TABLE I
NUMBER OF KTH-ORDER ZERO-DISPARITY CODEWORDS VERSUS CODEWORD LENGTH n AND K

n	K = 1	2	3	4
4	2	0	0	0
8	8	2	0	0
12	58	2	0	0
16	526	14	2	0
20	5448	48	0	0
24	61108	592	16	0
28	723354	2886	0	0
32	8908546	34888	78	2

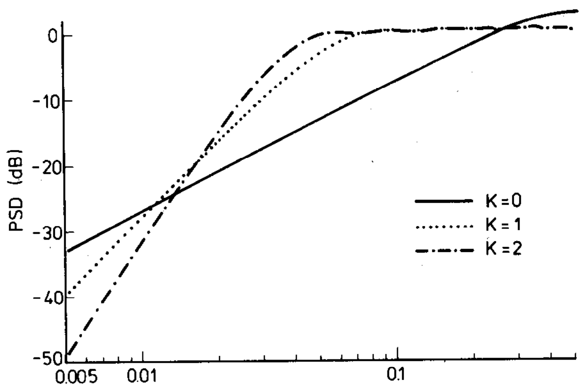

Fig. 1. Power density functions of codes with $K = 0, 1$, and 2 with rate 0.5.

proximately 1/2. Fig. 1 shows the spectra of these codes. Notice that for increasing values of K we obtain a more severe suppression of the power at very low frequencies.

III. The Minimum Distance of Higher Order Constrained Codes

Higher order constrained codes can also be used for error-correcting and -detecting purposes. In particular we will show that the minimum distance of a binary Kth-order zero-disparity code is at least $2(K+1)$. To that end let S be the set of all codewords of a binary Kth-order zero-disparity code of length n (n has to be a multiple of four, because otherwise S is empty). Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two different elements in S. Let $\boldsymbol{e}$ be given by

$$\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{x}. \tag{11}$$

Then it is easily seen that $e_i \in \{-2,0,2\}$ for all $1 \le i \le n$. As both $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfy (5), it readily follows that

$$\sum_{i=1}^{n} i^k e_i = 0, \qquad k \in \{0,1,2,\cdots,K\}. \tag{12}$$

In particular it follows that the number of occurrences of -2 in the vector $\boldsymbol{e}$ equals the number of occurrences of $+2$. Let $i_1,\cdots,i_q$ be the indices i for which $e_i = -2$ and let $j_1,\cdots,j_q$ be the indices i for which $e_i = +2$. Then we have to show that $q \ge K+1$. From (12), it follows that

$$\sum_{l=1}^{q} i_l^k = \sum_{l=1}^{q} j_l^k, \qquad k \in \{0,1,2,\cdots,K\}. \tag{13}$$

It is not difficult to show that for $q \le K$ it follows that $\{i_1,\cdots,i_q\} = \{j_1,\cdots,j_q\}$ [11, ch. 21]. However, since the two sets must be disjoint, we have shown that $q \ge K+1$. Hence the minimum distance of a Kth-order zero-disparity code is at least $2(K+1)$.

In particular, the properties of the $K = 1$, $n = 16$ code compare favorably with earlier codes. Ferreira [12] showed on an ad hoc basis the feasibility of a rate 9/16, minimum distance $d = 4$ dc-balanced code. From Table I we conclude that there are 526 first-order zero-disparity codewords, sufficient for a rate 9/16. Moreover, it is straightforward to show that if we omit from this code the lexicographically smallest and largest codeword, we obtain a first-order zero-disparity code that has maximum runlength (i.e., the maximum number of consecutive like symbols) equal to six. The cardinality of the new set is 524. In [8], a simple coding and decoding method for first-order zero-disparity codewords is presented.

IV. Conclusion

A new class of dc-constrained channel codes having higher order spectral zeros at dc was presented. The power spectral density function of these codes has, besides zero power at zero frequency, all low-order derivatives up to $2K+1$ equal to zero at $\omega = 0$. The additional constraints result in a higher rejection of the components in the low-frequency range than is possible with classical dc-balanced codes. Moreover, we have shown that the minimum distance of a Kth-order zero-disparity code is at least $2(K+1)$.

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Corrections and Additions to "Error Recovery for Variable Length Codes"

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***Abstract*—An error in the subject article is corrected. Additionally, a method of calculating the standard deviation of the expected span of errors is presented. For four example codes, the standard deviation of the subject article's reduced model is compared to the standard deviation of the subject article's complete model.**

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I. Corrections

In the above paper,[1] Section II, the reduction of the error state diagram is incorrect. The algebraic expressions $G(z)$ and $G'(z)$ are for code 1 while the diagram in Fig. 1 is for code 2. The correct probability transfer function can be shown by both Mason's gain formula [1] and by brute matrix reduction to be

$$G(z) = [71z^3 - 650z^2 + 700z]/11[11z^2 - 100z + 100].$$

The probability of eventual synchronization, $G(1)$, is still 1.0. The derivative of the correct probability generating function is

$$G'(z) = [781z^4 - 14200z^3 + 78600z^2 - 130000z + 70000] / 11[11z^2 - 100z + 100]^2,$$

therefore, the expected span of errors, $Es = G'(1) = 471/121 = 3.8926$ which is agreement with Maxted and Robinson's Table VI, code 2.

The probability generating function $G(z)$ and probability generating function's derivative $G'(z)$ presented by Maxted and Robinson are derived from the error state diagram for their code 1. This is shown in Tables I–IV and Fig. 1 of this correspondence.

A final correction to Maxted and Robinson may be offered. The expression for Es for the unstable family should have k replaced by $k-1$ and should read

$$2^{k-1} - [(k-2)/2] - [k/(2^{k+1}-2)].$$

TABLE I
INITIAL ENTRY PROBABILITIES FOR SINGLE BIT INVERSION ERROR, CODE 1

Codeword	Bit Inverted	Resultant State	Probability
01	1	S	4/22
01	2	S	4/22
00	1	10	2/22
00	2	S	2/22
11	1	S	2/22
11	2	10	2/22
100	1	0	1/22
100	2	0	1/22
100	3	S	1/22
101	1	1	1/22
101	2	1	1/22
101	3	S	1/22

Initial Entry Probabilities

$P(I\,S) = 4/22 + 4/22 + 2/22 + 2/22 + 1/22 + 1/22 = 7/11$

$P(I\,10) = 2/22 + 2/22 = 2/11$

$P(I\,0) = 1/22 + 1/22 = 1/11$

$P(I\,1) = 1/22 + 1/22 = 1/11$

TABLE II
TRANSITION PROBABILITIES FOR STATE 10, CODE 1

Error State	Codeword Appended	Resultant Bit String	Decoded Symbol and Error State
10	01	1001	D + 1 state
10	00	1000	D + 0 state
10	11	1011	E + 1 state
10	100	10100	EB + S state
10	101	10101	EA + S state

[1] J. C. Maxted and J. P. Robinson, *IEEE Trans. Inform. Theory*, vol. IT-31, pp. 794–801, Nov. 1985.

TABLE III
TRANSITION PROBABILITIES, NEXT DECODER STATE GIVEN AN ERROR STATE AND A FOLLOWING CODEWORD, CODE 1

Codeword	Code Probability	Error States 10	0	1
01	0.4	1	1	S
00	0.2	0	0	S
11	0.2	1	1	1
10	0.1	S	S	S
101	0.1	S	S	S

TABLE IV
TRANSITION PROBABILITY MATRIX FOR CODE 1

Current State	Next State S	0	1	10
I	0.6364	0.0909	0.0909	0.1818
0	0.2	0.2	0.2	0
1	0.8	0	0.2	0
10	0.2	0.2	0.6	0
S	1.0	0	0	0

TABLE V
STANDARD DEVIATION OF THE RECOVERY SPAN Es FOR SINGLE BIT INVERSION FAULTS

Code	Complete Model	Reduced Model	Pr
1	1.0955	1.2856	0.5000
2	6.5655	5.8900	0.1258

TABLE VI
STANDARD DEVIATION OF THE RECOVERY SPAN Es FOR SINGLE BIT INVERSION FAULTS OF EQUIVALENT HUFFMAN CODES

Code	Complete Model	Reduced Model	Pr
13	0.6086	0.6085	0.9000
14	5.8924	5.8610	0.1526

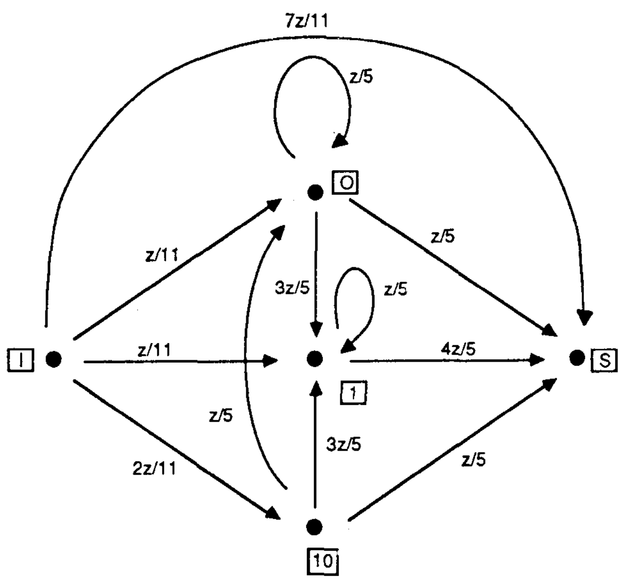

Fig. 1

II. Additions

By using the properties of factorial moments [2], it is possible to derive the standard deviation (SD) of the expected span of errors:

$$\operatorname{var}(Es) = G''(z) + G'(z) - (G'(z))^2;\ z = 1$$

$$\operatorname{SD}(Es) = [\operatorname{var}(Es)]^{1/2}.$$

The second derivative of codes 1 and 2's transfer functions are

$$G''(z) = (1750 + 5050z)/11(5 - z)^4$$

and

$$G''(z) = [3200z^3 + 30000z^2 - 360000z + 1000000] /11(11z^2 - 100z + 100)^3,$$

respectively.

Evaluated at $z = 1$, these second derivatives equal 2.4148 and 45.9805, respectively. The second derivative of Maxted and Robinson's reduced model probability transfer function, evaluated at $z = 1$, is

$$G''(1) = 2[1 - Ps]/Pr^2.$$

By substituting into the above expression the Pr values for codes 1 and 2 that are listed in Maxted and Robinson's Table VIII, and by noting that $Ps = 7/11 = 0.6364$ for both codes, it can be shown that $G''(1) = 2.9091$ and 45.9334 for the reduced model of codes 1 and 2, respectively.

With the values of $G''(1)$ for the complete and reduced model, it is now possible to compute the standard deviation for Maxted and Robinson's codes 1 and 2 (Table V of this correspondence). The standard deviation for Maxted and Robinson's special case codes, when k is large, are 0.5 and 2^{k-1} for the stable and unstable codes, respectively. The usefulness of Maxted and Robinson's reduced state model is very clear. By being able to derive Es and SD(Es) quickly it is possible to compare equivalent Huffman codes [3] for their error propagation properties. Table VI shows the results of comparing Maxted and Robinson's codes 13 and 14.

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