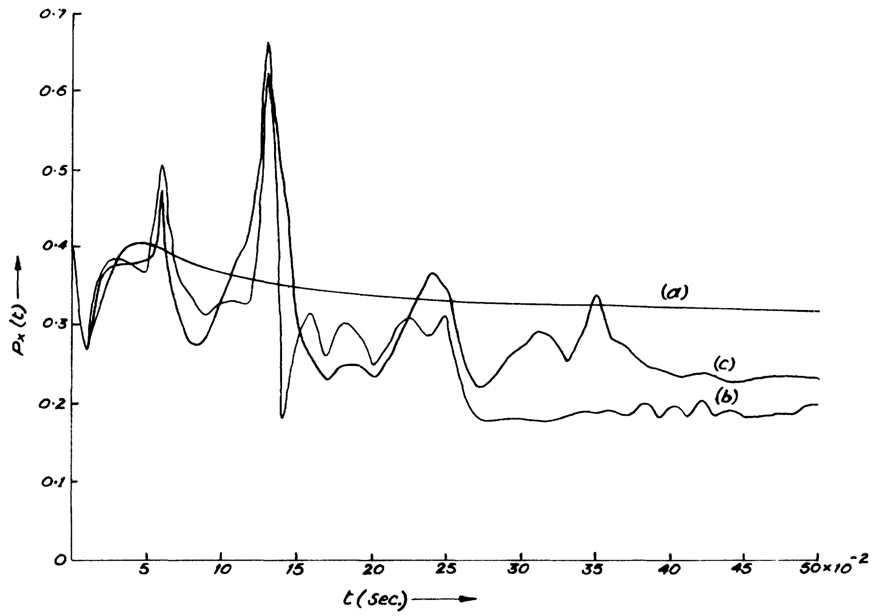

Fig. 2. Comparison of filtering error variance with constant bias (a), cubic correction (b), and quadratic correction (c) for continuous-time example.

The variances Q and R have been selected equal to 0.2 and 0.8, respectively, and the initial state has been selected as

$$x(0) = [0 \quad 0 \quad 1]^T.$$

The lower order model has been taken to be of the form

$$\dot{x}(t) = c_0 + c_p x^p + w(t), \qquad \text{where } x \text{ is scalar}$$

$$y(t) = x(t) + v(t).$$

The various estimate and variance equations were processed on the computer using a 0.01-s sampling period and the initial values $\hat{x}(0) = 0.9$, $\hat{c}_0(0) = 0$, $\hat{c}_p(0) = 0$, and

$$P(0) = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

To illustrate the type of results obtained, the plots of variance $P_x(t)$ of the error of state estimation for three different cases are shown in Fig. 2. These correspond to the case of constant bias correction, the case of nonlinear correction with $p = 2$, and the case of nonlinear correction with $p = 3$. As in the discrete-time case, it is evident that the addition of the nonlinear correction term leads to an improvement in state estimation. Furthermore, since the actual nonlinearity is cubic, the choice of $p = 3$ leads to a better modeling.

V. Conclusions

The problem of constructing lower order models for state estimation of nonlinear dynamical systems has been studied. It has been shown that though the use of a constant bias correction term to some extent compensates for the errors of modeling, better compensation can be achieved by adding a nonlinear correction term. It has been shown that the increase in the computational requirements due to the added nonlinear term is relatively small because the components of the variance equation of the augmented system are still decoupled.

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Uniform Asymptotic Stability of Discrete Large-Scale Systems

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***Abstract*—The purpose of this work is to develop algebraic conditions under which uniform asymptotic stability as well as uniform asymptotic connective stability of discrete large-scale systems are implied by uniform asymptotic stability of their subsystems. The stability properties of a discrete large-scale system are guaranteed by negative definiteness of a real symmetric matrix, the dimension of which is equal to the number of the subsystems.**

I. Introduction

Despite both the importance of the discrete system theory for the general system theory [1] and the widespread use of digital computers in the control of large-scale systems (such as electric power networks, transportation systems, and space vehicles) most of the papers on stability of large-scale systems as surveyed in [2] are devoted to continuous systems. Papers that treat Lyapunov-type stability of discrete large-scale systems are re-

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stricted either to systems with exponentially stable subsystems [3] or to a limited class of discrete systems with some such strong stability properties of the subsystems [4]. Removing the exponential stability requirement placed on the subsystems, the class of discrete large-scale systems can be significantly broadened [5]. Finite time stability as one of the most important non-Lyapunov types of stability of discrete composite systems is studied in [6].

A new concept of connective stability [7] has been recently introduced to treat continuous large-scale systems under structural perturbations [8]. It has been shown that algebraic conditions can be derived to guarantee exponential stability of a large-scale system despite on–off participation of the exponentially stable subsystems. Conditions for uniform asymptotic connective stability of large-scale systems characterized by their structure of a special type were developed in [9], in which sufficient conditions for uniform asymptotic stability of discrete large-scale systems of a special class were also obtained.

This work is concerned with investigations of stability properties of nonstationary nonlinear discrete dynamic large-scale systems composed of interconnected subsystems. Algebraic conditions for both uniform asymptotic stability and uniform asymptotic connective stability of large-scale systems are obtained without requiring Lyapunov functions to be of special form. It is shown that to achieve either uniform asymptotic stability or uniform asymptotic connective stability of a discrete large-scale system it suffices to assume uniform asymptotic stability of the subsystems and demonstrate negative definiteness of a real symmetric matrix provided that interactions fulfill weak algebraic inequalities presented in a general form. If the interactions do not satisfy the inequalities in the whole system state space, the stability properties do not hold in the whole. In this case a lower evaluation of the domain of uniform attraction as well as of the domain of uniform connective attraction is determined. Furthermore, if all subsystems are uniformly asymptotically stable in the whole and interactions satisfy the inequalities in the whole state space, the conditions guarantee uniform asymptotic stability in the whole and/or uniform asymptotic connective stability in the whole of the discrete large-scale system. It is to be noted that the conditions for the connective stability type ensure uniform asymptotic stability of the large-scale system under any structural perturbations.

The results obtained in this work also offer a computationally efficient reduction in dimensionality of stability problems in discrete large-scale systems.

The presented stability conditions are derived by using the (second) Lyapunov method [10], the concept of vector Lyapunov functions [11], [12], and the notion of comparison functions [13]. To illustrate an application of the obtained general results to discrete large-scale systems with nonstationary subsystems of Lur'e type, the Aizerman conjecture is proved for a class of nonstationary multinonlinear discrete systems. Two examples are also presented to illustrate the results on both uniform asymptotic stability and uniform asymptotic connective stability in the whole of discrete large-scale systems.

II. Uniform Asymptotic Stability of Composite Discrete Systems

Let a discrete large-scale system (S)

$$x(t_{k+1}) = f[t_k, x(t_k)] \qquad (S)$$

be composed of s interconnected subsystems S_i, which are described by

$$x_i(t_{k+1}) = g_i[t_k, x_i(t_k)] + h_i[t_k, x(t_k)], \qquad \forall i = 1,2,\cdots,s \quad (S_i)$$

where $x \in \mathscr{R}^n$ is the state of system (S), $x_i \in \mathscr{R}^{n_i}$ is the state of subsystem (S_i) and $f: \mathscr{T} \times \mathscr{R}^n \to \mathscr{R}^n$ is the system transition function. $\mathscr{T}$ is the infinite time interval $(\infty -, +\infty)$ and $\mathscr{T}_0 \subset \mathscr{T}$ is the semi-infinite time interval $[t_0, +\infty)$. The transition function of a free subsystem (S_i)

$$x_i(t_{k+1}) = g_i[t_k, x_i(t_k)], \qquad \forall i = 1,2,\cdots,s \qquad (S_i)$$

is denoted by $g_i: \mathscr{T} \times \mathscr{R}^{n_i} \to \mathscr{R}^{n_i}$. Functions $h_i: \mathscr{T} \times \mathscr{R}^n \to \mathscr{R}^{n_i}$, $\forall i = 1,2,\cdots,s$, represent interactions among subsystems. It is assumed that $f(t,x) = 0$, $\forall t \in \mathscr{T}$, if and only if $x = 0$, as well as $g_i(t,x_i) = 0$, $\forall t \in \mathscr{T}$, if and only if $x_i = 0$, $\forall i = 1,2,\cdots,s$, so that the origin $x = 0$ of the state space $\mathscr{R}^n$ is the unique equilibrium point of system (S) and the origin $x_i = 0$ of the state space $\mathscr{R}^{n_i}$ is the unique equilibrium point of subsystem (S_i), $\forall i = 1,2,\cdots,s$. In the sequel discrete time t_k will be

$$t_k = t_0 + k\Delta, \qquad k = 0,1,2,\cdots,n,\cdots \qquad (1)$$

where Δ is a real positive number.

Referring to the converse theorems on the uniform asymptotic stability of the equilibrium of subsystem (S_i), which are proved by Hahn [13, theorem 49.3] and Gordon [14, theorem 3], we conclude that uniform asymptotic stability of the equilibrium $x_i = 0$ implies existence of set $\mathscr{D}_i \subseteq \mathscr{R}^{n_i}$ and a positive-definite decrescent locally Lipschitzian function $V_i(t_k, x_i)$ whose forward difference along system solutions is negative definite

$$\phi_{i1}(\|x_i\|) \leq V_i(t_k, x_i) \leq \phi_{i2}(\|x_i\|), \qquad \forall (t_k, x_i) \in \mathscr{T} \times \mathscr{D}_i \quad (2)$$

$$\Delta V_i(t_k, x_i) \leq -\phi_{i3}(\|x_i\|), \qquad \forall (t_k, x_i) \in \mathscr{T} \times \mathscr{D}_i. \quad (3)$$

In (2) and (3) functions $\phi_{ij}: \mathscr{R}^1 \to \mathscr{R}^1$, $\phi_{ij} \in \mathscr{K}$, $j = 1,2,3$, are comparison functions of class $\mathscr{K}$ [13], and $\|x\| = (x^T x)^{1/2}$.

A purpose of this research is to demonstrate a possibility for testing uniform asymptotic stability of the equilibrium of composite system (S) by using only the minimal essential information about all its subsystems and their interactions g_i that are supposed to satisfy the following inequalities for some real numbers α_{ijl} satisfying $\alpha_{ijl} < 0$, if and only if $i = j = l$, $\forall i, j, l = 1,2,\cdots,s$,

$$\begin{aligned} \Delta V_i(t_k, x_i) &= V_i[t_{k+1}, g_i(t_k, x_i) + h_i(t_k, x)] - V_i(t_k, x_i) \\ &\leq \sum_{j=1}^{s} \sum_{l=1}^{s} \alpha_{ijl} \phi_j^{1/2}(\|x_j\|) \phi_l^{1/2}(\|x_l\|), \\ &\qquad \forall (t_k, x) \in \mathscr{T} \times \mathscr{D}, \quad \forall i = 1,2,\cdots,s \end{aligned} \quad (4)$$

where $\mathscr{D} \subseteq \mathscr{R}^n$

$$\mathscr{D} = \mathscr{D}_1 \times \mathscr{D}_2 \times \cdots \times \mathscr{D}_s. \qquad (5)$$

Functions ϕ_i can be either some of functions ϕ_{ij}, $j = 1,2,3$, or of other functions such that $\phi_i(\|x_i\|) > 0$, $\forall x_i \in \mathscr{D}_i$, $x_i \neq 0$; $\phi_i(0) = 0$. The required information is the knowledge of Lyapunov functions $V_i(t_k, x_i)$, comparison functions ϕ_{ij}, $\forall j = 1,2,3$, $\forall i = 1,2,\cdots,s$, and numbers α_{ijl}, $\forall i, j, l = 1,2,\cdots,s$ (4). To formulate simply a sufficient condition for uniform asymptotic stability of the equilibrium of composite system (S) let a real constant symmetric $s \times s$ matrix A be introduced

$$A = (a_{ij}), \qquad a_{ij} = \sum_{l=1}^{s} (\alpha_{lji} + \alpha_{lij}), \qquad \forall i, j = 1,2,\cdots,s. \quad (6)$$

Theorem 1: The equilibrium state $x = 0$ of composite system (S) is uniformly asymptotically stable if the matrix A with elements a_{ij} determined by (6) is negative definite. If it is also $\mathscr{D} = \mathscr{R}^n$ (5), then the equilibrium is uniformly asymptotically stable in the whole.

Proof: Let us introduce a candidate Lyapunov function $v: \mathscr{T} \times \mathscr{R}^n \to \mathscr{R}^1$,

$$v(t_k,x) = 2 \sum_{i=1}^{s} V_i(t_k,x_i)$$

which is a decrescent positive definite function (2)

$$2 \sum_{i=1}^{s} \phi_{i1}(\|x_i\|) \leq v(t_k,x) \leq 2 \sum_{i=1}^{s} \phi_{i2}(\|x_i\|), \qquad \forall (t_k,x) \in \mathscr{T} \times \mathscr{D}$$

and let $w: \mathscr{R}^n \to \mathscr{R}^s$ be

$$w(x) = (\phi_1^{1/2}(\|x_1\|)\ \phi_2^{1/2}(\|x_2\|) \cdots \phi_s^{1/2}(\|x_s\|))^T$$

so that the forward difference $\Delta v(t_k,x)$ along motions of composite system (S) is found by using (4) to satisfy

$$\Delta v(t_k,x) \leq w^T(x) A w(x), \qquad \forall (t_k,x) \in \mathscr{T} \times \mathscr{D}. \tag{7}$$

Let $\delta\mathscr{D}$ be the boundary of $\mathscr{D}$ and $\mathscr{D}_e \subseteq \mathscr{D}$ be

$$\mathscr{D}_e = \left\{ x: 2 \sum_{i=1}^{s} \phi_{i1}(\|x_i\|) < 2 \min_{x \in \delta\mathscr{D}} \sum_{i=1}^{s} \phi_{i1}(\|x_i\|) \right\}. \tag{8}$$

If $x_0 \in \mathscr{D}_e$, then (7) yields

$$\Delta v(t_0,x_0) \leq w^T(x_0) A w(x_0) \leq -\phi(\|x_0\|) \tag{9}$$

where $\phi: \mathscr{R}^1 \to \mathscr{R}^1$, $\phi \in \mathscr{K}$, is a comparison function of the form

$$\phi(\|x\|) = -\lambda_M(A)\|w(x)\|^2$$

and $\lambda_M(A)$ is the greatest eigenvalue of the matrix A. Since A is a negative definite matrix it follows that $\lambda_M(A) < 0$. Therefore, $\Delta v(t_k,x)$ (7) is also negative definite. From (9) we get

$$v[t_1,x(t_1; t_0, x_0)] \leq v(t_0,x_0)$$

which implies $x(t_1; t_0, x_0) \in \mathscr{D}_e$. Repeating this procedure and using the method of mathematical induction on $k = 0,1,2,\cdots, m,\cdots$, it is proved that

$$x(t_k; t_0, x_0) \in \mathscr{D}_e, \qquad \forall (t_k,t_0,x_0) \in \mathscr{T}_0 \times \mathscr{T} \times \mathscr{D}_e$$

as well as

$$\|x_0\| < \delta(\varepsilon) \Rightarrow \|x(t_k; t_0, x_0)\| < \varepsilon, \qquad \forall \varepsilon \in (0,\varepsilon_M] \tag{10}$$

where

$$\delta(\varepsilon) = 2 \min_{\delta\mathscr{S}_\varepsilon} \sum_{i=1}^{s} \phi_{i1}(\|x_i\|)$$

and $\delta\mathscr{S}_\varepsilon$ is the boundary of hypersphere

$$\mathscr{S}_\varepsilon = \{x: \|x\| < \varepsilon\}$$

provided that $\varepsilon_M > 0$ is selected so that $\mathscr{S}_{\varepsilon_M} \subseteq \mathscr{D}_e$. If $\delta(\varepsilon)$ is chosen to be $\delta(\varepsilon) = \delta(\varepsilon_M)$, $\forall \varepsilon \geq \varepsilon_M$, then relations (10) are valid $\forall \varepsilon > 0$. From (10) it results that the equilibrium $x = 0$ is uniformly stable. To prove that the equilibrium is uniformly attractive we suppose opposite, i.e., that there exists a real number $\zeta > 0$ such that

$$\|x(t_k; t_0, x_0)\| \to \zeta \text{ as } t_k \to +\infty, \qquad \text{for } (t_0,x_0) \in \mathscr{T} \times \mathscr{D}_e.$$

From (7) we get

$$V[t_1,x(t_1; t_0, x_0)] \leq V(t_0,x_0) - \phi(\|x_0\|)$$

or, in general,

$$\begin{aligned} V[t_{k+1},x(t_{k+1}; t_0, x_0)] &= V\{t_{k+1},x[t_{k+1}; t_k, x(t_k; t_0, x_0)]\} \\ &= V[t_k,x(t_k; t_0, x_0)] - \phi[\|x(t_k; t_0, x_0)\|], \\ &\qquad \forall (t_k,t_0,x_0) \in \mathscr{T}_0 \times \mathscr{T} \times \mathscr{D}_e. \end{aligned}$$

Therefore,

$$\begin{aligned} V[t_{k+1},x(t_{k+1}; t_0, x_0)] &= V(t_0,x_0) - \sum_{i=0}^{k} \phi(\|x(t_i; t_0, x_0)\|) \\ &\leq V(t_0,x_0) - (k+1)\phi(\zeta_M) \end{aligned}$$

where $\zeta_M = \min \{\inf_{t_k \in \mathscr{T}_0} \|x(t_k; t_0, x_0)\|; \zeta\} > 0$ since $\zeta > 0$ and $\|x(t_k; t_0, x_0)\|$ cannot be equal to zero for any finite k. Hence, for an arbitrary real number $\xi > 0$, we derive

$$V[t_{k+1},x(t_{k+1}; t_0, x_0)] \leq -\xi$$

as soon as

$$k \geq \phi^{-1}(\zeta_M)[V(t_0,x_0) + \xi] - 1$$

which is impossible since $V(t_k,x) \geq 0$ by definition. Therefore, $\zeta = 0$, i.e.,

$$\|x(t_{k+1}; t_0, x_0)\| \to 0 \text{ as } t_{k+1} \to +\infty, \qquad \forall (t_0,x_0) \in \mathscr{T} \times \mathscr{D}_e.$$

This result proves that $\mathscr{D}_e$ is a lower evaluation of the domain of attraction. Hence, the first part of the theorem is true. If $\mathscr{D} = \mathscr{R}^n$ then $\mathscr{D}_e = \mathscr{R}^n$, which proves the second part of the theorem.

It is to be noted that the proof of Theorem 1 enables us to evaluate the domain of attraction ($\mathscr{D}_a$) of composite system (S). Such an evaluation is given by set $\mathscr{D}_e$ (8), $\mathscr{D}_e \subseteq \mathscr{D}_a$,

$$\forall (t_0,x_0) \in \mathscr{T} \times \mathscr{D}_e \Rightarrow \|x(t_k; t_0, x_0)\| \to 0 \text{ as } t_k \to +\infty.$$

IV. Uniform Asymptotic Connective Stability

The classical stability theory is mainly concerned with initial condition, forcing function and parameter perturbations of dynamic systems which do not change their structural properties. On the higher hierarchical level, however, it is of interest to consider structural perturbations and use a notion of connective stability introduced and defined by Šiljak in [7], [8].

Structural perturbations of large-scale system (S) can be described by using an interconnection $s \times s$ matrix $E = (e_{ij})$ with elements e_{ij} that can take on values zero or one with the meaning

$$e_{ij} = \begin{cases} 1, & \boldsymbol{S}_j \text{ acts on } \boldsymbol{S}_i \text{ through } g_i \\ 0, & \boldsymbol{S}_j \text{ does not act on } \boldsymbol{S}_i. \end{cases} \tag{11}$$

An interconnection matrix is called the fundamental interconnection matrix E_f if all elements e_{ij} that correspond to existing or possible connections between subsystems are set one, and nonexisting interconnections are represented by invariant zero elements.

In this work a condition for connective stability will be derived so that all allowed structural perturbations, which may be arbitrary functions of the state $x(t_k) \in \mathscr{R}^n$ and/or time $t_k \in \mathscr{T}_0$, are described by interconnection matrices E. For this reason and by referring to papers [7], [8], the following definition is accepted.

Definition: The equilibrium $x = 0$ of a free discrete dynamic system (S) with subsystems (S_{ci})

$$x_i(t_{k+1}) = g_i[t_k,x_i(t_k)] + h_i[t_k,e_{i1}x_1(t_k),e_{i2}x_2(t_k),\cdots,e_{is}x_s(t_k)], \quad \forall i = 1,2,\cdots,s \quad (S_{ci})$$

is uniformly asymptotically connectively stable (in the whole) if and only if it is uniformly asymptotically stable (in the whole) for all interconnection matrices E.

Since the equilibrium $x = 0$ of system (S) should be uniformly asymptotically stable for all interconnection matrices the following statement is obtained [7], [8].

Statement: A necessary condition for connective stability of the equilibrium $x = 0$ of system (S) is that each subsystem (S_i), which can be completely disconnected, should possess the same stability properties as required from the entire system.

Referring to the statement, it is accepted that each subsystem (S_i) possesses uniformly asymptotically stable equilibrium $x_i = 0$ as well as that a set $\mathscr{D}_i \subseteq \mathscr{R}^{n_i}$ and a Lyapunov function $V_i(t_k,x_i)$ are used to prove the stability property. Comparison functions $\phi_{ij}(\|x_i\|)$, $\forall i = 1,2,\cdots,s$, $\forall j = 1,2,3$, (2), (3), are supposed to be known. Further, interactions h_i are required to satisfy the following inequalities for some real numbers $\alpha_{ijl} \geq 0$ and $\beta_{iii} > 0$:

$$\Delta V_i(t_k,x_i) \leq \sum_{j=1}^{s} \sum_{l=1}^{s} (-\beta_{ijl}\delta_{ij}\delta_{il} + e_{ij}e_{il}\alpha_{ijl})\phi_j^{1/2}(\|x_j\|)\phi_l^{1/2}(\|x_l\|), \quad \forall i = 1,2,\cdots,s, \quad \forall (t_k,x) \in \mathscr{D}, \forall E \quad (12)$$

where $\Delta V_i(t_k,x_i)$ is the forward difference of function $V_i(t_k,x_i)$ along solutions of (S_{ci}) and δ_{ij} is the Kronecker symbol. Conditions (12) imposed on system interactions are presented in a general form. It is obvious that values of α_{ijl} and β_{iii} depend crucially on a choice of Lyapunov functions V_i, comparison functions $\phi_{ij}(\|x_i\|)$ and functions $\phi_i(\|x_i\|)$ as well as on properties of interactions h_i. These values are required to be determined so that all elements a_{ij} of a real symmetric matrix A can be computed by using

$$a_{ij}(E) = \sum_{l=1}^{s} [-2\beta_{lij}\delta_{il}\delta_{jl} + e_{li}e_{lj}(\alpha_{lij} + \alpha_{lji})], \quad \forall i,j = 1,2,\cdots,s. \quad (13)$$

Matrix A may be viewed as a matrix with elements a_{ij} dependent on interconnection matrices E, (13). Since $\alpha_{ijl} \geq 0$, $\forall i,j,l = 1,2,\cdots,s$, and $e_{ij} \geq 0$, it follows that all elements $a_{ij}(E)$ of matrix $A(E)$ take on their maximal values for the fundamental interconnection matrix E_f

$$\max_E a_{ij}(E) = a_{ij}\, E_f), \qquad \forall i,j = 1,2,\cdots,s. \qquad (14)$$

Using this result the required condition for uniform asymptotic connective stability of large-scale system (S) can be formulated as follows.

Theorem 2: The equilibrium state $x = 0$ of large-scale system (S) with subsystems (S_{ci}) is uniformly asymptotically connectively stable if the matrix $A(E_f)$ with elements $a_{ij}(E_f)$ determined by (13) for $E = E_f$ is negative definite. If it is also $\mathscr{D} = \mathscr{R}^n$, (5) and (12), then the equilibrium is uniformly asymptotically connectively stable in the whole.

Proof: Repeating the proof of Theorem 1 we show that the equilibrium $x = 0$ of system (S) with subsystems (S_{ci}) is uniformly asymptotically stable for the fundamental interconnection matrix. Hence,

$$\Delta v(t_k,x)_{E_f} \leq -\phi(\|x\|)_{E_f}, \qquad \forall (t_k,x) \in \mathscr{T} \times \mathscr{D}_e \qquad (15)$$

where $\mathscr{D}_e$ is defined by (8), $v(t_k,x) = 2\sum_{i=1}^{s} V_i(t_k,x_i)$, and $\phi(\|x\|)_{E_f} = -\lambda_M[A(E_f)]\|w(x)\|^2$. Index E_f in the previous notations (15) shows that the corresponding quantities are determined for the fundamental interconnection matrix E_f. Furthermore, the use of (14) yields

$$\Delta v(t_k,x)_E \leq \Delta v(t_k,x)_{E_f} \leq -\phi(\|x\|)_{E_f}, \qquad \forall (t_k,x) \in \mathscr{T} \times \mathscr{D}_e, \quad \forall E.$$

From this result and uniform asymptotic stability of the equilibrium for the fundamental interconnection matrix E_f it follows that the equilibrium is uniformly asymptotically connectively stable. If it is also $\mathscr{D} = \mathscr{R}^n$ then $\mathscr{D}_e = \mathscr{R}^n$ and the equilibrium $x = 0$ is obviously uniformly asymptotically connectively stable in the whole, which proves the theorem completely.

Summarizing the proof of Theorem 2 we obtain an evaluation $(\mathscr{D}_e)$ of the domain of uniform connective attraction $(\mathscr{D}_{ac})$ of large-scale system (S) with subsystems (S_{ci}),

$$\forall (t_0,x_0) \in \mathscr{T} \times \mathscr{D}_e \Rightarrow \lim_{t_k \to +\infty} \|x(t_k;\, t_0,\, x_0)\|_E = 0, \qquad \forall E.$$

V. Subsidiary Results

To apply the results of Section III and Section IV to stability analysis of a given dynamic discrete large-scale system its Lyapunov function is required to be known. Proving results on absolute stability of nonstationary or time-invariant multinonlinear Lur'e type discrete systems in this section, we shall derive Lyapunov functions of these systems described by vector difference equations of the form

$$x(t_{k+1}) = A(t_k)x(t_k) + B(t_k)y[t_k,\sigma_1(t_k),\sigma_2(t_k),\cdots,\sigma_m(t_k)] \quad (16)$$

where $A: \mathscr{T} \to \mathscr{R}^{n^2}$ and $B: \mathscr{T} \to \mathscr{R}^{n+m}$ are matrices with time-dependent elements, $y: \mathscr{T} \times \mathscr{R}^m \to \mathscr{R}^m$ is a vector function with elements $\phi_i: \mathscr{T} \times \mathscr{R}^1 \to \mathscr{R}^1$, $\forall i = 1,2,\cdots,m$, satisfying sector conditions

$$0 \leq \frac{\phi_i(t_k,\sigma_i)}{\sigma_i} \leq K_i, \qquad \forall (t_k,\sigma_i) \in \mathscr{T}_0 \times \mathscr{R}^1, \quad \forall i = 1,2,\cdots,m. \qquad (17)$$

With $\sigma_i: \mathscr{T} \times \mathscr{R}^n \to \mathscr{R}^1$ has been denoted

$$\sigma_i(t_k,x) = c_i^T(t_k)x, \qquad \forall i = 1,2,\cdots,m \qquad (18)$$

and $c_i: \mathscr{T} \to \mathscr{R}^n$ is a vector function, $\forall i = 1,2,\cdots,m$. In this section $t_k \in \mathscr{T}$ indicate discrete values of time (k is an integer)

$$-\infty < \cdots < t_{k-1} < t_k < t_{k+1} < \cdots < +\infty, \qquad t_k \to +\infty \text{ as } k \to +\infty.$$

If all ϕ_i are linear functions, $\phi_i(t_k,\sigma_i) = \alpha_i\sigma_i$, $\alpha_i \in [0,K_i]$, where all α_i are real numbers, then system (16) is transformed in a linear system

$$x(t_{k+1}) = C(t_k,\Lambda)x(t_k) \qquad (19)$$

with matrix $\Lambda = \operatorname{diag}\{\alpha_1\alpha_2\cdots\alpha_m\}$ and matrix $C: \mathscr{T} \times \mathscr{R}^m \to \mathscr{R}^{n^2}$

$$C(t_k,\Lambda) = A(t_k) + B(t_k)D(t_k,\Lambda). \qquad (20)$$

Matrix $D: \mathscr{T} \times \mathscr{R}^m \to \mathscr{R}^{m\times n}$ is defined by

$$D = \Lambda(c_1(t_k)\; c_2(t_k) \cdots c_m(t_k))^T \qquad (21)$$

and it is said to belong to class $\mathscr{D}_0$ if and only if $\Lambda \in \mathscr{H}$, where

$$\mathscr{H} = \{\Lambda: \alpha_i \in [0,K_i], \forall i = 1,2,\cdots,m\}. \quad (22)$$

To present the required subsidiary results in a simple form a notation $F: \mathscr{T} \times \mathscr{R}^m \to \mathscr{R}^{n^2}$ is introduced

$$F(t_k,\Lambda) = C^T(t_k,\Lambda)C(t_k,\Lambda) - I \quad (23)$$

where I is the nth-order identity matrix.

Theorem 3: If matrix $F(t_k,\Lambda)$, (24), which is associated with linear system (19), is negative definite for every $\Lambda \in \mathscr{H}$, for every $t_k \in \mathscr{T}$, and for $t_k \to +\infty$, then nonlinear system (16) is absolutely stable on sectors $[0,K_i]$, (17), $\forall i = 1,2,\cdots,m$.

Proof: Let matrix $\Lambda(t_k,x)$ be introduced and defined by

$$\Lambda(t_k,x) = \text{diag}\,\{\alpha_1(t_k,x)\,\alpha_2(t_k,x)\cdots\alpha_m(t_k,x)\}$$

along with

$$\alpha_i(t_k,x) = \frac{\phi_i[t_k,\sigma_i(t_k,x)]}{\sigma_i(t_k,x)}, \qquad \forall i = 1,2,\cdots,m.$$

$D(t_k,x)$ will denote a matrix defined by

$$D(t_k,x) = \Lambda(t_k,x)(c_1(t_k)\,c_2(t_k)\cdots c_m(t_k))^T.$$

From (17) it follows that

$$D(t_k,x) \in \mathscr{D}_0, \qquad \forall (t_k,x) \in \mathscr{T}_0 \times \mathscr{R}^n. \quad (24)$$

Using matrix notation $C(t_k,x) = A(t_k) + B(t_k)D(t_k,x)$, system (16) may be rewritten in the form

$$x(t_{k+1}) = C[t_k,x(t_k)]x(t_k).$$

Let $F: \mathscr{T} \times \mathscr{R}^n \to \mathscr{R}^{n^2}$ be

$$F(t_k,x) = C^T(t_k,x)C(t_k,x) - I$$

and let $V(t_k,x) = \|x\|^2$ be a candidate Lyapunov function for system (16). The forward difference of function V determined along solutions $x(t_k; t_0, x_0)$ of system (16) is found to be

$$\Delta V(t_k,x) = x^T F(t_k,x)x, \qquad x = x(t_k; t_0, x_0).$$

From the condition of the theorem and result (24) it follows that $F(t_k,x)$ is negative definite for all $(t_k,x) \in \mathscr{T} \times \mathscr{R}^n$. Therefore, the supremum eigenvalue λ_s of matrix $F(t_k,x)$

$$\lambda_s = \sup_{\mathscr{T} \times \mathscr{R}^n} \lambda_M[F(t_k,x)]$$

is negative, which implies

$$\Delta V(t_k,x) \le -|\lambda_s| \cdot \|x\|^2, \qquad \forall (t_k,x) \in \mathscr{T}_0 \times \mathscr{R}^n, \quad \forall t_0 \in \mathscr{T}. \quad (25)$$

According to [15, theorem 1] the proof is completed with the previous result.

It has been shown that a generalized Aizerman conjecture is true for a class of the nth-order multinonlinear discrete systems (16) that fulfill conditions of Theorem 3. For these systems function $V = \|x\|^2$ may be accepted as a Lyapunov function. It is also significant to note that the systems are exponentially absolutely stable on all sectors $[0,K_i]$ with the degree of stability equal to $\ln(1 - \lambda_s)^{-1/2}$, which results from (25) and (12) of [16].

In [17], [18] the corresponding results are obtained with respect to open sectors for a class of time-invariant Lur'e type systems with one nonlinearity. As a conclusion of [18] it is noted that the results there can be generalized to multinonlinear time-invariant systems (26)

$$x(t_{k+1}) = Ax(t_k) + By[\sigma_1(t_k),\sigma_2(t_k),\cdots,\sigma_m(t_k)],$$
$$\sigma_i(t_k) = c_i^T(t_k)x(t_k), \qquad \forall i = 1,2,\cdots,m \quad (26)$$

which can be also derived for compact sectors $[0,K_i]$ as a direct consequence of Theorem 3 as follows. Matrices A and B as well as all vectors c_i are constant in (26).

Corollary: If matrix $F(\Lambda)$, which is associated to the linear version of system (26)

$$F(\Lambda) = C^T(\Lambda)C(\Lambda) - I, \qquad C(\Lambda) = A + BD(\Lambda)$$

is negative definite $\forall \Lambda \in \mathscr{H}$, then system (26) is exponentially absolutely stable on sectors $[0,K_i]$, $\forall i = 1,2,\cdots,m$.

VI. Application and Examples

It is known that for contractive both time-invariant [15] and nonstationary [19] discrete systems Lyapunov functions may be selected as positive definite quadratic forms

$$V_i(x_i) = x_i^T H_i x_i, \qquad \forall i = 1,2,\cdots,s \quad (27)$$

where $H_i = H_i^T$, $\forall i = 1,2,\cdots,s$, is a real constant positive definite $n_i \times n_i$ matrix. The same choice of Lyapunov functions is adequate if subsystems are of Lur'e type and Popov [20] like criteria [16], [21]–[26] are used to prove their (exponential [16], [26]) absolute stability, which can be verified either by using analytical tests of [27]–[29] or by applying general algebraic criteria for positive realness relative to the unit circle that have been recently proved by Šiljak [30]. A class of subsystems of discrete large-scale systems under consideration, for which Lyapunov functions can be chosen as quadratic forms (27), has been broadened by proving Theorem 3 for nonstationary multinonlinear Lur'e type subsystems.

Interconnections of composite system (S) are supposed to satisfy the following inequalities for some real numbers $\zeta_{ij} \ge 0$ and $\xi_{ijl} \ge 0$ provided that a Lyapunov function $V_i(t_k,x_i)$ of subsystem (S_i) is a quadratic form (27):

$$g_i^T(t_k,x_i)H_ih_i(t_k,x) \le \sum_{j=1}^{s} \zeta_{ij}\phi_i^{1/2}(\|x_i\|)\phi_j^{1/2}(\|x_j\|)$$

$$h_i^T(t_k,x)H_ih_i(t_k,x) \le \sum_{j=1}^{s}\sum_{l=1}^{s} \xi_{ijl}\phi_j^{1/2}(\|x_j\|)\phi_l^{1/2}(\|x_l\|)$$

$$\phi_i(\|x_i\|) = \phi_{i3}(\|x_i\|), \quad \forall (t_k,x) \in \mathscr{T} \times \mathscr{D},$$
$$\forall i = 1,2,\cdots,s. \quad (28)$$

If elements α_{ijl} of (4) are determined by

$$\alpha_{ijl} = -\delta_{il}\delta_{jl} + 2\zeta_{ij}\delta_{il} + \xi_{ijl}, \qquad \forall i,j,l = 1,2,\cdots,s \quad (29)$$

then elements a_{ij} of matrix A should be computed according to (6) and uniform asymptotic stability of the equilibrium of composite system (S) may be tested by verifying negative definiteness of the matrix (Theorem 1).

If system (S) is with variable structure and constituted of subsystems (S_{ci}) then interactions h_i are assumed to fulfill the following conditions for some real numbers $\zeta_{ij} \ge 0$ and $\xi_{ijl} \ge 0$ provided that the stability properties of subsystems (S_i) are proved by using quadratic forms (27) as their Lyapunov functions

$$g_i^T(t_k,x_i)H_ih_i[t_k,e_{i1}x_1(t_k),e_{i2}x_2(t_k),\cdots,e_{is}x_s(t_k)]$$
$$\le \sum_{j=1}^{s} e_{ij}\zeta_{ij}\phi_i^{1/2}(\|x_i\|)\phi_j^{1/2}(\|x_j\|),$$
$$\forall (t_k,x) \in \mathscr{T} \times \mathscr{D}, \quad \forall i = 1,2,\cdots,s \quad (30)$$

and

$$h_i^T[t_k,e_{i1}x_1(t_k),e_{i2}x_2(t_k),\cdots,e_{is}x_s(t_k)]H_ih_i[t_k,e_{i1}x_1(t_k),e_{i2}x_2(t_k),\cdots,e_{is}x_s(t_k)] \le \sum_{j=1}^{s}\sum_{l=1}^{s} e_{ij}e_{il}\xi_{ijl}\phi_j^{1/2}(\|x_j\|)\phi_l^{1/2}(\|x_l\|),$$
$$\forall(t_k,x)\in\mathscr{T}\times\mathscr{D},\quad \forall i = 1,2,\cdots,s. \quad (31)$$

Elements $a_{ij}(E)$ of matrix $A(E)$ are to be computed by using

$$a_{ij}(E) = \sum_{l=1}^{s}[-2\delta_{il}\delta_{jl} + 2\delta_{jl}(e_{ji}\zeta_{ji} + e_{ij}\zeta_{ij}) + e_{li}e_{lj}(\xi_{lij} + \xi_{lji}) \quad (32)$$

so that elements $a_{ij}(E_f)$ of matrix $A(E_f)$ are determined by (14), which enables uniform asymptotic connective stability of system (S) to be tested by applying Theorem 2.

Example 1: Let system (S) be composed of two subsystems (S_1,S_2) described by

$$g_1(t_k,x_1) = A_1(t_k)x_1(t_k) + b_1(t_k)\phi_1[\sigma_1(t_k)],\ \sigma_1(t_k) = c_1^T(t_k)x_1(t_k)$$
$$A_1(t_k) = \begin{pmatrix}\beta_1 \exp 2t_k - t_k^2) & \beta_2 \sin t_k\\ \beta_1 \exp(2t_k - t_k^2) & \beta_2 \sin t_k\end{pmatrix}$$
$$b_1(t_k) = \begin{pmatrix}\beta \exp[-2(2t_k - t_k^2)]\\ \beta \exp[-2(2t_k - t_k^2)]\end{pmatrix}$$
$$c_1(t_k) = \begin{pmatrix}\zeta_1 \exp[-3(t_k^2 - 2t_k)]\\ \zeta_2 \exp[-2(t_k^2 - 2t_k)]\sin t_k\end{pmatrix}$$
$$\phi_1(t_k,\sigma_1) = K_1 \operatorname{sat}\sigma_1$$
$$h_1(t_k,x) = \begin{pmatrix}\gamma_1 \sin 2(x_{11} + x_{12})\\ \gamma_1\|x_2\|(1 - \|x_2\|)\end{pmatrix}$$
$$\mathscr{D}_1 = \mathscr{R}^2 \quad (33)$$

and

$$g_2(t_k,x_2) = \begin{pmatrix}x_{21}^2 - x_{22}^2\\ 2x_{21}x_{22}\end{pmatrix}$$
$$h_2(t_k,x) = \begin{pmatrix}\gamma_2\|x_2\|(1 - \|x_2\|)\\ \gamma_2 \operatorname{sat}(x_{11} + x_{12})\end{pmatrix}$$
$$\mathscr{D}_2 = \{x_2\colon \|x_2\| \le 0.89\} \subset \mathscr{R}^2. \quad (34)$$

Real numbers β, β_1, β_2, ζ_1, ζ_2, and $K_1 > 0$ are supposed to satisfy

$$\sum_{i=1}^{2}[\max\{|\beta_i + K_1\beta\zeta_i|, |\beta_i - K_1\beta\zeta_i|\}]^2 \exp(2\delta_{11}) < 0.5. \quad (35)$$

Applying Theorem 3 and (35) to subsystem (S_1),

$$x_1(t_{k+1}) = A_1(t_k)x_1(t_k) + b_1(t_k)\phi_1[\sigma_1(t_k)],$$
$$\sigma_1(t_k) = c_1^T(t_k)x_1(t_k) \quad (S_1)$$

we conclude that it is exponentially absolutely stable on sector $[0,K_1]$ and that a function $V_1 = \|x_1\|^2$ may be taken as its Lyapunov function. From Theorem 3 it also follows that $\phi_1(\|x_1\|) = \phi_{13}(\|x_1\|)$ is given by

$$\phi_1(\|x_1\|) = \lambda_1^2\|x_1\|^2 \quad (36)$$

where

$$\lambda_1^2 = \{\max_{\substack{t_k\in\mathscr{T}\\ \alpha\in[0,K_1]}} \lambda_M[I - C^T(t_k,\alpha)C(t_k,\alpha)]\},$$
$$C(t_k,\alpha) = A_1(t_k) + \alpha b_1(t_k)c_1^T(t_k).$$

Hahn [12, p. 207] showed that $\mathscr{D}_2 = \{x_2\colon \|x_2\| < 1\}$ is the domain of attraction of subsystem (S_2)

$$x_2(t_{k+1}) = g_2(t_k,x_2) \quad (S_2)$$

and that $V_2(x_2) = \|x_2\|^2$ may be taken as its Lyapunov function with $\phi_2(\|x_2\|) = \phi_{23}(\|x_2\|)$

$$\phi_2(\|x_2\|) = \|x_2\|^2(1 - \|x_2\|^2). \quad (37)$$

It is to be noted that $\phi_2 \notin \mathscr{K}$, but $\phi_2(v) > 0$, $\forall v\in(0,1)$, and $\phi_2(0) = 0$.

In this example numbers α_{ijl} (4) are found to be

$$\alpha_{111} = -1 + 4\gamma_1\lambda_1^{-1}\lambda_2 + 4\gamma_1^2\lambda_1^{-2}$$
$$\alpha_{112} = \alpha_{121} = \gamma_1\lambda_1^{-1}\lambda_2 + 2\gamma_1^2\lambda_1^{-2}$$
$$\alpha_{122} = \gamma_1^2\lambda_1^{-2}$$
$$\alpha_{211} = 4\gamma_2$$
$$\alpha_{212} = \alpha_{221} = 2\gamma_2(2 + \gamma_2)$$
$$\alpha_{222} = -1 + 4\gamma_2 + \gamma_2^2$$

where

$$\lambda_2 = \max_{\substack{t_k\in\mathscr{T}\\ \alpha\in[0,K_1]}} \lambda_M[C^T(t_k,\alpha)C(t_k,\alpha)]$$

so that

$$a_{11} = -2(1 - 4\gamma_1\lambda_1^{-1}\lambda_2 - 4\gamma_1^2\lambda_1^{-2} - 4\gamma_2)$$
$$a_{12} = a_{21} = 2\gamma_1\lambda_1^{-1}\lambda_2 + 4\gamma_1^2\lambda_1^{-2} + 8\gamma_2 + 4\gamma_2^2$$
$$a_{22} = 2(1 - \gamma_1^2\lambda_1^{-2} - 4\gamma_2 - \gamma_2^2).$$

Applying Theorem 1 we conclude that the equilibrium $x = 0$ of system (S) described by (33) to (35) is uniformly asymptotically stable if

$$1 > \max\{(4\gamma_1\lambda_1^{-1}\lambda_2 + 4\gamma_1^2\lambda_1^{-2} + 4\gamma_2),(\gamma_1^2\lambda_1^{-2} + 4\gamma_2 + \gamma_2^2)\}$$

and

$$2(1 - \gamma_1^2\lambda_1^{-2} - 4\gamma_2 - \gamma_2^2)(1 - 4\gamma_1\lambda_1^{-1}\lambda_2 - 4\gamma_1^2\lambda_1^{-2} - 4\gamma_2) > (\gamma_1\lambda_1^{-1}\lambda_2 + 2\gamma_1^2\lambda_1^{-2} + 4\gamma_2 + 2\gamma_2^2)^2.$$

A lower evaluation $\mathscr{D}_e$ of the domain of uniform attraction of system (S) is given by

$$\mathscr{D}_e = \{x\colon \|x\| \le 0.89\} \subset \mathscr{R}^4.$$

For example, if $\gamma_1\lambda_1^{-1} = \gamma_2 = 0.05$ then the equilibrium $x = 0$ of system (S) is uniformly asymptotically stable for any $\lambda_2 \in (0,1)$.

Example 2: Let system (S) with variable structure be composed of three subsystems (S_1,S_2,S_3), interconnections of which are described by interconnection matrices E

$$E = \begin{pmatrix}e_{11} & e_{12} & e_{13}\\ e_{21} & e_{22} & e_{23}\\ 0 & e_{32} & e_{33}\end{pmatrix}.$$

The fundamental interconnection matrix E_f is obtained as

$$E_f = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}.$$

Subsystem (S_1) is described by

$$x_1(t_{k+1}) = A_1(t_k)x_1(t_k) + b_1(t_k)\phi_1[\sigma_1(t_k)] + h_1[t_k,e_{11}x_1(t_k),e_{12}x_2(t_k),e_{13}x_3(t_k)] \quad (38)$$

with

$$\sigma_1(t_k) = c_1^T(t_k)x_1(t_k)$$

and $A_1(t_k),b_1(t_k),\phi_1(\sigma_1),\mathscr{D}_1$ are given by (33). Interaction h_1 is defined as

$$h_1 = \begin{pmatrix} \zeta_1 \operatorname{sat}(e_{11}x_{11} + e_{13}x_{31}) \\ \zeta_1 \operatorname{sat}(e_{11}x_{11} + e_{12}x_{21}) \end{pmatrix}. \quad (39)$$

Function $\phi_1 = \lambda_1^2\|x_1\|^2$ was already obtained for subsystem (S_1) (Example 1). Subsystems (S_{ci}), $\forall i = 1,2$, are described by

$$x_i(t_{k+1}) = A_ix_i(t_k) + b_i\phi_i[\sigma_i(t_k)] + h_i[t_k,e_{i1}x_1(t_k),e_{i2}x_2(t_k),e_{i3}x_3(t_k)], \quad \forall i = 2,3 \quad (40)$$

where $\sigma_i = c_i^Tx_i$ and

$$h_2 = \begin{pmatrix} \zeta_2 \operatorname{sat}(e_{22}x_{21} + e_{23}x_{31}) \\ \zeta_2 \operatorname{sat}(e_{22}x_{21} + e_{21}x_{11}) \end{pmatrix}$$

$$h_3 = \begin{pmatrix} \zeta_3 \operatorname{sat}(e_{33}x_{31} + e_{32}x_{21}) \\ \zeta_3 \operatorname{sat}(e_{33}x_{31} + e_{32}x_{22}) \end{pmatrix} \quad (41)$$

with $\zeta_2,\zeta_3 > 0$. Nonlinearities $\phi_i(\sigma_i)$, $\forall i = 2,3$, are assumed to satisfy sector conditions

$$0 \le \frac{\phi_i(\sigma_i)}{\sigma_i} \le K_i, \qquad \forall i = 2,3 \quad (42)$$

pairs (A_i,b_i) and (A_i,c_i^T) are supposed to be completely controllable and completely observable, respectively, matrix A_i is required to be stable, that is, that all its eigenvalues λ_{ij} satisfy $|\lambda_{ij}| < 1$, $\forall i = 2,3$. Furthermore, subsystems (S_i)

$$x_i(t_{k+1}) = A_ix_i(t_k) + b_i\phi_i[\sigma_i(t_k)], \qquad \forall i = 2,3 \qquad (S_i)$$

are selected to satisfy Popov-like frequency criterion

$$K_i^{-1} + \operatorname{Re}\chi_i(z) > 0, \qquad \forall z: |z| = 1, \quad \forall i = 2,3 \quad (43)$$

where z is a complex number, and

$$\chi_i(z) = c_i^T(A_i - zI_i)^{-1}b_i.$$

I_i is the identity matrix of order n_i.

Following references [16], [23], [31] it is proved that condition (43) is necessary and sufficient for the existence of real numbers $\varepsilon_i > 0$ and γ_i, a real constant vector g_i of order n_i, and real constant, positive definite $n_i \times n_i$ matrices $H_i = H_i^T$ and $Q_i = Q_i^T$ such that

$$A_i^TH_iA_i - H_i = -\varepsilon_iQ_i - g_ig_i^T$$

$$2A_i^TH_ib_i + c_i = -2\gamma_ig_i$$

$$K_i^{-1} - b_i^TH_ib_i = \gamma_i^2. \quad (44)$$

Then, a quadratic form $V_i(x_i) = x_i^TH_ix_i$ is one of the Lyapunov functions of subsystem (S_i) [16], [23], [31] whose forward difference along motions of (S_i) is given by

$$\Delta V_i = -x_i^T(A_i^TH_iA_i - H_i + g_ig_i^T)x_i - [\gamma_i\phi_i(\sigma_i) + x_i^Tg_i]^2 - \Omega_i$$

where

$$\Omega_i = [\sigma_i - \phi_i(\sigma_i)K_i^{-1}]\phi_i(\sigma_i) \ge 0, \qquad \forall\sigma_i$$

so that it can be estimated either by

$$\Delta V_i \le -\varepsilon_ix_i^TQ_ix_i$$

or by

$$\Delta V_i \le -\phi_{i3}(\|x_i\|). \quad (45)$$

Comparison function ϕ_i is selected as

$$\phi_i(\|x_i\|) = \phi_{i3}(\|x_i\|) = \lambda_i^2\|x_i\|^2, \qquad \forall i = 2,3 \quad (46)$$

where

$$\lambda_i^2 = \varepsilon_i\lambda_{mi}(Q_i)$$

and $\lambda_{mi}(Q_i) > 0$ is the minimum eigenvalue of matrix Q_i, $\forall i = 2,3$.

Determining elements $a_{ij}(E_f)$ of matrix $A(E_f)$, which are denoted simply by a_{ij}, according to (32) we find

$$a_{11} = 2[-1 + \lambda_1^{-2}(4\zeta_1\|C_1\|_M + 2\zeta_1^2 + 2\zeta_2^2)]$$

$$a_{12} = a_{21} = \lambda_1^{-1}\lambda_2^{-1}(\zeta_1\|C_1\|_M + \zeta_1^2 + \zeta_2\|C_2\|_M + \zeta_2^2)$$

$$a_{13} = a_{31} = \lambda_1^{-1}\lambda_3^{-1}\zeta_1(\|C_1\|_M + \zeta_1)$$

$$a_{22} = 2[-1 + \lambda_2^{-2}(\zeta_1^2 + 4\zeta_2\|C_2\|_M + 2\zeta_2^2 + 2\zeta_3^2)]$$

$$a_{23} = a_{32} = \lambda_2^{-1}\lambda_3^{-1}(\zeta_2\|C_2\|_M + \zeta_2^2 + \zeta_3\|C_3\|_M + 2\zeta_3^2)$$

$$a_{33} = 2[-1 + \lambda_3^{-2}(\zeta_1^2 + \zeta_2^2 + 2\zeta_3 + 2\zeta_3^2)] \quad (47)$$

where

$$\|C_1\|_M = \max_{\substack{t_k \in \mathscr{T} \\ \alpha_1 \in [0,K_1]}} \|C_1(t_k,\alpha_1)\|,$$

$$C_1(t_k,\alpha_1) = A_1(t_k) + \alpha_1b_1(t_k)c_1^T(t_k)$$

$$\|C_i\|_M = \max_{\alpha_i \in [K_{i1},K_{i2}]} \|C_i(\alpha_i)\|, \qquad C_i = A_i + \alpha_ib_ic_i^T, \quad \forall i = 2,3.$$

Applying Theorem 2 we conclude that the equilibrium $x = 0$ of large-scale system (S) with subsystems (38), (40) is uniformly absolutely connectively stable if elements $a_{ij} = a_{ij}(E_f)$ (47) of matrix $A(E_f)$ satisfy

$$a_{11} < 0, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0.$$

VII. Conclusion

Stability properties of discrete large-scale systems have been studied. It has been shown how both uniform asymptotic and uniform asymptotic connective stability of large-scale systems can be tested by using algebraic conditions that have been derived without assuming a special form of Lyapunov functions of subsystems. Results are valid for dynamic nonstationary nonlinear discrete large-scale systems. The stability properties of a system are ensured by the negative definiteness of a real constant symmetric matrix. The dimension of the matrix is equal to the number of the subsystems, which is the most important advantage of the use of the vector Lyapunov function concept and the decomposition principle. This advantage consists in reduction of the matrix order.

The obtained results are applied to a class of the systems, which has been broadened by proving a generalized Aizerman conjecture as a subsidiary result. Furthermore, the results enable to estimate the domain of uniform attraction and can be also used to study different types of practical stability. The main results of the paper broaden application of Popov-like frequency criteria to a wider class of the systems including such large-scale systems with subsystems whose Lyapunov functions are of the Lur'e type and which satisfy the criteria. When Lyapunov functions of the subsystems are of type "quadratic form plus integral of nonlinearity" then the application of the results is a matter of simple but long algebraic manipulations based on those of references [22], [23], [25].

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Comments on "On Arranging Elements of a Hierarchy in Graphic Form"

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1) In the above paper[1] Warfield states:

a) "In reviewing the literature related to hierarchies, it is rather surprising that it has not been possible to find a work that deals explicitly with methods for forming hierarchies.... [The methods] are not prominently displayed either in the theoretical works that discuss hierarchies or in papers that involve applications of hierarchical concepts."

b) "It seems appropriate to describe methods for arranging a hierarchy in graphic form."

2) As far as statement a) is concerned, it seems to us that the mathematical theory of partially ordered sets [1], [2] often deals with problems concerning hierarchies quite explicitly. In any case, this theory has been extensively employed to derive useful criteria for forming hierarchies. We have recently proposed an approach on the adequacy of models, based on hierarchical concepts [3], [4], and are at present working along the same line. We have also suggested applying these concepts to automated medical diagnoses [5] and are at present working, with similar techniques, on the problem of computer-aided design.

3) We agree with the statement b). We do, however, wish to recall that the problem of arranging hierarchies in graphic form has been extensively studied [1] (Hasse diagrams). We have recently proposed a modified diagram to better suit special display requirements [5].

4) In conclusion, we do not agree completely with statements a) and b). Nevertheless, the algorithms proposed by Warfield to graphically display ordered sets are certainly useful in practical applications.

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Author's Reply[2,3]

My interpretation of the correspondence of Reggiani and Marchetti is that it deals with the difference between analysis and synthesis. The theory of partially ordered sets and lattices is clearly relevant to the construction of hierarchies and multilevel graphs that involve feedback. I believe that this theory had not, until very recently, been translated into practical procedures whereby such structures can be systematically synthesized.

Many people are interested in hierarchies and multilevel systems who are in careers rather remote from the theory of

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