TABLE II
ENTROPY OF PICTURES FOR DIFFERENT VALUES OF FUZZIFIERS

PICTURE X	ENTROPY H(X)					
	$F_e=2$			$F_e=3$		
	$F_d=20$	$F_d=30$	$F_d=40$	$F_d=20$	$F_d=30$	$F_d=40$
Fig.4	0.749	0.896	0.963	0.474	0.679	0.812
Fig.6(a)	0.639	0.778	0.854	0.398	0.561	0.682
Fig.6(b)	0.642	0.783	0.860	0.399	0.564	0.686
Fig.6(c)	0.640	0.783	0.862	0.395	0.562	0.685
Fig.7(b)	0.644	0.785	0.862	0.400	0.566	0.688
Fig.8(b)	0.639	0.782	0.861	0.393	0.560	0.683
Fig.10	0.825	0.867	0.864	0.681	0.793	0.842

(Fig. 4), a large number of levels near the crossover points and it is these levels which cause an increase in $(p_{mn} \cap \bar{p}_{mn})$ value. But the case is different for $F_e = 2$ and $F_d = 40$, where the crossover point becomes lower than all the others and the number of pixels having intensity below this point therefore becomes smaller than that in the input picture. The index value is thus decreased. Outputs in Fig. 9 do possess a minimum γ_1 value due to the T_3 operation, which reduces the ambiguity by further increasing/decreasing the property values which are greater/smaller than 0.5.

In a part of the experiment, these γ values were compared with those of "entropy" $H(X)$ (14) of the pictures. Table II shows the H values for some of the images (as typical cases for illustration) with the same values of F_e and F_d as used for $\gamma_1(X)$. The nature of variation of entropy with F_e and F_d is seen to conform to that of the linear index of fuzziness; only the effective values are larger.

VII. Conclusion

The concept of the fuzzy set is found to be applied successfully to the problems of grey-tone image enhancement. The addition of a smoothing algorithm between primary and final enhancement operations resulted in an improved performance. The three different smoothing techniques considered here are defocussing, averaging, and max–min rule over the neighbors of a pixel. All these techniques are seen to be almost equally effective (as measured by the amount of fuzziness present) in enhancing the image quality. The performance of this system in enhancing an image is also compared with that of the histogram equalization technique, an existing method and is seen to be much better as far as ambiguity is concerned. The linear index of fuzziness $\gamma_1(X)$ and entropy $H(X)$ of an image reflect a kind of quantitative measure of its quality and are seen to be reduced with enhancement. The amount of ambiguity is found to be minimum when the T_3 rule is adopted in the enhancement algorithm. $H(X)$ provides higher effective values of fuzziness as compared to $\gamma_1(X)$ but the nature of their variation among the different images with respect to F_e and F_d is identical.

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The Weibull Distribution as a Human Performance Descriptor

GAYLE L. BERRY

Abstract—**Results which support the contention that the Weibull distribution is a better fit to human task performance times than the Gaussian are shown. A method for estimating the Weibull parameters is shown to be accurate.**

Introduction

There are many situations in which it is necessary to predict the performance of people doing complex tasks in human–machine systems. Control of vehicles (aircraft, ships, space shuttle, etc.), control of processes (refineries, nuclear power stations, chemical plants, etc.), and communications, command, control, and intelligence (C^3I) situations (tactical or strategic) are some that come to mind which have in common that consequences of incorrect or untimely performances can be disastrous. To be able to predict the likelihood of correct and timely human actions in these and many more mundane situations would assist in the design of systems which would increase the probability of successful operation.

Computer simulation of human–machine systems is one valuable way of gaining insight into the performance of the entire system and the interactions of the human and machine components within the system. Monte Carlo simulations, however, require certain types of data to be input, one of which is information about the distribution of performance times for the individual tasks which comprise the network of interaction. Clearly, the probability distribution used to model individual task performance times is crucial to this method. There is little point in acquiring (at significant expense) accurate data about performance times if it is only used to derive parameters for an

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incorrect distribution. What little evidence exists indicates that the Gaussian (or normal) distribution is not the correct distribution for modeling task performance time despite its common use and simplicity.

Background

Individual researchers have generally acquired empirical data on human performance times where possible, and made whatever approximations that could best be made where no data were available. In either case, some assumption was made about the nature of the distribution of the data; generally, that it was Gaussian. Other choices, of lower popularity in simulation practice, have been the uniform, triangular, and log normal distributions [1], [2].

A uniform distribution is obviously not appropriate for modeling the variability of task performance times, but has the advantage of simplicity. A triangular distribution, formed from estimates of the most likely, minimum, and maximum performance times is a step above the uniform distribution. It is still quite simple but provides a better representation of performance time variability. The log-normal distribution can represent skewness and has been used to model performance times for repair tasks [1]. For larger or composite tasks such as repair tasks, then, the shape of the log normal is not so affected by extremely long, though uncommon performance times as the Gaussian or triangular distributions would be. In the case of elemental tasks (of which composite tasks such as repair tasks are composed) performed by trained personnel, however, such outliers are not common.

Despite the importance of accurately representing task performance times in Monte Carlo simulation, little empirical information is available to guide the choice from among these and other distributions. Although the Gaussian distribution is still widely used to describe human task performance times, its applicability has been questioned for some time [2], [3]. Mills and Hatfield [3] investigated three probability density functions for describing human task performance times. Examining data from six independent tasks and a larger task constructed from the same six elements, they concluded that the normality (Gaussian) assumption should be rejected for describing human task performance times. Furthermore, of the three probability density functions tested, the Weibull distribution produced the best overall results.

Use of the Weibull distribution as a general assumption for human task performance time description would be more appropriate, according to Mills and Hatfield, than the Gaussian. However in many situations empirical data are not available from which to determine the appropriate Weibull parameters. Instead, only the mean and standard deviation (or expert judgments of the mean and standard deviation) are known. By these two parameters, the Gaussian distribution is defined; the Weibull is not. It is necessary to be able to conveniently estimate the Weibull parameters (α and λ) from mean and standard deviation estimates, in order for the Weibull distribution to come into wider use.

Weibull Distribution

The Weibull two-parameter probability distribution function is given by

$$f(t) = \alpha\lambda t^{\alpha-1}e^{-\lambda t^{\alpha}}, \qquad \alpha > 0,\ \lambda > 0,\ t > 0. \tag{1}$$

The value of α is a "shape" parameter, and λ is a "scale" parameter.

If empirical task performance time data are available, an iterative procedure defined from the maximum likelihood method [3] can be employed to determine α and λ. In this procedure, α and λ are estimated by

$$\hat{\lambda} = \frac{n}{\sum t_i^{\hat{\alpha}}} \tag{2}$$

and

$$\hat{\alpha} = \frac{n}{\hat{\lambda}\sum t_i^{\hat{\alpha}}\ln t_i - \sum \ln t_i} \tag{3}$$

where n is the number of observations of the performance time for the task, and the t_i are the observed performance times.

Given an initial value of $\hat{\alpha}$, $\hat{\lambda}$ can be computed from (2). $\hat{\alpha}$ is then recomputed from (3) into which the previously obtained value of $\hat{\lambda}$ has been substituted. Since the process of iteration will almost assuredly converge for any initial value of $\hat{\alpha}$ [3], selection of a starting $\hat{\alpha}$ is not of great importance.

If, as frequently is the case, empirical human performance task times are neither available nor possible to obtain (for reasons of cost or time constraints, or because the task has been conceptually defined but not yet operationalized), the distribution parameters must be based on judgments. Experts can generally provide estimates of means and standard deviations for performance times, since these parameters are directly meaningful. This is not the case for α and λ, the Weibull parameters. The following procedure provides a way to estimate these parameters from a known mean and standard deviation.

The mean and standard deviation for the Weibull distribution are given by [4]

$$\mu = \Gamma\left(\frac{1}{\alpha}+1\right)\lambda^{-1/\alpha} \tag{4}$$

$$\sigma = \left\{\lambda^{-2/\alpha}\left[\Gamma\left(\frac{2}{\alpha}+1\right) - \left(\Gamma\left(\frac{1}{\alpha}+1\right)\right)^2\right]\right\}^{1/2}. \tag{5}$$

Rearranging,

$$\alpha = \frac{\ln\lambda}{\ln\left[\mu/\Gamma\left(\frac{1}{\alpha}+1\right)\right]} \tag{6}$$

and

$$\lambda = \exp\left[-\frac{\alpha}{2}\left[\ln\sigma^2 - \ln\left(\Gamma\left(\frac{2}{\alpha}+1\right) - \left\{\Gamma\left(\frac{1}{\alpha}+1\right)\right\}^2\right)\right]\right]. \tag{7}$$

By selecting a starting value, $\hat{\alpha}$, and using the expert estimate for σ in (7), an estimate of $\lambda, \hat{\lambda}$, is obtained. Using the current estimates of $\hat{\alpha}$ and $\hat{\lambda}$, as well as the expert estimate of μ in 6, a new estimate of $\hat{\alpha}$ is found. This procedure is repeated until a desired accuracy is attained.

Results

Performance time data from seventeen tasks were collected. The tasks were components of a computer-based game which involved two human operators using a graphics cathode ray tube (CRT) terminal with keyboard and joystick (Tektronix 4051 graphics computer), an electronic timer, and manual recording equipment. Tasks included reading aloud numeric and short phrase information, keying in short alpha or numeric data strings, using the joystick to match a fixed point on the graphics CRT, deciding among up to three given alternatives, reading information from an electronic timer, entering time or short numeric data manually in a log, and locating information in a manual. Both manual and cognitive tasks are included. The complete game and tasks are described in Berry [5]. Four pairs of operators, all college students familiar with the computer equipment employed, performed the game. The students, two females and six males, ranging in age from 20 to 34, were videotaped, and performance times collected from the videotapes. Because some of the tasks were performed much more often than others, the total number of observations varied from 13 for task three to 85 for task one.

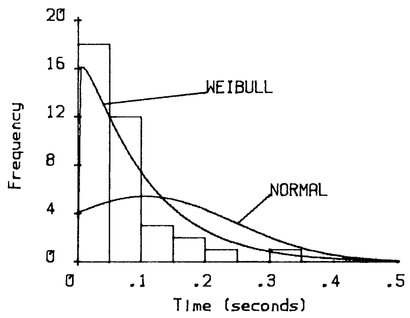

Fig. 1. Comparison of Weibull and Gaussian distributions with observed performance times.

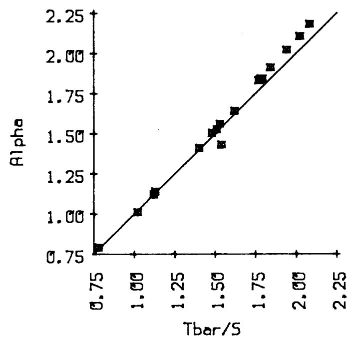

Fig. 2. Final estimation of α versus initial estimate of $\bar{t}/s$.

The sample mean and standard deviation were computed for each task and the Weibull parameters α and λ determined from (2) and (3) (this was possible since the raw data were available). Fig. 1 shows the Gaussian and Weibull distributions superimposed on the raw data for a sample task, task six. This figure illustrates one natural advantage of the Weibull over the Gaussian distribution in representing human performance times: the Weibull distribution is nonnegative. Gaussian distributions are not so restricted, of course, and when used in a Monte Carlo simulation to represent performance times, must be artificially truncated.

The Weibull parameters were recomputed using only the sample mean and standard deviations and (6) and (7). The iterative procedure using these two equations was very sensitive to the starting value of $\hat{\alpha}$ selected. Using only the values of the sample mean and standard deviation, the ratio $\bar{t}/s$ was used as an estimator of α, the "shape" parameter of the Weibull distribution. As can be seen from Fig. 2, this provided an excellent first approximation which very closely matched the final α value.

Since it would normally only be necessary to use this procedure when raw data are not available, and the mean and standard deviation are estimations of experts, a stringent convergence criterion was not used for the iteration procedure. The procedure was considered to be converged when two successive values of $\hat{\alpha}$ and of $\hat{\lambda}$ each differed by no more than three percent.

Table 1 contains the values of α and λ as computed from the raw data, using (2) and (3) and as computed from the sample mean and standard deviation using (6) and (7). The maximum relative difference between α values computed for the same task is for task six (25.3 percent) and between λ values is for task nine (42.7 percent). Despite these apparently wide differences, the graphs of the two Weibull distributions for these (as for the other) tasks are quite similar.

Goodness of Fit

A chi-square goodness-of-fit test was performed for each of the seventeen tasks, with the Gaussian, Weibull from raw data, and Weibull from sample mean and standard deviation. At the five percent level of confidence, the Gaussian distribution provided a satisfactory fit to the data in only three of the 17 cases; the Weibull from raw data was satisfactory in 13 out of 17 cases; and, the Weibull from sample mean and standard deviation was satisfactory in 15 out of 17 cases. Table II summarizes these results.

The Weibull distribution computed in either manner is a much better choice than the Gaussian distribution for modeling performance times for these 17 tasks. Besides providing a better fit to the data in so many instances, there were no tasks for which the Gaussian distribution was acceptable and the Weibull not. In addition, the method for estimating α and λ from the sample mean and standard deviation was at least as satisfactory as that which used all the available raw data. This method using (6) and (7) was unacceptable in only two instances, neither of which was acceptably modeled by the other methods either. (One of these tasks turned out to be essentially bimodal and so is not expected to be matched by these distributions.)

TABLE I
WEIBULL PARAMETERS COMPUTED FROM RAW DATA AND FROM SAMPLE MEAN AND STANDARD DEVIATION

	Weibull Parameters			
	Raw Data		Sample Mean and Standard Deviation	
Task	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
1	1.61	20.3	1.50	17.0
2	1.88	13.3	1.83	12.7
3	1.88	204.3	1.84	183.8
4	1.13	4.11	1.12	4.02
5	1.44	1.51	1.43	1.45
6	1.06	11.5	.792	7.66
7	1.63	48.1	1.53	36.8
8	1.38	11.2	1.41	12.0
9	1.23	34.1	1.01	19.5
10	2.25	1035.	2.18	857.9
11	1.77	33.4	1.64	26.4
12	0.99	2.84	1.14	3.19
13	1.62	67.1	1.56	57.1
14	2.17	4234.5	2.11	3265.6
15	1.99	57.8	1.91	51.6
16	2.09	19.6	2.02	17.2
17	1.82	99.3	1.84	104.8

TABLE II
CHI-SQUARE GOODNESS-OF-FIT TEST FOR 17 TASKS AND THREE PROBABILITY DISTRIBUTIONS AT FIVE PERCENT LEVEL OF CONFIDENCE*

		Weibull	
Task	Normal	Raw Data	Mean and Standard Deviation
1	*		
2			
3			
4	*		
5	*	*	
6	*		
7	*		
8			
9	*	*	
10	*		
11	*		
12	*	*	*
13	*		
14	*	*	*
15	*		
16	*		
17	*		

*Reject the hypothesis that data fit the given distribution, at the five percent level of confidence.

Summary and Conclusions

The Gaussian distribution has been widely used to model human task performance times. Previous studies indicated that the Gaussian distribution was a poor model for human task performance time, and that the Weibull distribution was significantly more accurate. The results of this experiment with 17 tasks strongly supports both these positions.

To make use of the Weibull distribution practical a relatively simple method to estimate the Weibull parameters α and λ was used which requires only knowledge of (or estimates of) task performance time mean and standard deviation. The chi-square goodness-of-fit test showed the Weibull distribution derived in this manner to be at least equal in performance to the Weibull distribution derived directly from the complete raw data.

If actual performance data are available the maximum likelihood method (2) and (3) for estimating α and λ would be preferred. Not only does this procedure employ all the (unaggregated) data, but it is also computationally simpler and more robust. The computations used here were performed in Basic on a Tektronix 4051 graphics computer, and any microcomputer with Basic (or other high level language) capabilities would be equally satisfactory.

The second procedure (using (6) and (7)) is more sensitive, especially to the starting value chosen for α. It also requires the use of the gamma function, which is generally only available in larger computing facilities. The computations for this correspondence were performed on a Univac 1100/45 computer. Smaller computers could be used, provided a gamma function routine was available. However this procedure requires only values for the mean and standard deviation as input.

Although the choice of procedure for estimating α and λ depends on the data available, the choice of the Weibull over the Gaussian distribution for modeling human performance times is clearly in favor of the Weibull. Using this distribution in Monte Carlo simulations of human-machine systems will provide a more accurate representation of the performance times of the component tasks.

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A Mathematical Model for the Diffusion of Innovation

PETER MARKOWICH

***Abstract*—A model for the temporal diffusion of technological innovation is presented. This model is derived by describing mathematically the production behavior of technologies and the buying behavior of the average consumer. It has the structure of a three-dimensional dynamical system, and its state variables are the production quantities of the innovative and the old technology and the market share of the innovative technology in terms of sales. Unit prices and external capital flows of both technologies are regarded as external variables. The focus of the model is the exploration and description of the interaction of the technological and societal aspects which drive the diffusion process.**

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I. Introduction

Modeling the diffusion of technological innovation has been a research topic for many years. Since the evolution of a new technology is an extremely complex process depending on various economical, environmental, and political parameters [1], [2] and on the kind of innovation we are dealing with, there is a large diversity of models which differ in methodology, goal, and focus. Used methodologies range from verbal argumentation to high level mathematical sophistication. The goals vary from pure description to policy orientation using optimization techniques [3], [4]. An overview of used approaches is given in [5].

There are models describing the spatial spread of innovation which regard the spread of information as the basic factor for the spatial diffusion. These models take into account geographical factors which may very well inhibit or favor the spread of information. Some of them also describe the temporal diffusion of a new technology. The basic ideas of spatial and spatial-temporal diffusion models are explained in [6]-[9].

An important temporal diffusion model is the well-known empirical Fisher and Pry logistic substitution model [10] which is based on the observation that market shares of (successful) technologies admit roughly an s-shaped pattern. This leads to the model equation

$$\dot{F} = cF(1 - F), \quad F(t_0) = F_0$$

where $F = F(t)$ represents the market share (in terms of production) the new technology has obtained at time t and the parameters F_0, c have to be estimated from historical data. Many investigations were carried out in order to refine the Fisher-Pry model [11]-[14]. These more sophisticated empirical models tried to do better curve-fitting to historical data and, therefore, did not give much information on the underlying system.

Recently, Peterka [15] used a more systematic and formal approach to investigate the competition of n technologies for the same market. Using basic economic considerations which describe the supply (or technological) aspect of the diffusion process, he derived a model which reduces to the Fisher and Pry model in the case of two competing technologies. This model was extended by Spinrad [16].

Also the demand (or societal) aspect of the (temporal) diffusion process has been modeled extensively. Mansfield [17] pointed out the importance of imitation for the diffusion and Stover [18] used these ideas in setting up a decision-theoretical diffusion model. Peterka and Fleck *et al.* [19] used similar techniques in analyzing the buying behavior of the average consumer. It turns out that the model they derived is qualitatively consistent with the Fisher-Pry model. The adoption behavior of potential customers has been investigated extensively using stochastic-decision-theoretical methods [20]-[22]. The models described in [23] particularly focus on the interaction of adopters and non-adopters and on the impact of advertisement on the adoption process. However not much work has been done in order to model the interaction of supply and demand aspect, which together drive the diffusion process (one model having features of both aspects can be found in [24]).

A differential equation model with production quantities and market shares in terms of sales as state variables is derived here by taking into account basic economic considerations determin-