[3] G. F. Groner, J. F. Heafner, and T. W. Robinson, "On-line computer classification of handprinted Chinese characters as a translation aid," *IEEE Trans. Electron. Comput.*, vol. EC-16, pp. 856–860, Dec. 1967.

[4] H. Terai and K. Nakata, "On-line, real-time recognition of handwriting Chinese characters and Japanese katakana sylabary," *Trans. Inst. Electron. Commun. Eng. Japan*, vol. 56-D, pp. 312–319, May 1973, in Japanese.

[5] H. Arakawa and I. Masuda, "On-line recognition of handprinted characters," *Trans. Inst. Electron. Commun. Eng. Japan*, vol. 59-D, pp. 809–816, Nov. 1978, in Japanese.

[6] G. M. Miller, "On-line recognition of hand-generated symbols," in *Proc. Fall Joint Computer Conference*, pp. 399–412, 1969.

[7] S. Fujiwara, Y. Tominaga, and T. Kiyono, "On-line recognition of handwritten characters by general comtour vector sequence method," *J. Inform. Process. Japan*, vol. 17, pp. 191–199, Mar. 1973, in Japanese.

[8] N. Aoki and S. Inokuchi, "On-line recognition of handwritten hirakana characters," *Trans. Inst. Electron. Commun. Eng. Japan*, vol. J60-D, July, 1977, in Japanese.

[9] K. Odaka, H. Arakawa, and I. Masuda, "On-line recognition of handwritten characters by approximating each stroke with several points," *Trans. Inst. Electron. Commun. Eng. Japan*, vol. 63-D, pp. 153–160, Feb. 1980, in Japanese.

[10] I. Masuda, "On-line recognition of hand-printed characters," *J. Inst. Electron. Commun. Eng. Japan*, vol. 61, pp. 125–128, Feb. 1978, in Japanese.

[11] G. F. Groner, "Real-time recognition of handprinted text," in *Proc. Fall Joint Computer Conf.*, pp. 591–601, 1966.

Linear Regression Analysis with Fuzzy Model

HIDEO TANAKA, MEMBER, IEEE, SATORU UEJIMA, AND KIYOJI ASAI

***Abstract*—In modeling some systems where human estimation is influential, we must deal with a fuzzy structure of the system considered. This structure is represented as a fuzzy linear function whose parameters are given by fuzzy sets. The fuzzy linear functions are defined by Zadeh's extension principle. Considering a fuzzy linear function as a model of fuzzy structure of the system, a fuzzy linear regression analysis is formulated. In the usual regression model, deviations between the observed values and the estimated values are supposed to be due to measurement errors. Here, on the contrary, these deviations are assumed to depend on the indefiniteness of the system structure. We regard these deviations as the fuzziness of the parameters of the system. Thus these deviations are reflected in a linear function with fuzzy parameters. As an example of this problem, the fuzzy linear model of the price mechanism of prefabricated houses is obtained. The fuzzy parameter of this model means a possibility distribution. The estimated values are obtained as fuzzy sets which represent the fuzziness of the system structure, while the conventional confidential interval is related to the observation errors. This fuzzy linear regression model might be very useful for finding a fuzzy structure in an evaluation system.**

I. INTRODUCTION

Fuzziness must be considered in systems where human estimation is influential. Since we deal with the phenomenon originated from a fuzzy structure, a model of such a vague phenomenon might be represented as a fuzzy system equation which can be described by the fuzzy functions [1]–[4] defined by Zadeh's extension principle. The idea that a vague phenomenon should be identified by a fuzzy function is natural.

Manuscript received September 16, 1981; revised April 15, 1982.

H. Tanaka and K. Asai are with the Department of Industrial Engineering, College of Engineering, University of Osaka Prefecture, Umemachi 4-804, Sakai, Osaka 591, Japan.

S. Uejima is with Department of Research and Development, 4th Division of Fundamental Software, Nippon Electric Corporation, Nishincho 1-10, Fuchu 183, Japan.

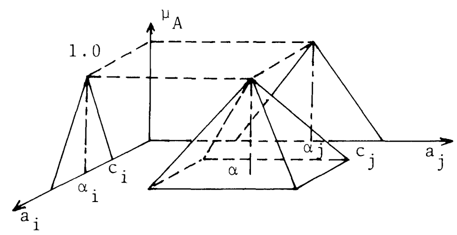

Fig. 1. Fuzzy set of parameter A: $A \triangleq$ "approximate α."

In this correspondence, we are concerned with the application of fuzzy linear function to a regression analysis in a vague phenomenon. In the usual regression model, deviations between the observed values and the estimated values are supposed to be due to observation errors. We assume, on the contrary, that these deviations depend on the indefiniteness of the system structure. We regard these deviations as the fuzziness of system parameters. Thus they are reflected in a fuzzy linear function which represents a vague phenomenon. As an example of this problem, we have obtained the fuzzy linear model of a price mechanism of prefabricated houses. The fuzzy parameter of the linear model obtained means a possibility distribution which corresponds to the fuzziness of the system. The fuzzy parameters studied in this correspondence are restricted to a class of "triangular" membership functions. This fuzzy regression model might be very useful for finding a fuzzy structure in an evaluation system.

II. FUZZY FUNCTION WITH FUZZY PARAMETERS

Let us consider two sets X and Y and a function $f(x, a)$ which is a mapping from X into Y. If parameters are given by fuzzy sets A, the function is called a fuzzy function, denoted by $f(x, A)$. When an x is given, the fuzzy set $Y = f(x, A)$ mapped from the fuzzy set A can be defined as follows.

Definition 1: The fuzzy function is denoted by

$$f: X \rightarrow \mathcal{F}(y); \quad Y = f(x, A) \tag{1}$$

where $\mathcal{F}(y)$ is the set of all fuzzy subsets on Y. The fuzzy set Y is defined by the membership function

$$\mu_Y(y) = \begin{cases} \max\limits_{\{a \mid y = f(x,a)\}} \mu_A(a), & \{a \mid y = f(x,a)\} \neq \varnothing \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where A is a fuzzy set on the product space of parameters whose membership function is denoted by $\mu_A(a)$.

Definition 1 is a natural extension of the concept of mapping sets. Given x, a fuzzy set A is mapped into Y, and its image of A is given by Definition 1. The fuzzy parameters considered here are assumed limited to the type of the following fuzzy sets.

Definition 2: Fuzzy parameters are defined by the fuzzy sets as illustrated in Fig. 1. This fuzzy set can be represented as

$$\mu_A(a) = \min_j \left[\mu_{A_j}(a_j)\right] \tag{3}$$

$$\mu_{A_j}(a_j) = \begin{cases} 1 - \dfrac{|\alpha_j - a_j|}{c_j}, & \alpha_j - c_j \leqslant a_j \leqslant \alpha_j + c_j \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

where $c_j > 0$.

The fuzzy parameter of Definition 2 means "approximate α," described by the center α and the width c. Hence fuzzy parameters $A = (A_1, \cdots, A_n)$ can be denoted in the vector form of

$$A = \{\alpha, c\}, \quad \alpha = (\alpha_1, \cdots, \alpha_N)^t, \quad c = (c_1, \cdots, c_n)^t. \tag{5}$$

0018-9472/82/1100-0903$00.75 ©1982 IEEE

TABLE I
INPUT–OUTPUT DATA

Sample number	Output y	Inputs x
1	y_1	$x_{11},\cdots,x_{1n}$
.	.	.
.	.	.
.	.	.
N	y_N	$x_{N1},\cdots,x_{Nn}$

Note that a fuzzy set A_i is defined on the real line R^1, and A defined on the vector space R^n can be considered a Cartesian product.

To understand Definitions 1 and 2, consider the following fuzzy linear function

$$Y = A_1x_1 + A_2x_2 \tag{6}$$

where $A = \{(1,4),(2,1)\}$. Given $x = (1,1)^t$, the fuzzy set $Y = (5,3)$ is obtained by these definitions. This example means that "approximate **1**" plus "approximate **4**" is "approximate **5**". This idea is generalized to get the following proposition.

Proposition 1: Given fuzzy parameter $A = \{\alpha, c\}$, the fuzzy linear function

$$Y = A_1x_1 + \cdots + A_nx_n = Ax \tag{7}$$

is obtained as the following membership function:

$$\mu_Y(y) = \begin{cases} 1 - \dfrac{|y - x^t\alpha|}{c^t|x|}, & x \neq 0 \\ 1, & x = 0,\ y = 0 \\ 0, & x = 0,\ y \neq 0 \end{cases} \tag{8}$$

where $|x| = (|x_1|,\cdots,|x_n|)^t$ and $\mu_Y(y) = 0$, when $c^t|x| \leqslant |y - x^t\alpha|$. The proof of this proposition is given in the Appendix.

III. FORMULATION OF THE FUZZY LINEAR REGRESSION MODEL

A regression problem has two basic aspects to be dealt with: 1) what is the most appropriate mathematical model, and 2) how do we determine the best fitting model for the data shown in Table I. Here y_i is called an output or an observation for the ith sample, and x_{ij} is called a jth input or a jth independent variable for the ith sample.

In this correspondence we confine ourselves to the linear regression problem. Letting the linear regression model be $y = \alpha^t x$, the deviation between the observed value and the estimated value $y_i^* = \alpha^t x_i$

$$y_i - y_i^* = \epsilon_i, \qquad i = 1,\cdots,N \tag{9}$$

is generally regarded as the observation error ϵ_i which is a random variable with zero mean. On the contrary, we assume that the deviations depend on the fuzziness of the system structure. In other words, the deviations are closely related to fuzziness of system parameters rather than observation errors.

With the above view, we consider a fuzzy linear function

$$Y = A_1x_1 + \cdots + A_nx_1 = Ax \tag{10}$$

where A_i is a fuzzy set defined by Definition 2. When we have nonfuzzy data in Table I, the problem of the fuzzy linear regression model is to determine fuzzy parameters A^* such that the fuzzy output set $Y_i^* = A^*x_i$ contains y_i with more than h degree (Fig. 2). The dispersion of data represents the fuzziness of the phenomenon underlying a model. We consider our data as input–output relations whose vagueness is derived from the existence of fuzzy parameters. In our model, the deviations among data are explained as the vagueness of the system structure expressed by fuzzy parameters.

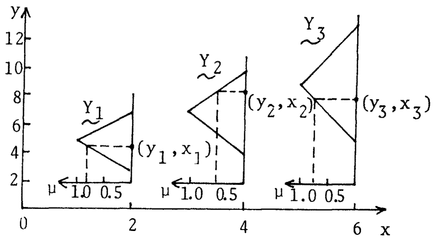

Fig. 2. Explanation of fuzzy linear regression model $Y_i = A_0 + A_1x_i$; $A_0 = (3,1)$, $A_1 = (1,0.5)$, $h = 0.5$.

TABLE II
INPUT–FUZZY OUTPUT DATA

Sample number	Fuzzy Output Y	Inputs x
1	$Y_1 = (y_1, e_1)$	$x_{11},\cdots,x_{1n}$
.	.	.
.	.	.
.	.	.
N	$Y_N = (y_N, e_N)$	$x_{n1},\cdots,x_{Nn}$

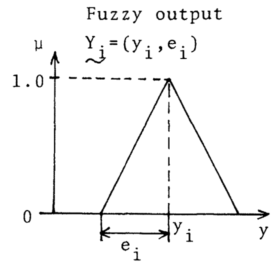

Fig. 3. Fuzzy output.

For the nonfuzzy input–output data such as those in Table I, we have already formulated the fuzzy linear regression model [5]. We deal with fuzzy output data denoted by $Y_i = (y_i, e_i)$, where y_i is a center and e_i is a width. The input–output data are shown in Table II and Fig. 3. The membership function of Y_i is given by

$$\mu_{Y_i}(y) = 1 - \frac{|y_i - y|}{e_i}. \tag{11}$$

To formulate a fuzzy linear regression model, the following are assumed to hold.

1) The data can be represented by a fuzzy linear model:

$$Y_i^* = A_1^*x_{i1} + \cdots + A_n^*x_{in} \triangleq A^*x_i, \tag{12}$$

where the type of fuzzy parameter A_i is given by Definition 2. Given x_i, Y_i^* can be obtained from Proposition 1 as

$$\mu_{Y_i^*}(y) = 1 - \frac{|y_i - x_i^t\alpha|}{c^t|x_i|}. \tag{13}$$

2) The degree of the fitting of the estimated fuzzy linear model $Y_i^* = A^*x_i$ to the given data $Y_i = (y_i, e_i)$ is measured by the following index $\bar{h}_i$ which maximizes h subject to $Y_i^h \subset Y_i^{*h}$, where

$$Y_i^h = \{y \mid \mu_{Y_i}(y) \geqslant h\}$$
$$Y_i^{*h} = \{y \mid \mu_{Y_i^*}(y) \geqslant h\} \tag{14}$$

which are h-level sets. This index $\bar{h}_i$ is illustrated in Fig. 4. The degree of the fitting of the fuzzy linear model to all data $Y_1,\cdots,Y_N$ is defined by $\min_j[\bar{h}_j]$.

3) The vagueness of the fuzzy linear model is defined by

$$J = c_1 + \cdots + c_n. \tag{15}$$

The problem is explained as obtaining fuzzy parameters A_i^* which minimize J subject to $\bar{h}_i \geqslant H$ for all i, where H is chosen

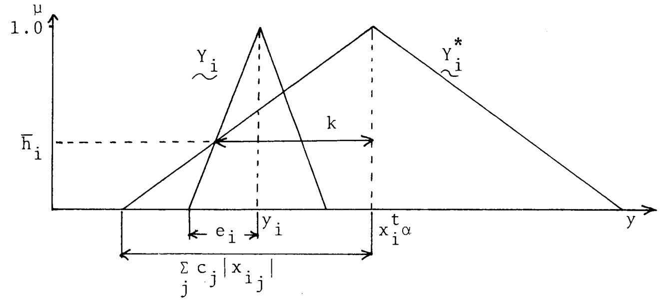

Fig. 4. Degree of fitting of Y_i^* to given fuzzy data Y_i.

as the degree of the fitting of the fuzzy linear model by the decisionmaker. The $\bar{h}_i$ can be obtained as

$$\bar{h}_i = 1 - \frac{|y_i - x_i^t\alpha|}{\sum_j c_j|x_{ij}| - e_i}. \tag{16}$$

This is derived from the following relation in Fig. 4:

$$1 : (1 - \bar{h}_i) = \left(\sum_j c_j|x_{ij}|\right) : k \tag{17}$$

where

$$k = |y_i - x'\alpha| + e_i(1 - \bar{h}_i). \tag{18}$$

More specifically, our problem is to find out the fuzzy parameters $A_i^* = (\alpha_i, c_i)$ which are the solution of the following linear programming problem:

$$\min_{\alpha, c} J = c_1 + \cdots + c_n$$

subject to $c \geqslant 0$ and

$$\alpha^t x_i + (1 - H)\sum_j c_j|x_{ij}| \geqslant y_i + (1 - H)e_i$$

$$-\alpha^t x_i + (1 - H)\sum_j c_j|x_{ij}| \geqslant -y_i + (1 - H)e_i,$$

$$i = 1, \cdots, N. \tag{19}$$

We can obtain the best fitting model for the given data by solving the conventional linear programming problem (19). In general, the number of constraints $2N$ is much larger than the number of variables n. Therefore, solving the dual problem of (19) is easy compared with solving the primal problem (19). Since the variables α are not necessarily nonnegative, the new variables $\alpha' \geqslant 0$ are introduced to have

$$\alpha = \alpha' + d, \tag{20}$$

where $d^t = (d, \cdots, d)$ and d is assumed to be a sufficiently small negative number so that the variables α' are always positive.

Using $\beta = (\beta_1, \cdots, \beta_N)^t$ and $D = (D_1, \cdots, D_N)^t$ as the dual variables, the dual problem of (19) can be written in the form of

$$\left.\begin{aligned} &\max_{\beta, D} J = \sum_{i=1}^{N} \beta_i \left[y_i + (1 - H)e_i - d\sum_{j=1}^{n} x_{ij} \right] \\ &\qquad - \sum_{i=1}^{N} D_i \left[y_i - (1 - H)e_i - d\sum_{j=1}^{n} x_{ij} \right] \\ &\text{subject to } \beta \geqslant 0, D \geqslant 0 \text{ and} \\ &(1 - H)\sum_{i=1}^{N} \beta_i|x_{ij}| + (1 - H)\sum_{i=1}^{N} D_i|x_{ij}| \leqslant 1 \\ &\sum_{i=1}^{N} \beta_i x_{ij} - \sum_{i=1}^{N} D_i x_{ij} \leqslant 0, \qquad j = 1, \ldots, n. \end{aligned}\right\} \tag{21}$$

TABLE III
INPUT–OUTPUT DATA CONCERNING HOUSE PRICES

NO.	$Y = (y, e)$	x_0	x_1	x_2	x_3	x_4	x_5
1	(6060 , 550)	1	1	38.09	36.43	5	1
2	(7100 , 50)	1	1	62.10	26.50	6	1
3	(8080 , 400)	1	1	63.76	44.71	7	1
4	(8260 , 150)	1	1	74.52	38.09	8	1
5	(8650 , 750)	1	1	75.38	41.40	7	2
6	(8520 , 450)	1	2	52.99	26.49	4	2
7	(9170 , 700)	1	2	62.93	26.49	5	2
8	(10310 , 200)	1	2	72.04	33.12	6	3
9	(10920 , 600)	1	2	76.12	43.06	7	2
10	(12030 , 100)	1	2	90.26	42.64	7	2
11	(13940 , 350)	1	3	85.70	31.33	6	3
12	(14200 , 250)	1	3	95.27	27.64	6	3
13	(16010 , 300)	1	3	105.98	27.64	6	3
14	(16320 , 500)	1	3	79.25	66.81	6	3
15	(16990 , 650)	1	3	120.50	32.25	6	3

Y_i: ith fuzzy house price (1000 yen). x_0: constant. x_1: rank of material. x_2: first floor space (m^2). x_3: second floor space (m^2). x_4: number of rooms. x_5: number of Japanese-style rooms.

TABLE IV
FUZZY PARAMETER A^* ($H = 0.5$)

Fuzzy Parameter	A_0^*	A_1^*	A_2^*	A_3^*	A_4^*	A_5^*
Center α_i	11040	1810	2140	870	-540	-180
Width α_i	820	0	370	0	0	0

Since this dual problem diminishes the number of constraints because of $N \geqslant n$, solving (21) is more efficient than solving (19).

IV. APPLICATION AND DISCUSSION

This fuzzy linear regression model is applied to the price mechanism of prefabricated houses. The input–output data shown in Table III, except e_i, are obtained from the catalogue issued by some corporation. The values of e_i are assumed by the authors. In the column x_1 of Table III, three ranks of material are indicated: low grade (1), medium grade (2), and high grade (3). From these data, we decide the fuzzy linear system $Y = A_0 + A_1x_1 + \cdots + A_5x_5$, which is the best fitting model for the data given.

The results of fuzzy parameters A are given in Table IV, where $H = 0.5$. α_i denotes a center of a fuzzy parameter, and c_i shows fuzziness of its parameter. The calculations to obtain the fuzzy parameters A have been done by transforming the original data into deviations from the mean of each of the variables x except for fuzzy output data Y. The estimated values $Y_2^*, Y_4^*, \cdots, Y_{14}^*$ are shown in Fig. 5. For the example of number ten, the estimated value is obtained as $Y_{10}^* = (12464, 1056)$ while the given data is $Y_{10} = (12030, 100)$. Note that the solution satisfies the relation $Y_i^h \subset Y_i^{*h}$ with $H = 0.5$. The following results are reached.

1) The fuzzy mean value of house prices can be explained by $A_0 = (11040, 820)$.

2) The vagueness of house price can be represented as the fuzziness of the constant parameter A_0 and first floor space A_2. By this fuzziness of parameters the dispersion of the given data can be explained. In this case, the fuzziness of A is $J = 1190$.

3) The fact that A_4 and A_5 are negative, depends on the strong correlations between variables x_4 and x_5. In the case of a fixed floor space, the larger the number of rooms, the lower the price, since the small rooms diminish the price.

Last, let us describe the characteristics of the fuzzy linear regression model compared with the conventional one. Since the estimated price of a new model can be obtained as a fuzzy set, the decisionmaker can choose a price out of the estimated fuzzy set at his disposal. As regards the width of fuzzy set, it is evident from Fig. 2 that the larger the x, the larger the width of $Y = A_0 + A_1x$. Letting Y be a fuzzy price, a high price has more fuzziness than a low price. For example, suppose that the fuzziness is about ten percent of the price. Then the price of 100 yen has the fuzziness of approximately 10 yen, and 10 000 yen has the

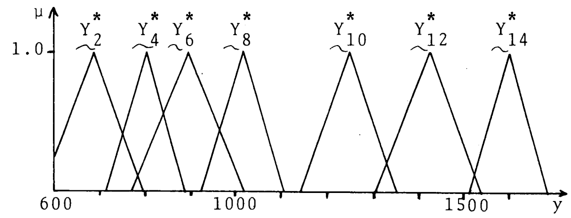

Fig. 5. Estimation Y_i^* obtained from fuzzy regression model, $i = 2, 4, \cdots, 14$.

fuzziness of approximately 100 yen. The fuzzy linear model can explain the above fuzziness. This is an essential point of departure from the conventional regression model. The confidential interval in the conventional regression model seems to estimate the upper and lower limits of the observation errors.

V. Concluding Remarks

We have applied the fuzzy linear function to the linear regression model and established a method of system representation for a vague phenomenon. As shown in the example, the vagueness of the evaluation system is represented by the fuzzy linear function. If nonfuzzy output data are given, the same procedure is available with $e_i = 0$. The fuzzy function with fuzzy parameters might be very widely applied in various fields, as described in [6]–[9]. Although only one type of fuzzy sets is discussed here, another type can be treated by the same approach.

Appendix

Proof of Proposition 1

We prove Proposition 1 in the case of $x \neq 0$, since the other cases are self-evident. It follows from (1)–(4) that

$$\mu_Y(y) = \max_{\left\{ \begin{array}{c} t \\ a_3 \\ \vdots \\ a_n \end{array} \middle| \; y = t + \sum_{j=3}^n a_j x_j \right\}} \left[\min_{j \in \{3, \cdots, n\}} \left[1 - \frac{|\alpha_j - a_j|}{c_j} \right] \Lambda \mu_T(t) \right] \tag{22}$$

where

$$\begin{aligned} \mu_T(t) &= \max_{\{a_1, a_2 | t = a_1 x_1 + a_2 x_2\}} \left[\left(1 - \frac{\alpha_1 - a_1}{c_1} \right) \Lambda \left(1 - \frac{\alpha_2 - a_2}{c_2} \right) \right] \\ &= \max_{a_2} \left[\left(1 - \frac{|\alpha_1 x_1 + a_2 x_2 - t|}{c_1 |x_1|} \right) \Lambda \left(1 - \frac{|\alpha_2 - a_2|}{c_2} \right) \right]. \end{aligned} \tag{23}$$

The fuzzy set T is shown in Fig. 6 where it is clear that $\mu_T(t) = h$. It follows from Fig. 6 that

$$1 : (1 - h) = c_1 \frac{|x_1|}{|x_2|} : \left(\frac{|t - \alpha_1 x_1 - \alpha_2 x_2|}{|x_2|} - c_2 (1 - h) \right). \tag{24}$$

Then we have

$$h = 1 - \frac{\left| t - \sum_{j=1}^{2} \alpha_j x_j \right|}{\sum_{j=1}^{2} c_j |x_j|}. \tag{25}$$

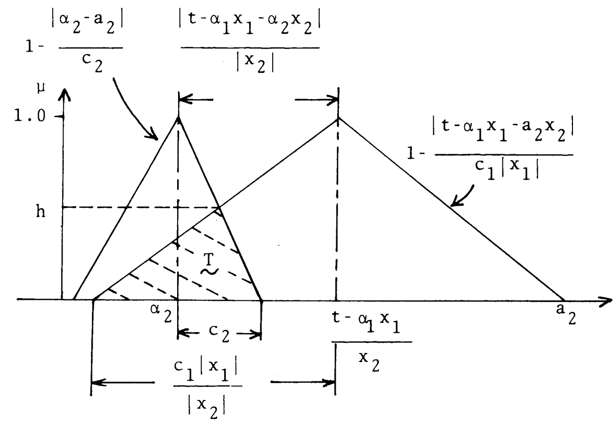

Fig. 6. Explanation for calculating $\mu_T(t)$ (fuzzy set T).

Hence (22) can be rewritten in the form

$$\mu_Y(y) = \max_{\left\{ \begin{array}{c} t' \\ a_4 \\ \vdots \\ a_n \end{array} \middle| \; y = t' + \sum_{j=4}^n a_j x_j \right\}} \cdot \left[\min_{j \in \{4, \cdots, n\}} \left(1 - \frac{|\alpha_j - a_j|}{c_j} \right) \Lambda \max_{\left\{ \begin{array}{c} t \\ a_3 \end{array} \middle| \; t' = t + a_3 x_3 \right\}} \cdot \left(1 - \frac{|\alpha_3 - a_3|}{c_3} \right) \Lambda \left(1 - \frac{\left| t' - \sum_{j=1}^{2} \alpha_j x_j \right|}{\sum_{j=1}^{2} c_j |x_j|} \right) \right]. \tag{26}$$

Applying just the same operation to (26), we have

$$\mu_Y(y) = \max_{\left\{ \begin{array}{c} t' \\ a_4 \\ \vdots \\ a_n \end{array} \middle| \; y = t' + \sum_{j=4}^n a_j x_j \right\}} \cdot \left[\min_{j \in \{4, \cdots, n\}} \left(1 - \frac{|\alpha_j - a_j|}{c_j} \right) \Lambda \left(1 - \frac{\left| t' - \sum_{j=1}^{3} \alpha_j x_j \right|}{\sum_{j=1}^{2} c_j |x_j|} \right) \right]. \tag{27}$$

Repeating this procedure, we have

$$\mu_Y(y) = 1 - \frac{\left| y - \sum_{j=1}^{n} \alpha_j x_j \right|}{\sum_{j=1}^{n} c_j |x_j|}. \tag{28}$$

References

[1] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning—I," *Inform. Sci.*, vol. 8, pp. 199–249, 1975.

[2] C. V. Negoita and D. A. Ralescu, *Application of Fuzzy Sets to Systems Analysis*. Birkhauser Verlag, 1975, pp. 12–24.

[3] R. R. Yager, "On solving fuzzy mathematical relationships," *Inform. Contr.*, vol. 41, pp. 29–55, 1979.
[4] D. Dubois and H. Prade, "Operations on fuzzy number," *Int. J. Syst. Sci.*, vol. 9, pp. 613–626, 1978.
[5] H. Tanaka, S. Uejima, and K. Asai, "Fuzzy linear regression model," presented at the Int. Congr. on Applied Systems Research and Cybernetics, Acapulco, Mexico, Dec. 12–15, 1980.
[6] D. Dubois and H. Prade, "Systems of linear fuzzy constraints," *Fuzzy Sets Syst.*, vol. 3, pp. 37–48, 1980.
[7] H. Prade, "Operations research with fuzzy data," in *Fuzzy Sets Theory and Applications to Policy Analysis and Information Systems*, P. P. Wang and S. K. Chang, Eds. New York: Plenum, 1980, pp. 155–170.
[8] H. Tanaka and K. Asai, "Fuzzy linear programming based on fuzzy functions," presented at the IFAC 8th Triennial World Congr., Kyoto, Japan, Aug. 24–28, 1981.
[9] H. Tanaka and K. Asai, "Fuzzy linear programming with fuzzy parameters," presented at the Int. Conf. on Policy Analysis and Information Systems, Taiwan, Aug. 19–22, 1981.

A Neural Network Model of Pain Mechanisms: Functional Properties of the Network Cells Providing the Power Law

HARUYUKI MINAMITANI

Abstract—**Model simulation of a neural network providing the conduction mechanisms of pain and tactile sensations was undertaken, and the functional relations between neural activities of the network cells and stimulus strength applied on peripheral receptive fields were obtained. The functional relations of thalamocortical cells fall close to a straight line indicating a simple power law. The exponents of the power function are more than three for pain perceptive neural cells, while those for tactile sensation are about 1.5, which agrees well with the counterpart physiological data. The results are instructive for covering the gap between physiologically and psychophysically investigated somatosensory functions.**

I. Introduction

Studies elucidating somatosensory mechanisms are undertaken by two fundamental means. The electrophysiological method is the proper way to investigate neural receptive fields and neural pathways of somatic sensations such as touch and pain. On the other hand, the psychophysical method has served to obtain the subjective functions between any applied stimulus and the consequent sensibility perceived in human subjects, which present no inner mechanism of neural connections. Boundary studies covering the gap between physiological and psychological observations have not been largely reported, except to correlate the psychophysical power law with primary afferent or central neural activities [1]–[7]. Although some models and theories concerned with pain and tactile sensations have been proposed in conformity with the physiological aspects, none of them deal with the mathematical analysis and the synthesis as well. Model simulation may be expected to establish quantitatively the interrelation of both studies, viz., the numerical description connecting the psychophysical sensory mechanisms with the counterpart physiological evidence.

In a previous paper [8], a neural network model of pain mechanisms was proposed, and the model simulation was carried out to investigate inner mechanisms essential for pain and tactile sensations. The modeling was based on various assumptions drawn from the results of physiological and anatomical studies in the literature [8]. The overall simplified model was constructed of 18 neural units, three afferent neural fibers (A_β, A_δ, and C fibers), six spinal cord cells, three brainstem cells, three thalamic cells, and three cortical cells, the response activities of which are provided by solving 18 simultaneous differential equations with two sets of linear differential equations for the adaptation effect of peripheral afferents. Thalamic ventral posterolateral nuclei (VPL) on the dorsal column-medial lemniscus tract (DCLT), being organized as a large myelinated A_β fiber and rapid tempo system, serves tactile sensation and terminates the axons in the first and second somatic sensory area (SI, SII) of the cerebral cortex. On the other hand, the extralemniscal reticular tract (ELRT) system plays an essential role in protopathic sensory modalities such as pain and thermal sensation, the evoked responses of which are transmitted by small A_δ and C fibers. The centromedian parafascicular complex (CM-Pf) on ELRT serves the perception of slow burning pain and projects directly as well as indirectly on the associated area with SI and SII in the celebral cortex (H). Additionally, the thalamic posterior nuclei group (PO) on the neospinothalamic tract (NSTT) plays an important role in pain as well as mechano-perception, of which information are mainly transmitted by small myelinated A_δ fibers. PO projects the information on the somatic sensory area of celebral cortex, probably on SII and partly on SI. The temporal modality of the neural responses mimics the somatic sensation as follows.

Fast elevation of the initial bursts of VPL and PO cells are elicited by the A_β fibers' facilitation but are decreased drastically by the adaptation of A_β fibers, inhibition from the adjacent neural cells, and the negative feedback from the upper brain. This modality at the *initial phase* ($\sim$ 50 ms) seems to be the warning signal for the noxious stimuli succeeding the rapid reflex withdrawal movement. After the initial phase, activities of VPL and PO cells are again facilitated by the succeeding impulses transmitted on A_β and A_δ fibers. Particularly, the firing rate of PO cells increases up to the second maximum at about 75 ms, which suggests that the fast stinging pain may be evoked in PO cells and projected on the celebral cortex. When low stimulus intensity is applied, the firing rate of PO cells does not increase, and no fast pain but only tactile sensation occurs at this *second phase* (50–250 ms). If stimulus intensity is considerably high, CM-Pf cells are also facilitated by the transmitted impulses of C fibers at the *third phase* (250 ms $\sim$) and the sensation of slow burning pain occurs and is projected on the cortical H and SII cells. Therefore, pain sensibility can be estimated by the firing activities of PO and CM-Pf cells as well as cortical SII and H cells, while tactile sensibility is estimated by those of VPL and cortical SI cells.

Corresponding to a previous paper [8], the response characteristics of the neural cells to stimulus intensity and the frequency of repetitive pulse sequences are investigated. Some stimulus–response relations characterized by the Stevens' power law are obtained from the specific features of temporal modality and compared with those of physiological data. As can be seen, this model simulation provides valuable information for elucidating the central neural mechanisms for sensory magnitude estimation, which may in turn present the analytical cue to discussing the physiological correlates of psychophysical functions. Computer simulation of the overall model has been made on a HITAC 8700/8800 digital computer (Hitachi Corp.), using the fourth-order Runge–Kutta method to solve the nonlinear differential neural equations. The simulation program has been written in Fortran. Parameter determination has been carried out on a trial-and-error basis by using the iterative method of model simulation. The precise procedure and the parameters have been presented in [8].

Manuscript received December 1, 1981; revised April 13, 1982.
The author is with the Department of Electrical Engineering, Biomedical Engineering Division, Faculty of Science and Technology, Keio University, Hiyoshi 3-14-1, Kohoku-ku, Yokohama 223, Japan.