was extended to infinite state systems by Gluss [9]. A more general method for decision making, though only qualitative in nature, has also been proposed by Zadeh [4]. However, none of these approaches is applicable to the problems stated above having either fuzzy knowledge about the system state, or fuzzy utilities associated with various alternatives, or both.

A decision method is presented here for such fuzzy decision situations for three possible situations: a) knowledge about the system state is fuzzy, with nonfuzzy utilities, b) fuzzy utilities for a nonfuzzy state, and c) fuzzy knowledge about the state with fuzzy utilities. Any of these types of fuzziness results in fuzzy utilities associated with each alternative. The decision about the best alternative is taken on the basis of the fuzzy utility associated with various alternatives. The proposed method uses the concept of maximizing sets [13] for assigning the final grades of membership to utilities associated with an alternative. The fuzzy set representing the optimal alternative is then determined using some simple operations. This set gives the grade of membership of each alternative on the basis of the known fuzzy or nonfuzzy state and the values of the fuzzy or nonfuzzy utilities associated with the alternatives. This set of optimal alternative gives not only the information about the best alternative, but also tells us the relative merit of all alternatives.

In this paper, a fuzzy set A^f will be denoted by

$$A^f = \{(f_A(a_1),a_1),(f_A(a_2),a_2),\cdots,(f_A(a_n),a_n)\} \tag{1}$$

where $f_A(a_i)$ represents the grade of membership of a_i in A^f. We consider f_A to represent the membership function $f_A\colon U \to [0,1]$, where U is the universe of discourse. The set $S(A) \subseteq U$ which is given by

$$S(A) = \{a_1,\cdots,a_i,\cdots,a_n\} \tag{2}$$

is called the support of fuzzy set A^f and is the set of points in U at which $f_A(a_i)$ is positive.

It should be mentioned that in this paper the support of fuzzy sets is always taken to be a set, not an interval. In other words, it is assumed that fuzzy variables can take only discrete values.

II. The Problem Formulation

Let the system under consideration have n states, which are governed by some parameter setting. In this paper by a state of the system we shall mean that parameter setting which results in the state under consideration. Thus the set X, which is

$$X = \{x_1,x_2,\cdots,x_j,\cdots,x_n\} \tag{3}$$

actually gives the parameter values, but it may be considered as representing the set of states of the system:

$$A = \{a_1,a_2,\cdots,a_i,\cdots,a_m\}. \tag{4}$$

It is known that if the alternative a_i is selected and the state of the system is x_j, then the utility of the system is u_{ij}. For the system having n states and m available alternatives, the utilities of the system for various alternatives are given by an $m \times n$ matrix U, which is

$$U = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ u_{m1} & & u_{mn} \end{bmatrix}. \tag{5}$$

If the state of the system is known to be $x_j \in X$, then the problem of selecting the best alternative is reduced to the problem of finding the alternative having highest utility for this state. Thus we have to find the utility

$$u_0 = \bigvee_{i=1}^{m} u_{ij} \tag{6}$$

where $\vee$ is the max operator.

The alternative which gives this utility is then selected as the optimal alternative. However, if either the knowledge about the system state is fuzzy, or the utilities associated with alternatives are known only fuzzily, then this simple procedure cannot be applied. In this case no alternative may be considered as the best alternative. Since various alternatives are known fuzzily, our aim should be to find a fuzzy set

$$A_o^f = \{(f_{Ao}(a_i),a_i)\} \tag{7}$$

representing the optimal alternative. The set A_o^f should be so formed that the relative merit of an alternative is reflected by the grade of membership of this alternative in this set.

In the following sections, methods are presented for the determination of the set A_o^f, when either the knowledge about the state, the utilities associated with alternatives, or both are fuzzy.

III. Fuzzy Knowledge About the State of the System

Let the state of the system be

$$X^f = \{(f_X(x_k),x_k)\}, \qquad x_k \in X \tag{8}$$

With this fuzzy knowledge about the state of the system, the utility associated with each alternative cannot be determined exactly. However, the information about the state of the system may be utilized in determining the fuzzy utility associated with the alternative $a_i \in A$. This utility is given by

$$U_i^f = \{(f_{Ui}(u_k),u_k)\} \tag{9}$$

where

$$u_k = u_{ik} \tag{10}$$

and

$$f_{Ui}(u_k) = f_X(x_k). \tag{11}$$

The decisionmaker is now faced with the problem of selecting an optimal alternative on the basis of the fuzzy utilities associated with each strategy. From the set U_i^f, one may be tempted to make a decision either on the basis of the maximum utility associated with alternatives or on the basis of the utilities having the maximum grade of membership in the sets U_i^fs. However, both of these may lead to the selection of the improper alternative as the optimal alternative. If the optimal alternative is selected on the basis of the maximum utility associated with alternatives, then it may happen that the grade of membership of the maximum utility of the optimal alternative a_o, on the basis of which a_o is determined to be optimal, may be very small, and the other utilities having higher grades of membership may be very small in value. This may result in a very bad choice, since the optimal alternative may result in very poor utility. On the other hand, if the optimal alternative is selected on the basis of the utilities having the highest grades of membership, then effectively we are considering the state known nonfuzzily, to be that state which has the highest grade of membership in the set X^f. This has the defect of neglecting all other states of X^f, some of which may have a grade of membership comparable to the highest grade.

The balanced approach for the selection of the optimal alternative should consider both the maximum utility associated with various alternatives and the grade of membership of the utilities, which are same as the grade of membership of the states in X^f. This is accomplished by utilizing the concept of maximizing sets. The maximizing set $M(f)$ for a function f on Y is a fuzzy subset of Y such that the grade of membership of a point y in $M(f)$ represents the degree to which $f(y)$ approximates to sup f in some specified sense [13]. This definition of the maximizing set is for a function f. For the present purpose we define the maximizing set for the given set. We define the maximizing set $M(Y)$ of a set Y as a fuzzy set such that the grade of membership of a point $y \in Y$ in $M(Y)$ represents the degree to which y approximates to sup Y in some specified sense.

In order to use this concept, first a set Y is formed, which gives all possible utility values resulting for the given fuzzy state. The set Y will be

$$Y = \bigcup_{i=1}^{m} S(U_i). \tag{12}$$

Now we determine the maximizing set for alternative $a_i \in A$, which is the fuzzy set representing the membership of $S(U_i)$ in the maximizing set corresponding to Y. This maximizing set for the alternative a_i is denoted by U_{im}^f and is

$$U_{im}^f = \{(f_{U_{im}}(u_k),u_k)\} \tag{13}$$

where

$$f_{U_{im}}(u_k) = [u_k/u_{\max}]^n \tag{14}$$

and

$$u_{\max} = \sup Y. \tag{15}$$

n is an integer and may be selected depending on the applications.

Now a fuzzy set U_{i0}^f is formed by combining the information available in the sets U_{im}^f and U_i^f. This set is formed by considering the fuzzy intersection of U_{im}^f and U_m^f, and hence is characterized by the grade of membership

$$f_{U_{i0}}(u_k) = f_{U_{im}}(u_k) \wedge f_{U_i}(u_k) \tag{16}$$

where $\wedge$ denotes the min operator.

Thus, in forming the set U_{i0}^f, the information about the state of the system has been considered in assigning the grade of membership to each utility value u_k. At the same time the relative value of each utility is also considered in assigning this grade of membership. The set U_{i0}^f may now be considered as representing the grades of membership of each utility resulting when alternative a_i is selected. The grade of membership of this alternative a_i, in the fuzzy set A_o^f representing the fuzzy optimal alternative, may now be taken as

$$f_{A0}(a_i) = \bigvee_k f_{Ui0}(u_k). \tag{17}$$

In other words, a_i is compared with other alternatives on the basis of the maximum value of the grade of membership in the set U_{i0}^f. It should be noted that by this selection criterion, the alternatives are judged on the basis of that utility associated with them which has fairly high value and also corresponds to the state having a fairly high grade of membership.

Once the grade of membership $f_{A_o}(a_i)$ of each $a_i \in A$ is determined, the fuzzy set

$$A_o^f = \{(f_{A_o}(a_i),a_i)\} \tag{18}$$

gives a clear idea about the optimality of each alternative. The optimal alternative a_o is the alternative having the highest grade of membership in the set A_o^f. Formally, the optimal alternative a_o is that alternative which has

$$f_{A_o}(a_o) = \bigvee_i f_{A_o}(a_i). \tag{19}$$

It should be noted that the set of A_o^f gives a clear picture of the effectiveness of each alternative for the given state. If this set is normalized by dividing each grade of membership by $f_{Ao}(a_o)$, then the grade of membership of an alternative gives a clear idea about its closeness to the optimal alternative. This information may be very useful in many practical cases.

Before considering an example to illustrate the computation procedure, let us consider a special situation. Sometimes it may happen that during computation a certain element U_p appears in a fuzzy set more than once with the same or different grades of membership. The grade of membership of such an element should be determined on the basis of all its appearances. This may be done using the method [12] given in brief below.

If U_p appears K times with grades of membership $f_1, f_2, \cdots, f_k$, then the grade of membership of U_p should be f, given by

$$f = f_1 \oplus f_2 \oplus \cdots \oplus f_k \tag{20}$$

where

$$f_1 \oplus f_2 = f_1 + f_2 - f_1 \cdot f_2. \tag{21}$$

It should be noted that this method considers only equality of elements. No importance has been given to the distances between element values. This is because the variables are considered as taking discrete values. If the variables are allowed to take continuous values, then these variables will be represented by membership functions. The computation method should be modified for such cases. However, in this paper we shall consider only discrete variables.

Example 1: In a system, let the set of alternatives be

$$A = \{a_1,a_2,a_3\}.$$

A parameter of the system may have any setting from set X, which is

$$X = \{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}\}.$$

This parameter setting represents the state of the system. The utility matrix for this system is

$$U = \begin{bmatrix} 9 & 7 & 2 & 2 & 3 & 1 & 7 & 8 & 8 & 4 \\ 2 & 1 & 7 & 8 & 1 & 7 & 6 & 4 & 3 & 8 \\ 6 & 4 & 3 & 4 & 5 & 6 & 8 & 5 & 2 & 3 \end{bmatrix}.$$

The state of the system is known to be

$$X^f = \{(0.4,x_3),(0.8,x_4),(1.0,x_5),(0.7,x_6),(0.3,x_7)\}.$$

For determination of the set of optimal alternatives, the first step is to determine fuzzy utilities associated with each alternative. For the known state these are

$$U_1{}^f = \{(0.4,2),(0.8,2),(1.0,3),(0.7,1),(0.3,7)\}$$
$$= \{(0.88,2),(1.0,3),(0.7,1),(0.3,7)\}$$
$$U_2{}^f = \{(0.4,7),(0.8,8),(1.0,1),(0.7,7),(0.3,6)\}$$
$$= \{(0.82,7),(0.8,8),(1.0,1),(0.3,6)\}$$
$$U_3{}^f = \{(0.4,3),(0.8,4),(1.0,5),(0.7,6),(0.3,8)\}.$$

The set Y is now determined:

$$Y = S(U) \cup S(U_2) \cup S(U_3)$$
$$= \{2,3,1,7\} \cup \{7,8,1,6\} \cup \{3,8,4,6,5\}$$
$$= \{1,2,3,4,5,6,7,8\}.$$

Choosing $n = 1$ in (14), the maximizing sets for various alternatives are determined. These are

$$U_{1m}{}^f = \{(0.25,2),(0.375,3),(0.125,1),(0.875,7)\}$$
$$U_{2m}{}^f = \{(0.875,7),(1,8),(0.125,1),(0.75,6)\}$$
$$U_{3m}{}^f = \{(0.375,3),(0.5,4),(0.625,5),(0.75,6),(1.8)\}$$

Using (16), the following sets are obtained:

$$U_{1o}{}^f = \{(0.25,2),(0.375,3),(0.125,1),(0.3,7)\}$$
$$U_{2o}{}^f = \{(0.82,7),(0.8,8),(0.125,1),(0.3,6)\}$$
$$U_{3o}{}^f = \{(0.375,3),(0.5,8),(0.625,5),(0.7,6),(0.3,8).$$

This results in

$$f_{A_o}(a_1) = \vee\ (0.25,0.375,0.125,0.3) = 0.375.$$

Similarly,

$$f_{A_o}(a_2) = 0.82$$
$$f_{A_o}(a_3) = 0.7.$$

Hence the set of optimal alternatives is

$$A_o{}^f = \{(0.375,a_1),(0.82,a_2),(0.7,a_3)\}.$$

The best alternative is thus a_2, having the highest grade of membership in this set.

IV. Fuzzy Utilities

Now let us consider that the utility U_{ij} associated with the alternative a_i for the state x_j is fuzzy and is given by

$$u_{ij}{}^f = \{(f_{U_{ij}}(u_k),u_k)\}. \tag{22}$$

The utility matrix is, in this case,

$$U = \begin{bmatrix} U_{11}{}^f & \cdots & U_{1n}{}^f \\ \vdots & & \vdots \\ U_{m1} & \cdots & U_{mn}{}^f \end{bmatrix}. \tag{23}$$

If the state of the system is known to be $x_j \in X$, then the selection procedure for the optimal alternative is as discussed in the previous section. From the fuzzy knowledge about the state of the system, the fuzzy utility associated with the alternative $a_i \in A$ was obtained, and it is given by the set $U_i{}^f$. This set $U_i{}^f$ was then utilized for the determination of the grade of membership of this alternative in the optimal alternative $A_o{}^f$. In the present case of a known nonfuzzy state x_j with fuzzy utilities, $U_i{}^f$ is

$$U_i{}^f = U_{ij}{}^f. \tag{24}$$

Starting with this set, the fuzzy optimal alternative $A_o{}^f$ may be obtained.

Example 2: Now let us consider that in this example the set of alternatives A and the set of states X are same as in Example 1, but the utility matrix is given as

$$U = \begin{bmatrix} VH & H & L & L & M & VL & H & VH & VH & M \\ L & VL & H & VH & VL & H & H & M & M & VH \\ H & M & L & M & M & H & VH & M & L & L \end{bmatrix}$$

where L, M, H, VL and VH stand for low, medium, high, very low and very high, respectively. The values assigned to these linguistic variables are fuzzy and are given by the following fuzzy sets:

$$\text{low} = \{(0.4,1),(1.0,2),(0.5,3)\}$$
$$\text{very low} = \{(1.0,1),(0.4,2)\}$$
$$\text{medium} = \{(0.4,3),(0.7,4),(1.0,5),(0.7,6),(0.4,7)\}$$
$$\text{high} = \{(0.5,7),(1.0,8),(0.5,9)\}$$
$$\text{very high} = \{(0.5,9),(1.0,10)\}.$$

In this example, the state of the system is known to be x_9. Thus, using (22), the fuzzy utilities of various alternatives are

$$U_1{}^f = \text{very high} = \{(0.5,9),(1.0,10)\}$$
$$U_2{}^f = \text{medium} = \{(0.4,3),(0.7,4),(1.0,5),(0.7,6),(0.4,7)\}$$
$$U_3{}^f = \text{low} = \{(0.4,3),(1.0,2),(0.5,3)\}.$$

As is obvious in this example, the best alternative is a_1, which has very high utility. The fuzzy optimal alternative in this case is

$$A_o{}^f = \{(1.0,a_1),(0.6,a_2),(0.3,a_3)\}$$

It is clear that the relative ratings assigned to the alternatives are sound according to the judgment of any human being.

V. Fuzzy State—Fuzzy Utility

In this section, we consider the situation in which the utility matrix of the system is fuzzy, as given by (23), and the knowledge about the state of the system is also fuzzy, as given by (8). By using the fuzzy knowledge about the state of the system, the utility associated with the alternative a_i, denoted by $U_i{}^{f*}$, may be obtained. This is given by

$$U_i{}^{f*} = \{(f_{U_i}*(U_k{}^f),U_k{}^f\}, \tag{25}$$

where

$$U_k{}^f = U_{ik}{}^f \tag{26}$$

and

$$f_{Ui}*(U_k{}^f) = f_X(x_k). \tag{27}$$

It should be noted that the fuzzy set $U_i{}^{f*}$ as given in (25) is the set of fuzzy sets. This set should be reduced to a fuzzy set which gives grades of membership of nonfuzzy utilities associated with the alternative a_i. In reducing U_i^{f*}, we should consider the grade of membership of set U_k^f in it and modify the grades of membership of each element of U_k^f. An element of the set U_i^f is

$$(f_{U_i}*(U_k^f),U_k^f) = (f_{U_i}*(f_{U_{ik}}(u_1),u_1)(f_{U_{ik}}(u_1),u_1)). \tag{28}$$

Our aim is to determine the grade of membership of various nonfuzzy utilities u_1 in the set U_i^{f*}. The grade of membership of u_1 should be determined on the basis of its membership in U_{ik}^f and the membership of U_{ik}^f in U_i^{f*}. Thus

$$f_{U_i}{}^{k*}(u_1) = f_X(x_k) \wedge f_{U_{ik}}(u_1). \tag{29}$$

In order to determine the fuzzy utilities associated with a_i, the above procedure is to be repeated for all values of k for which $x_k \in S(X^f)$. The resulting set U_i^{f*} now contains utility values u_1 and their grades of membership. Sometimes it may happen that a certain element u_p may appear more than once in this set, with the same or different grades of membership. The grade of membership of such an element u_p should be determined on the basis of all its appearances due to different values of k, using (20) and (21).

After using the above procedure, the set U_i^{f*}, which was originally a fuzzy set of fuzzy sets has now been reduced to fuzzy set U_{ir}^{f*}, given by

$$U_{ir}^{f*} = (f_{U_{ir}}*(u_1),u_1). \tag{30}$$

The set U_{ir}^{f*} now contains various utilities and their grades of membership for the alternative a_i. Thus we may define

$$U_i^f = U_{ir}^{f*}, \tag{31}$$

and then, starting with this set, the fuzzy optimal alternative A_o^f may be obtained.

Example 3: In this example we consider the case when the state of the system of example 2 is fuzzy and is the same as that of example 1. For this case, we obtain

$$\begin{aligned} U_1^{f*} &= \{(0.4,L),(0.8,L),(1.0,M),(0.7,VL),(0.3,H)\} \\ &= \{(0.88,L),(1.0,M),(0.7,VL),(0.3,H)\} \\ U_2^{f*} &= \{(0.4,H),(0.8,VH),(1.0,VL),(0.7,H),(0.3,H)\} \\ &= \{(0.864,H),(0.8,VH),(1.0,VL)\} \\ U_3^{f*} &= \{(0.4,L),(0.8,M),(1.0,M),(0.7,H),(0.3,VH)\} \\ &= \{(0.4,L),(1.0,M),(0.7,H),(0.3,VH)\}. \end{aligned}$$

Let us consider the set U_1^{f*}. On substituting the values of L, M, VL, and H, we obtain

$$\begin{aligned} U_1^{f*} = \{&(0.88,[(0.4,1),(1.0,2),(0.5,3)]),(1.0,[(0.4,3),(0.7,4),\\ &(1.0,5),(0.7,6),(0.4,7)]),(0.7,[(1.0,1),(0.4,2)]),\\ &(0.3,[0.5,7),(1.0,8),(0.5,9)])\}. \end{aligned}$$

Using (27), this is simplified to

$$\begin{aligned} U_1^{f*} &= \{(0.4,1),(0.88,2),(0.5,3),(0.4,3),(0.7,4),(1.0,5),(0.7,6),\\ &\quad (0.4,7),(0.7,1),(0.4,2),(0.3,7),(0.3,8),(0.3,9)\} \\ &= \{(0.4,1),(0.88 \oplus 0.4,2),(0.5 \oplus 0.4,3),(0.7,4),(1.0,5),\\ &\quad (0.7,6),(0.4 \oplus 0.3,7),(0.3,8),(0.3,9)\} \\ &= \{(0.4,1),(0.928,2),(0.7,3),(0.7,4)(1.0,5),(0.7,6),\\ &\quad (0.58,7),(0.3,8),(0.3,9)\end{aligned}$$

Similarly,

$$\begin{aligned} U_2^{f*} &= \{(1.0,1),(0.4,2),(0.5,7),(0.864,8),(0.75,9),(0.8,10)\} \\ U_3^{f*} &= \{(0.4,1),(0.4,2),(0.64,3),(0.7,4),(1.0.5),(0.7,6),\\ &\qquad (0.7,7),(0.7,8),(0.65,9),(0.3,10)\}. \end{aligned}$$

Using these as the alternatives, the fuzzy set of optimal alternatives may be derived. This set is

$$A_o^f = \{(0.6,a_1),(0.8,a_2),(0.7,a_3)\}.$$

Thus the best alternative is a_2, which has highest grade of membership in this set.

VI. Discussion

In this paper a method is presented for decisionmaking in fuzzy situations by representing the ill-defined quantity as a fuzzy set. This method may be useful in selecting the best alternative when the state of the system and/or utilities associated with alternatives are known in terms of a linguistic variable. Although in this paper such ill-defined terms are considered to be represented as a fuzzy set, the representation of linguistic variables derived from these alomic terms may be obtained [4] and then may be used to represent the state and/or utility. Thus, if the representation of only high, medium, and low is available, then very high, very very high, not high, not low, and not medium may be deduced from these and may be used for the selection of the best alternative.

The method presented here selects that alternative which has the best compromise in the value of the utility and its grade of membership. This is analogous to the method employed by many human beings. Another advantage of the method is that the fuzzy set representing the optimal alternative gives information about the relative merits of all alternatives. This may be desirable in many situations where one may be willing to use some suboptimal alternative, due to constraints imposed by some practical considerations, if it is not very poor compared to the optimal alternative.

In this paper we have considered all variables as taking discrete values and hence representable using fuzzy sets. If we consider variables as taking continuous values, then membership functions should be used. The method presented here may be extended to cover such a situation.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[2] ——, "Fuzzy algorithms," *Information and Control*. vol. 12, pp. 94–102, 1968.

[3] R. E. Bellman and L. A. Zadeh, "Decision making in a fuzzy environment," *Mgmt. Science*, vol. 17, no. 4, B-141–B-164, 1970.

[4] L. A. Zadeh, "Outline of a new approach to the analysis of complex systems and decision processes," *IEEE Trans. Systems, Man, and Cybernetics*, vol. SMC-3, no. 1, pp. 28–44, 1973.

[5] S. S. L. Chang and L. A. Zadeh, "Fuzzy mapping and control," *IEEE Trans. Systems, Man, and Cybernetics*, vol. SMC-2, no. 1, pp. 30–35, 1972.

[6] R. E. Bellman, R. Kalaba, and L. A. Zadeh, "Abstraction and pattern classification," *J. Math., Anal., Appl.*, vol. 13, pp. 1–7, 1966.

[7] S. K. Chang, "On the execution of fuzzy programs using finite state machines," *IEEE Trans. Computers*, vol. C-2, no. 3, pp. 241–253, 1972.

[8] A. D. Allen, "A Method for evaluating technical journals on the basis of published comments through fuzzy implications," *IEEE Trans. Systems, Man, and Cybernetics*, vol. SMC-3, no. 4, pp. 422–425, 1973.

[9] B. Gluss, "Fuzzy multistage decision making and terminal regulators and their relationship to nonfuzzy quadratic state and terminal regulators," *Int. J. Control*, vol. 17, no. 1, pp. 177–192, 1973.

[10] S. Tamura, S. Niguchi, and K. Tanaka, "Pattern classification based on fuzzy relation," *IEEE Trans. Systems, Man, and Cybernetics*, vol. SMC-1, no. 1, pp. 61–66, 1971.

[11] A. Kandel, "On the properties of fuzzy switching functions," *J. of Cybernetics*, vol. 4, no. 1, pp. 119–126, 1947.

[12] R. Jain, "Outline of an approach for the analysis of fuzzy systems," *Int. J. Control*, to be published.
[13] L. A. Zadeh, "On fuzzy algorithms," Memo. No. ERL-M325, Electronics Research Laboratory, University of California, 1972.
[14] E. H. Mamdani and S. Assilian, "An experiment in linguistic synthesis with a fuzzy logic controller," *Int. J. Man-Machine Studies*, vol. 7, pp. 1–13, 1975.
[15] J. C. Dunn, "A graph theoretic analysis of pattern classification via Tamura's fuzzy relation," *IEEE Trans. Systems, Man, and Cybernetics*, SMC-3, pp. 310–313, 1974.

Fourier Texture Features: Suppression of Aperture Effects

CHARLES R. DYER AND AZRIEL ROSENFELD, FELLOW, IEEE

Abstract—**When texture features based on the discrete Fourier power spectrum are used for pattern classification, their performance has been found to be poorer than that of features based on space-domain gray-level statistics. This may be due in part to the presence of aperture effects in the spectrum, resulting from the fact that the discrete Fourier transform treats the input as though it were periodic. As a check on this, a method of removing the aperture effects, described by W. B. Schaming of RCA, was used; but it was found that the features computed on the resulting spectra still did not perform as well as space-domain features.**

In [1], various texture features were computed for a set of 180 samples of a Band 6 LANDSAT image of Kentucky, representing three geological terrain types: Mississipian limestone and shale, Lower Pennsylvanian shale, and Pennsylvanian sandstone and shale. One feature set was derived from the discrete Fourier power spectra of the samples (see below); the others were sets of space-domain features (see [1] for their exact definitions). It was found that the Fourier-based features did not do as well as the space-domain features in classifying the samples using a Fisher linear discriminant program (see [1]).

The specific Fourier-based features used were defined as follows. Let $f(x,y)$ be the given digital picture; then the discrete Fourier transform of f is defined by

$$F(u,v) = \frac{1}{n^2} \sum_{x,y=0}^{n-1} f(x,y) e^{-2\pi j(ux+vy)/n}$$

where f and F are n-by-n arrays ($n = 64$ in our studies). The Fourier features were of the form

$$\sum_{\substack{r_1^2 \le u^2+v^2 < r_2^2 \\ \theta_1 \le \tan^{-1}(v/u) < \theta_2}} |F(u,v)|^2$$

for the following values of $r_1, r_2, \theta_1, \theta_2$:

r_1	r_2	θ_1	θ_2
2	4	$-22\frac{1}{2}°$	$22\frac{1}{2}°$
4	8	$22\frac{1}{2}°$	$67\frac{1}{2}°$
8	16	$67\frac{1}{2}°$	$112\frac{1}{2}°$
16	32	$112\frac{1}{2}°$	$157\frac{1}{2}°$

Thus a total of 16 Fourier features were used (four (r_1,r_2) pairs combined with four (θ_1,θ_2) pairs). The Fisher classification results obtained using each of these 16 features are given in Table I. The results obtained using each of the 120 pairs of these features are shown in Table II (where the feature numbering is as in Table I).

TABLE I
NUMBERS OF PICTURES CORRECTLY CLASSIFIED USING INDIVIDUAL FOURIER FEATURES

Feature No.	(r_1,r_2)	θ*	No. Correctly Classified
1	(16,32)	0	130
2	(16,32)	45	117
3	(16,32)	90	116
4	(16,32)	135	133
5	(8,16)	0	108
6	(8,16)	45	81
7	(8,16)	90	92
8	(8,16)	135	102
9	(4,8)	0	94
10	(4,8)	45	94
11	(4,8)	90	111
12	(4,8)	135	101
13	(2,4)	0	77
14	(2,4)	45	95
15	(2,4)	90	90
16	(2,4)	135	86

* θ is the central direction of the spatial frequency sector.

It was suggested in [1] that the Fourier-based features might be performing poorly because the power spectra are subject to aperture effects. The discrete Fourier transform treats its input as though it were periodic; i.e., as though the leftmost column of the input were repeated immediately to the right of the rightmost column and the top row immediately below the bottom row. Since these columns and rows are usually quite different, it is as though abrupt edges were present in the input. These edges may strongly influence the spectra, showing up as prominent cross-shaped patterns centered at the origin, and this may degrade the usefulness of texture features that are computed from the spectra.

Various methods of removing these aperture effects have been suggested (see [2]). W. B. Schaming of RCA has described one such method [2], which is based on reflecting the given image f in the x and y axes to obtain a $2n$-by-$2n$ image $\bar{f}$, as illustrated in Fig. 1. It will be noted that in $\bar{f}$ the top and bottom rows are the same and the left and right columns are the same; thus if we treat $\bar{f}$ as periodic, abrupt edges are not introduced. It was felt that if Fourier-based features were computed on such reflected image samples, rather than on the original samples, the classification performance might be improved.[1]

Manuscript received August 5, 1975; revised April 21, 1976. This work was supported by the Division of Engineering, National Science Foundation, under Grant ENG74-22006.
The authors are with the Computer Science Center, University of Maryland, College Park, MD 20742.

[1] In the reflected images $\bar{f}$, some directional discriminability is lost; for example, if f contains stripes oriented at 45°, then $\bar{f}$ contains stripes oriented at both 45° and 135° (see Fig. 1). Thus the Fourier features computed for $\theta = 45°$ and $\theta = 135°$ have the same values in the case of the reflected images. However, such lack of directional discrimination was not felt to be important in our particular texture classification problem, where the best feature pairs generally did not involve the diagonal directions (see [1]). The discrete Fourier transform of $\bar{f}$ is closely related to the discrete cosine transform [3] of f; this was pointed out to us by Prof. R. M. Haralick.