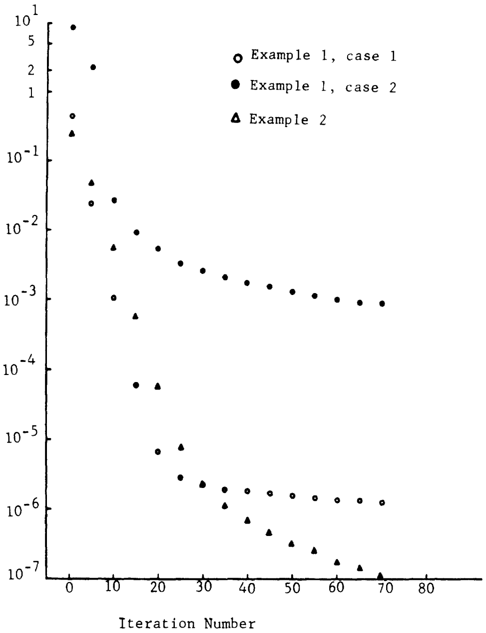

Fig. 3. Change of performance indexes in examples 1 and 2.

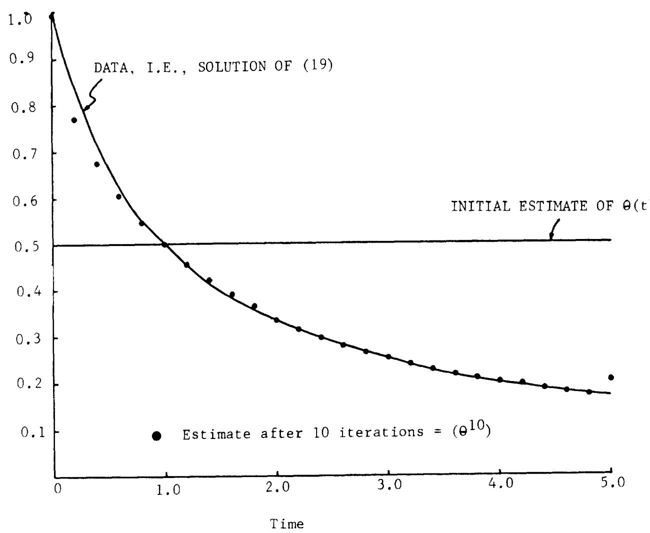

Fig. 4. Example 2; $Q(t) = 1$, $R(t) = 0.01$.

illustrate this viewpoint data were generated for the following system:

$$\dot{x} + x^2 = 0 \qquad x(0) = 1. \tag{19}$$

The solution $x(t)$ for $t \in [0,5]$ is plotted in Fig. 4.

Next it was assumed that the model was

$$\dot{y} + a(t)y = 0 \qquad y(0) = 1 \tag{20}$$

and the parameter $a(t)$ was identified starting with an initial estimate of $a(t) = 0.5$ for $t \in [0,5]$. The estimates of $a(t)$ after the tenth iteration are also shown in Fig. 3. It is next noted that after just ten iterations the final estimate of $a(t)$ and $x(t)$ for $t \in [0,5]$ are very similar. This suggests that the model should be

$$\dot{y} + y^2 = 0 \qquad y(0) = 1 \tag{21}$$

which is of the form of (19). Consequently, in this case it has been possible to predict the system structure based on the result of the parameter identification scheme.

V. Conclusions

A method for the identification of a large class of systems has been proposed. It has been assumed that the system may be modeled by an ordinary differential equation. The form of the equation must be known, although the structure of the parameter need not be known. Examples have illustrated the applicability of the procedure. It is emphasized that this procedure may be utilized as an aid in determining system structure.

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A Simple Partial Fraction Procedure

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***Abstract*—A simple procedure is presented for obtaining the partial fraction expansion of the ratio of two polynomials having real coefficients. The procedure avoids the use of complex numbers and appears to be more general and more accurate than other such procedures.**

Consider the polynomial ratio

$$P(s) = \frac{N(s)}{D(s)} = \frac{a_0 s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1}}{s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n} \tag{1}$$

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and suppose the denominator has been factored into first-order and second-order factors

$$D(s) = \prod_{i=1}^{m} (s + \lambda_i)^{r_i} \prod_{i=1}^{k} (s^2 + p_i s + q_i)^{t_i} \qquad (2)$$

where

$$\sum_{i=1}^{m} r_i + 2 \sum_{i=1}^{k} t_i = n.$$

It is desired to obtain a partial fraction expansion of $P(s)$ of the form

$$P(s) = \sum_{i=1}^{m} \sum_{j=1}^{r_i} \frac{K_{ij}}{(s + \lambda_i)^j} + \sum_{i=1}^{k} \sum_{j=1}^{t_i} \frac{A_{ij}s + B_{ij}}{(s^2 + p_i s + q_i)^j}. \qquad (3)$$

Since existing procedures are adequate for evaluating the K_{ij}, attention will be focused on the evaluation of the A_{ij} and B_{ij}. The procedure that is developed, however, is also applicable to the evaluation of the K_{ij} and in some instances may be simpler than other procedures.

Let $(s^2 + ps + q)$ be a second-order factor of $D(s)$ which is repeated t times. Then

$$P(s) = \frac{N(s)}{D_1(s)(s^2 + ps + q)^t} \qquad (4)$$

and the partial fraction expansion is

$$P(s) = S(s) + \frac{A_1 s + B_1}{(s^2 + ps + q)} + \frac{A_2 s + B_2}{(s^2 + ps + q)^2} + \cdots + \frac{A_t s + B_t}{(s^2 + ps + q)^t} \qquad (5)$$

where $S(s)$ is the sum of the terms of the partial fraction expansion associated with the other factors of $D(s)$. Now

$$D_1(s)|_{\mathrm{mod}\,(s^2+ps+q)} = c_0 s + c_1 \qquad (6)$$

and $N(s)$ can be expanded as

$$N(s) = (s^2 + ps + q)(a_0' s^{n-3} + a_1' s^{n-4} + \cdots + a_{n-4}' s + \beta_t) + (c_0 s + c_1)(\gamma_t s + \delta_t). \qquad (7)$$

Then, from (6) and (7),

$$(s^2 + ps + q)^t P(s)|_{s^2+ps+q=0} = \gamma_t s + \delta_t \qquad (8)$$

and by comparison with (5), it is obvious that $\gamma_t = A_t$ and $\delta_t = B_t$. Subtracting the partial fraction term just obtained, we have

$$P(s) - \frac{A_t s + B_t}{(s^2 + ps + q)^t} = \frac{N_{t-1}(s)}{D_1(s)(s^2 + ps + q)^{t-1}}$$

where

$$N_{t-1}(s) = \frac{N(s) - D_1(s)(A_t s + B_t)}{(s^2 + ps + q)}. \qquad (9)$$

Now expand $N_{t-1}(s)$ as

$$N_{t-1}(s) = (s^2 + ps + q)(a_0'' s^{n-5} + a_1'' s^{n-6} + \cdots + a_{n-6}'' s + \beta_{t-1}) + (c_0 s + c_1)(\gamma_{t-1} s + \delta_{t-1}). \qquad (10)$$

Again, by comparison with (5), it is clear that $\gamma_{t-1} = A_{t-1}$ and $\delta_{t-1} = B_{t-1}$. The recursive procedure for finding the A_i and B_i is now obvious.

1) Obtain $D_1(s)|_{\mathrm{Mod}\,(s^2+ps+q)} = c_0 s + c_1$.

2) Starting with $N_t(s) = N(s)$, repeat steps 3)–5) for $i = 0, 1, 2, \cdots, t-1$.

3) Divide $N_{t-i}(s)$ by $(s^2 + ps + q)$ to obtain

$$\frac{N_{t-i}(s)}{(s^2 + ps + q)} = (\alpha_{0_i} s^{n-3-2i} + \alpha_{1_i} s^{n-4-2i} + \cdots + \alpha_{(n-4-2i)_i} s) + \frac{d_{1_i} s^2 + d_{2_i} s + d_{3_i}}{s^2 + ps + q}. \qquad (11)$$

4) Determine A_{t-i} and B_{t-i} from

$$d_{1_i} = \beta_{t-i} + c_0 A_{t-i}$$
$$d_{2_i} = p\beta_{t-i} + c_1 A_{t-i} + c_0 B_{t-i}$$
$$d_{3_i} = q\beta_{t-i} + c_1 B_{t-i}.$$

These equations were obtained by equating the coefficients of the powers of s in (11) with the coefficients of the corresponding powers of s in (7), (10), and the similar equations representing the expansions of N_{t-i}, $i = 2, \cdots, t-1$.

5) For $i < t - 1$, obtain

$$N_{t-i-1}(s) = \frac{N_{t-i}(s) - D_1(s)(A_{t-i}s + B_{t-i})}{(s^2 + ps + q)}.$$

The procedure for evaluation of the k_{ij} is obvious by analogy.

Example

Consider a case of a repeated second-order factor:

$$P(s) = \frac{2s^5 + 6s^4 + 11s^3 + 12s^2 + 7s + 1}{s(s + 1)(s^2 + s + 1)^2}.$$

In this case,

$$D_1(s) = s^2 + s \qquad D_1(s)|_{\mathrm{mod}\,(s^2+s+1)} = -1$$

$$\frac{N_2(s)}{(s^2 + s + 1)} = 2s^3 + 4s^2 + 5s + \frac{3s^2 + 2s + 1}{(s^2 + s + 1)}.$$

The equations to be solved for A_2 and B_2 are then

$$3 = \beta_2 \qquad 2 = \beta_2 - A_2 \qquad 1 = \beta_2 - B_2$$

and the solution is $\beta_2 = 3$, $A_2 = 1$, and $B_2 = 2$. Then

$$N_1(s) = \frac{N(s) - (s^2 + s)(s + 2)}{s^2 + s + 1} = 2s^3 + 4s^2 + 4s + 1$$

$$\frac{N_1(s)}{s^2 + s + 1} = 2s + \frac{2s^2 + 2s + 1}{s^2 + s + 1}.$$

The equations to be solved for A_1 and B_1 are

$$2 = \beta_1 \qquad 2 = \beta_1 - A_1 \qquad 1 = \beta_1 - B_1$$

and the solution is $\beta_1 = 2$, $A_1 = 0$, and $B_1 = 1$. These results agree with the expansion of $P(s)$, which is

$$P(s) = \frac{1}{s} + \frac{1}{s + 1} + \frac{1}{s^2 + s + 1} + \frac{s + 2}{(s^2 + s + 1)^2}.$$

This example demonstrates the simplicity of the proposed procedure.

Note: The quantitative comparison of this and other procedures for partial fraction expansion remains to be performed.

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