Using (4), a proof of property 3 given in [2] has been found and is detailed in the next section. A different approach has been necessary to establish the validity of property 4.

Proof of Property 3

This property is

$$x_{in} = |x_{i1}| = \binom{n-1}{i-1}.$$

Equation (4) gives

$$x_{in} = \sum_{k=1}^{i} (-1)^{i+k} 2^{k-1} \binom{n-k}{i-k}\binom{n-1}{k-1}. \quad (5)$$

Using the identity

$$\binom{n-k}{i-k}\binom{n-1}{k-1} = \binom{n-1}{i-1}\binom{i-1}{k-1},$$

(5) becomes

$$x_{in} = (-1)^{i-1}\binom{n-1}{i-1}\sum_{k=1}^{i}(-1)^{k-1} \cdot 2^{k-1}\binom{i-1}{k-1}. \quad (6)$$

The terms behind the summation sign in (6) are recognized as the coefficients in the binomial expansion of $(s-2)^{i-1}$, which sum to $(-1)^{i-1}$. Property 3 is thus established.

Proof of Property 4

This property states that

$$x_{ij} = x_{i,j+1} - x_{i-1,j+1} - x_{i-1,j}, \quad (i = 2, \cdots, n \quad j = 1, \cdots, n-1).$$

The upper limit of j was inadvertently given as n in [2]—this case is, of course, covered by property 3.

To prove this property, it has been convenient to describe ${}_1\boldsymbol{G}$ and $\boldsymbol{F}_1$ by the following generating rules, taken from [1]:

$$({}_1\boldsymbol{G})_{ij} = ({}_1\boldsymbol{G})_{i+1,j+1} - ({}_1\boldsymbol{G})_{i,j+1},$$

with

$$({}_1\boldsymbol{G})_{nj} = (-1)^{n-j}$$

and

$$({}_1\boldsymbol{G})_{i,n+1} \triangleq 0; \quad (7)$$

$$(\boldsymbol{F}_1)_{ij} = (\boldsymbol{F}_1)_{i,j-1} + 2(\boldsymbol{F}_1)_{i-1,j-1}$$

with

$$(\boldsymbol{F}_1)_{1j} = 1$$

and

$$(\boldsymbol{F}_1)_{0j} \triangleq 0. \quad (8)$$

In (7) and (8) the limits on i and j are 1 to n.

It is convenient to take the four elements involved in property 4 in two groups of two. Thus, we have

$$x_{i,j+1} - x_{ij} = \sum_{k=1}^{n} ({}_1\boldsymbol{G})_{ik}[(\boldsymbol{F}_1)_{k,j+1} - (\boldsymbol{F}_1)_{kj}].$$

Applying (8) to the bracketed terms in this expression, we obtain

$$x_{i,j+1} - x_{ij} = 2\sum_{k=1}^{n} ({}_1\boldsymbol{G})_{ik}(\boldsymbol{F}_1)_{k-1,j}. \quad (9)$$

We also have

$$x_{i-1,j+1} + x_{i-1,j} = \sum_{k=1}^{n} ({}_1\boldsymbol{G})_{i-1,k}[(\boldsymbol{F}_1)_{k,j+1} + (\boldsymbol{F}_1)_{kj}].$$

Again applying (9), we obtain

$$x_{i-1,j+1} + x_{i-1,j} = 2\sum_{k=1}^{n} ({}_1\boldsymbol{G})_{i+1,k}[(\boldsymbol{F}_1)_{kj} + (\boldsymbol{F}_1)_{k-1,j}]. \quad (10)$$

The extension of the upper limit of k to n in (9) and (10) has been done for notational convenience in the sequel. Combining (9) and (10) and applying (7) in the form

$$({}_1\boldsymbol{G})_{i,k} - ({}_1\boldsymbol{G})_{i-1,k} = ({}_1\boldsymbol{G})_{i-1,k-1}$$

results in

$$\tfrac{1}{2}(x_{i,j+1} - x_{ij} - x_{i-1,j+1} - x_{i-1,j}) = \sum_{k=1}^{n} ({}_1\boldsymbol{G})_{i-1,k-1}(\boldsymbol{F}_1)_{k-1,j} - \sum_{k=1}^{n} ({}_1\boldsymbol{G})_{i-1,k}(\boldsymbol{F}_1)_{kj}. \quad (11)$$

Since we have $(\boldsymbol{F}_1)_{0j} \triangleq 0$ and $(\boldsymbol{F}_1)_{nj} = 0$ for $j = 1, \cdots, n-1$, the right-hand side of (11) may be written as

$$\sum_{k=2}^{n} ({}_1\boldsymbol{G})_{i-1,k-1}(\boldsymbol{F}_1)_{k-1,j} - \sum_{k=1}^{n-1} ({}_1\boldsymbol{G})_{i-1,k}(\boldsymbol{F}_1)_{kj} = 0.$$

Property 4 is thus proved.

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Active Transmission Lines

***Abstract*—Active transmission lines have historical significance and current importance, model many physical processes, and have characteristics unattainable with passive lines. These points are discussed with respect to recent literature.**

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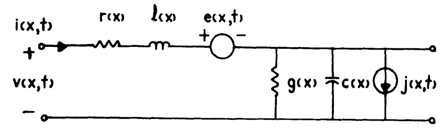

Fig. 1. Active transmission line differential section.

The active transmission line is a generalization of the passive transmission line, and is characterized by the parameters

- r: the series resistance per unit length (ohms/meter)
- l: the series inductance per unit length (henries/meter)
- e: the series voltage per unit length (volts/meter)
- g: the shunt conductance per unit length (mhos/meter)
- c: the shunt capacitance per unit length (farads/meter)
- j: the shunt current per unit length (amperes/meter).

The differential section of the active transmission line is shown in Fig. 1. Parameters r, l, g, and c are time invariant and may be functions of distance. The distributed voltage and current sources are independent and/or dependent sources which may have both time and spatial dependencies.

The equations relating voltage and current along the active line are readily expressed as

$$-\frac{\partial v(x,t)}{\partial x} = e(x,t) + r(x)i(x,t) + l(x)\frac{\partial i(x,t)}{\partial t} \quad (1)$$

$$-\frac{\partial i(x,t)}{\partial x} = j(x,t) + g(x)v(x,t) + c(x)\frac{\partial v(x,t)}{\partial t}. \quad (2)$$

The two-port response of active lines when $e(x, t)$ and $j(x, t)$ are independent sources has been examined elsewhere [1]. It was shown that such sources could be removed when accompanied by appropriate changes in port conditions.

This is not the case when $e(x, t)$ and $j(x, t)$ are dependent sources; here the port conditions remain invariant but the two-port parameters of the active line are modified. Various dependencies may be assumed depending on the problem to be analyzed. Many problems require linear dependencies where

$$1)\ e(x,t) = K_i(x)i(x,t) \quad (3)$$

or

$$2)\ e(x,t) = K_v(x)v(x,t) \quad (4)$$

and

$$3)\ j(x,t) = L_v(x)v(x,t) \quad (5)$$

or

$$4)\ j(x,t) = L_i(x)i(x,t). \quad (6)$$

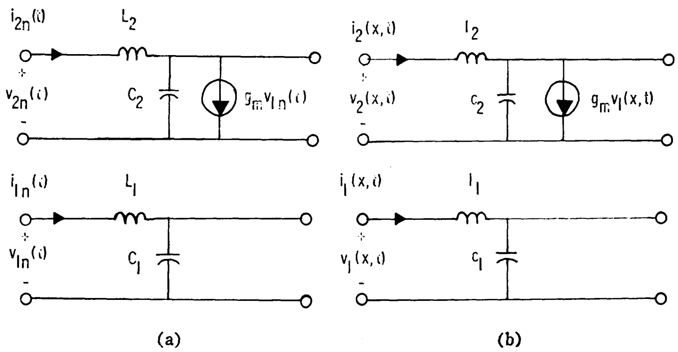

Fig. 2. Equivalent circuit of (a) the nth section of an artificial distributed amplifier, and (b) the differential section of a distributed amplifier.

K_i, K_v, L_v, and L_i are real-valued functions. The analysis of such active lines has been outlined elsewhere [2].

From one point of view, active transmission lines are natural outgrowths of distributed amplifiers, although active lines are required for their characterization. Artificial distributed amplifiers, patented by Percival in 1936, consisted of two artificial delay lines unilaterally actively coupled using pentodes. The equivalent circuit of a single section is shown in Fig. 2(a). Transistor artificial distributed amplifiers have also been investigated [3], [4], but with less successful results due to significant capacitive couplings and the low input and output impedances of transistors. MOSFET (and to a lesser extent JFET) transistors may be used to advantage because of their small capacitive couplings and high input and output impedances (e.g., see [5]).

In an effort to realize true distributed amplifiers, and amplifiers capable of operating at higher frequencies, McIver proposed the traveling-wave transistor [6]. The equivalent circuit of a differential section is shown in Fig. 2(b); two delay lines are unilaterally actively coupled using strip-type transmission lines on doped semiconductors. Others have considered the effect of capacitive coupling between lines [7], [8]. For both artificial and "nonartificial" distributed amplifiers, the phase velocities of the two lines are matched. This requirement and the presence of capacitive coupling between lines may be eliminated when "single-line" distributed amplifiers, such as the active transmission line, are utilized.

Active transmission lines have not heretofore been extensively considered. Recently the class of artificial active lines having $j(x, t) = L_v v(x, t)$ for L_v constant have been examined [9]. Such characteristics can be obtained using the small-signal negative conductance of tunnel diodes. Extensive bulk-effect investigations devoted to Gunn and IMPATT diode traveling wave amplifiers are leading to active line realizations having $j(x, t) = L_v(x)v(x, t)$ [10].

Active transmission lines model many physical processes [11]. These include heat conduction in an internally heated material, a vibrating string pressure waves in gas, neutron diffusion and fission, and semiconductor photodetection [12]. Problems involving diffusion of solutes undergoing ionization and deionization in a solvent have similar analogs. Neuristors have been modeled assuming $j(x, t) = L_v(v)v(x, t)$ where $L_v(v)$ is a nonlinear function of voltage (e.g., see [13]).

It is an interesting and unifying result that transmission line response to independent distributed sources (e.g., initial condition response), artificial and nonartificial distributed amplifiers, a class of single-line distributed (or lumped) amplifiers, and analogs of many physical processes may be characterized and described using active transmission lines.

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An Inversion Method for Transforms with Multiple Poles

***Abstract*—Algorithms suitable for digital computation are developed for obtaining the partial fraction expansion of a Laplace transform with multiple poles. The method is a generalization of a known method in which the usual differentiation technique is avoided.**

A method for obtaining the partial fraction expansion of a Laplace transform with multiple poles has been outlined by Kuo.[1] This method was suggested to him by the late Prof. Leonard O. Goldstone of the Polytechnic Institute of Brooklyn, and is very appealing because it replaces the usual differentiation technique by a long division of two polynomials written in ascending powers. Prof. Kuo's presentation indicates the procedure but does not calculate the coefficients explicitly. The following is a general development in which the coefficients are displayed for the general case in a form suitable for digital computation.

Let the transform with a pole at $s=a$ of multiplicity r be given by

$$H(s) = \frac{P(s)}{(s-a)^r Q(s)} \tag{1}$$

where

$$P(s) = \sum_{k=0}^{m} \alpha_k s^k$$
$$Q(s) = \sum_{k=0}^{n} \beta_k s^k. \tag{2}$$

Letting $s=p+a$ and dividing $P(p+a)$ by $Q(p+a)$, with both polynomials written in ascending powers of p, results in

$$\frac{P(p+a)}{Q(p+a)} = \frac{\sum_{k=0}^{n} a_k p^k}{\sum_{k=0}^{m} b_k p^k} \tag{3}$$

$$= \sum_{k=0}^{r-1} c_k p^k + \frac{p^r \sum_{k=0}^{N} d_k p^k}{\sum_{k=0}^{m} b_k p^k} \tag{4}$$

where $N = \max(n-r,\ m-1)$. The division is terminated and the remainder is computed when the power $r-1$ of p is obtained in the quotient. The coefficients a_k and b_k are the coefficients of p^k under the translation $s=p+a$, and the coefficients c_k and d_k are obtained as a result of the division. It should be noted that b_0 is not zero because $Q(a) \neq 0$, and c_0 is not zero because it is the coefficient of $(s-a)^{-r}$ in the partial fraction expansion of $H(s)$. The equality of (3) and (4) results in

$$\sum_{k=0}^{n} a_k p^k = \left(\sum_{k=0}^{r-1} c_k p^k\right)\left(\sum_{k=0}^{m} b_k p^k\right) + p^r \sum_{k=0}^{N} d_k p^k.$$

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